PACIFIC JOURNAL OF MATHEMATICS
Vol. 74, No 1, 1978

# MOMENT SEQUENCES OBTAINED FROM RESTRICTED POWERS

RODNEY NILLSEN

**Let $(m_n)_{n=1}^\infty$ be an increasing divergent sequence of positive numbers. Then we are interested in characterising those sequences $(\alpha_n)_{n=1}^\infty$ for which $\alpha_n = \int_0^1 x^{m_n} f(x)dx$ for $n = 1, 2, \cdots$ and some $f \in L^2([0, 1))$. It is shown that if $(m_n)_{n=1}^\infty$ diverges sufficiently rapidly, then $\sum_{n=1}^\infty |\alpha_n|^2 < \infty$ if and only if $\alpha_n = \sqrt{m_n} \int_0^1 x^{m_n} f(x)dx$ for $n = 1, 2, \cdots$ and some $f \in L^2([0, 1))$. It is also shown that if $(m_n)_{n=1}^\infty$ is a lacunary sequence of integers then the Hilbert subspace of $L^2([0, 1))$ generated by the functions $x^{m_n}$ $(n = 1, 2, \cdots)$ has a reproducing kernel.**

1. Introduction. Let $C([0, 1])$ denote all the complex valued continuous functions on $[0, 1]$. If $\alpha \geqq 0$ let $e_\alpha$ be the function in $C([0, 1])$ given by $x \to x^\alpha$. Throughout the paper, $S$ will be a given sequence $(m_n)_{n=1}^\infty$ of positive real numbers so that $0 \leqq m_1 < m_2 < m_3 < \cdots$ and $\lim_{n\to\infty} m_n = \infty$. The subspace of $C([0, 1])$ obtained by taking the uniform closure of the vector space generated by $\{e_{m_i}: i = 1, 2, \cdots\}$ will be denoted by $M(S)$.

A classical result due to Müntz and Szasz (see, for example, [2] p. 272) says that if $m_1 = 0$, $M(S) = C([0, 1])$ if and only if $\sum_{i=2}^\infty 1/m_i = \infty$. In the case where $\sum_{i=2}^\infty 1/m_i < \infty$ it can be shown that $e_\alpha$ is not in $M(S)$ unless $\alpha = m_i$ for some $i$ ([7], p. 305). It follows from this, using the Hahn-Banach theorem, that if $\sum_{i=2}^\infty 1/m_i < \infty$ and if $j$ is given, then there is a measure $\mu$ so that $\int_0^1 x^{m_i} d\mu(x) = 0$ if $i \neq j$ and $\int_0^1 x^{m_j} d\mu(x) \neq 0$. Among the results of this paper, it is shown that if $S$ satisfies a certain stronger condition than $\sum_{i=2}^\infty 1/m_i < \infty$, the measure $\mu$ can be chosen to be absolutely continuous with respect to Lebesgue measure and at the same time be supported by $[0, \delta]$, where $\delta > 0$ is preassigned.

We also let $L^2$ be the Hilbert space of all square integrable functions on $[0, 1)$ and we denote by $A(S)$ the subspace of $L^2$ obtained by taking the closure, in the $L^2$ norm, of the vector space generated by $\{e_{m_n}: n = 1, 2, \cdots\}$. Any function in $M(S)$, if restricted to $[0, 1)$, belongs to $A(S)$. $S = (m_n)_{n=1}^\infty$ is said to be *lacunary* (or a *Hadamard set*) if there is $\alpha > 1$ so that $m_{n+1} > \alpha m_n$, for $n = 1, 2, 3, \cdots$. Lacunary sets are well known in complex analysis ([8], pp. 314-316) and in harmonic analysis ([7], pp. 100-118). We show that if $S$ is lacunary then $A(S)$ has a reproducing kernel.

Finally we shall be concerned with characterizing those sequences

$(\alpha_n)_{n=1}^{\infty}$ which are of the form $\alpha_n = \int_0^1 f(x)x^{m_n}dx$ for $n = 1, 2, \cdots$, for some $f \in L^2$. As noted in [3], vol. II, pp. 139-140, any such sequence $(\alpha_n)_{n=1}^{\infty}$ belongs to $\ell^2$, the Hilbert space of all square summable sequences on the positive integers. It is shown that, regardless of what $S$ is, it is never possible to obtain all of $\ell^2$ simply by taking different functions $f$ in $L^2$. However it *is* possible to obtain all of $\ell^2$ in this way if we consider instead sequences $(\alpha_n)_{n=1}^{\infty}$ of the form $\alpha_n = \sqrt{m_n}\int_0^1 x^{m_n}f(x)dx$, provided the sequence $S$ diverges rapidly enough.

The basic idea underlying a number of our results is that provided there are "sufficiently large" gaps between $m_i$ and $m_{i+1}$ for $i = 1, 2, \cdots$, then the functions $\{e_{m_i}: i = 1, 2, \cdots\}$ are "sufficiently orthogonal" for them to be treated (in a certain sense) as orthogonal functions.

2. **Properties of $A(S)$.** In $L^2$ the Gram-Schmidt process can be applied to the functions $e_{m_1}, e_{m_2}, \cdots$ to obtain an orthonormal sequence $p_1, p_2, \cdots$. Of course we can write $p_n = \sum_{j=1}^{n} a_{nj}e_{m_j}$ or

$$p_n(x) = \sum_{j=1}^{n} a_{nj}x^{m_j}\,, \qquad \text{for} \quad 0 \leqq x < 1 \quad \text{and} \quad n = 1, 2, 3, \cdots . \tag{2.1}$$

If the inner product in $L^2(0, 1)$ is denoted by $(\cdot, \cdot)$ then also we have

$$(p_n, p_m) = 0 \quad \text{if} \quad m \neq n\,, \quad \text{and} \quad 1, \text{ if} \quad m = n\,. \tag{2.2}$$

Then functions in $A(S)$ are precisely those functions of the form $\sum_{n=1}^{\infty} \alpha_n p_n$ for some sequence $(\alpha_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$, where the series is to be interpreted in terms of the $L^2$ norm. We shall assume that the Gram-Schmidt process has been carried out so that $a_{nn} > 0$ for all $n$, in which case the constants $a_{nj}$ in (2.1) are uniquely determined. If $S$ is a sequence of integers, it should be noted that the $p_n$ are polynomials and can be regarded as being defined on $D = \{\lambda: |\lambda| < 1\}$.

LEMMA 2.1.

$$a_{11} = \sqrt{2m_1 + 1} \quad \textit{and} \quad a_{21} = -\sqrt{2m_2 + 1}\left[\frac{2m_1 + 1}{m_2 - m_1}\right]. \tag{2.3}$$

*If $n > 2$ and $j < n$ we have*

$$a_{nj} = (-1)^{n+j}\sqrt{2m_n + 1}\left[\frac{2m_j + 1}{m_n - m_j}\right]\prod_{\substack{i=1 \\ i \neq j}}^{n-1}\left[\frac{m_i + m_j + 1}{|m_i - m_j|}\right]. \tag{2.4}$$

*If $n > 1$ we have*

$$a_{nn} = \sqrt{2m_n + 1} \prod_{i=1}^{n-1} \left[ \frac{m_n + m_i + 1}{m_n - m_i} \right]. \tag{2.5}$$

*Proof.* If $f_1, f_2, \cdots, f_n \in L^2([0, 1))$, let $g(f_1, f_2, \cdots, f_n)$ be the determinant of $((f_i, f_j))_{1 \leqq i, j \leqq n}$ (the Gramian determinant, see [2], p. 177). Then by [2], p. 183 we have

(2.6)

$$p_n = \frac{1}{\sqrt{g(e_{m_1}, e_{m_2}, \cdots, e_{m_n}) g(e_{m_1}, e_{m_2}, \cdots, e_{m_{n-1}})}} \begin{vmatrix} (e_{m_1}, e_{m_1}) & \cdots & (e_{m_n}, e_{m_1}) \\ \vdots & & \vdots \\ (e_{m_1}, e_{m_{n-1}}) & \cdots & (e_{m_n}, e_{m_{n-1}}) \\ e_{m_1} & \cdots & e_{m_n} \end{vmatrix}.$$

Since $(e_{m_i}, e_{m_j}) = 1/(m_i + m_j + 1)$, it is possible to use (2.6) to find the $a_{nj}$ and we find that (2.3), (2.4), and (2.5) are true. The calculation is tedious but straight forward and is similar to one used in one proof of the Müntz-Szasz theorem (see [2], pp. 270-271).

LEMMA 2.2. *$S$ is lacunary if and only if there is a number $C$ so that for all $n \geqq 2$,*

$$\left| \prod_{\substack{i=1 \\ i \neq j}}^{n} \left( \frac{m_i + m_j + 1}{m_i - m_j} \right) \right| \leqq C, \qquad \text{when} \quad j \leqq n. \tag{2.7}$$

*Proof.* Each term of the product in (2.7) is greater than 1, so that if (2.7) holds, $C > 1$. In this case, for $n > 1$ we will have $(m_n + m_{n-1} + 1) \leqq C(m_n - m_{n-1})$ so that $m_n \geqq (C + 1)/(C - 1) m_{n-1}$, and $S$ must be lacunary.

Conversely, if $S$ is lacunary and $m_1 > 0$, choose $\alpha > 1$ so that $m_{i+1} > \alpha m_i$ for $i = 1, 2, \cdots$. Then if $i > j$ we have

$$m_i > \alpha^{i-j} m_j. \tag{2.8}$$

If $x \geqq 0$ then $1 + x \leqq e^x$ so that if $j \leqq n - 1$,

$$\prod_{i=j+1}^{n} \left( \frac{m_i + m_j + 1}{m_i - m_j} \right) = \prod_{i=j+1}^{n} \left( 1 + \frac{2m_j + 1}{m_i - m_j} \right) \leqq e^{s_{j,n}}, \tag{2.9}$$

where

$$s_{j,n} = \sum_{i=j+1}^{n} \left( \frac{2m_j + 1}{m_i - m_j} \right).$$

If $j > 1$, we also have

$$\prod_{i=1}^{j-1} \left( \frac{m_i + m_j + 1}{m_j - m_i} \right) = \prod_{i=1}^{j-1} \left( 1 + \frac{2m_i + 1}{m_j - m_i} \right) \leqq e^{t_j}, \tag{2.10}$$

where

$$t_j = \sum_{i=1}^{j-1} \left( \frac{2m_i + 1}{m_j - m_i} \right) .$$

Now

$$\begin{aligned} s_{j,n} &= 2 \sum_{i=j+1}^{n} \left( \frac{1}{m_i/m_j - 1} \right) + \sum_{i=j+1}^{n} \left( \frac{1}{m_i - m_j} \right), \qquad \text{by (2.9)} , \\ &< 2 \sum_{i=j+1}^{\infty} \frac{1}{\alpha^{i-j} - 1} + \sum_{i=j+1}^{\infty} \frac{1}{m_j(\alpha^{i-j} - 1)} , \qquad \text{by (2.8)} , \\ &< 2 \sum_{i=1}^{\infty} \frac{1}{\alpha^i - 1} + \frac{1}{m_1} \left( \sum_{i=1}^{\infty} \frac{1}{\alpha^i - 1} \right) , \qquad \text{and} \end{aligned}$$

$$\begin{aligned} t_j &= 2 \sum_{i=1}^{j-1} \frac{1}{m_j/m_i - 1} + \sum_{i=1}^{j-1} \frac{1}{m_i(\alpha^{j-i} - 1)} , \qquad \text{by (2.8) and (2.10)} , \\ &< 2 \sum_{i=1}^{j-1} \frac{1}{\alpha^{j-i} - 1} + \sum_{i=1}^{j-1} \frac{1}{m_i(\alpha^{j-i} - 1)} , \\ &< 2 \sum_{i=1}^{\infty} \frac{1}{\alpha^i - 1} + \frac{1}{m_1} \sum_{i=1}^{\infty} \frac{1}{\alpha^i - 1} . \end{aligned}$$

These inequalities for $s_{j,n}$ and $t_j$ are sufficient to deduce that (2.7) holds if for $C$ we take

$$C = e^{2(2+1/m_1)\sum_{i=1}^{\infty} 1/(\alpha^i - 1)} .$$

If $m_1 = 0$ a similar argument suffices to deduce the conclusion.

THEOREM 2.3. *Let $S$ be lacunary, consist of integers and for $z_1, z_2 \in \boldsymbol{D}$ let*

$$K(z_1, z_2) = \sum_{n=1}^{\infty} p_n(z_1) p_n(z_2) . \tag{2.11}$$

*Then the series in* (2.11) *converges absolutely and uniformly on compact subsets of $\boldsymbol{D} \times \boldsymbol{D}$. Also $A(S)$ is a Hilbert space of analytic functions on* $[0, 1)$ *which has a reproducing kernel given by the restriction of $K$ to* $[0, 1) \times [0, 1)$.

*Proof.* Let $0 \leqq \delta < 1$ and consider $p_n(z)$ where $z \in \boldsymbol{D}$ and $|z| < \delta$. Let $C$ be chosen so that (2.7) holds and use (2.1), (2.3), (2.4), and (2.5) to obtain

$$\begin{aligned} |p_n(z)| &\leqq C\sqrt{2m_n + 1} \left( \sum_{j=1}^{n-1} \frac{2m_j + 1}{m_n - m_j} \delta^{m_j} + \delta^{m_n} \right) , \\ &\leqq C \frac{\sqrt{2m_n + 1}}{m_n} \left( \sum_{j=1}^{n-1} \frac{2m_j + 1}{1 - m_j/m_n} \delta^{m_j} + m_n \delta^{m_n} \right) , \end{aligned}$$

$$< C\frac{\sqrt{2+\frac{1}{m_n}}}{\sqrt{m_n}}\Big(\sum_{j=1}^{n-1}\frac{2m_j+1}{1-\alpha^{j-n}}\delta^{m_j}+m_n\delta^{m_n}\Big)\,,$$

$$< \frac{C_0}{\sqrt{m_n}}\Big(\sum_{j=1}^{\infty}(2m_j+1)\delta^{m_j}\Big)\,,$$

for some $C_0$, since $\lim_{n\to\infty} m_n = \infty$ .

Hence we have for all $k$,

$$\sum_{n=k}^{\infty}|p_n(z)|^2 \leqq C_0^2\Big(\sum_{j=1}^{\infty}(2m_j+1)\delta^{m_j}\Big)^2\Big(\sum_{n=k}^{\infty}\frac{1}{m_n}\Big) < \infty\,. \tag{2.12}$$

If we now use (2.12), which is true for all $|z| < \delta$, an application of Schwartz's inequality proves that the series in (2.11) has the stated convergence properties. We also see that if $\sum_{n=1}^{\infty}|\alpha_n|^2 < \infty$ and if we let $F(z) = \sum_{n=1}^{\infty}\alpha_n p_n(z)$ for $|z| < 1$, then this series converges absolutely and uniformly on compact subsets of $\boldsymbol{D}$ so that $F$ is analytic in $\boldsymbol{D}$. From this we see that $A(S)$ consists of analytic functions on $[0, 1)$ and that if $0 \leqq x < 1$ and $f \in A(S)$, $f(x) = \sum_{n=1}^{\infty}\alpha_n p_n(x)$, where $\alpha_n$ is the $n$th Fourier coefficient of $f$ with respect to the orthonormal system $(p_n)$. We now see that $K$ is a reproducing kernel for $A(S)$, for (2.12) shows that if $z \in \boldsymbol{D}$ then $x \to K(z, x)$ belongs to $A(S)$, and if $f = \sum_{n=1}^{\infty}\alpha_n p_n \in A(S)$, where $\sum_{n=1}^{\infty}|\alpha_n|^2 < \infty$, then

$$\begin{aligned}\int_0^1 K(z, x)f(x)dx &= \lim_{n\to\infty}\int_0^1\Big(\sum_{k=1}^{n}p_k(z)p_k(x)\Big)\Big(\sum_{k=1}^{n}\alpha_k p_k(x)\Big)dx\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\alpha_k p_k(z)\,, \qquad \text{by (2.2)}\,,\\ &= f(z)\,, \qquad \text{if} \quad z \in [0, 1)\,.\end{aligned}$$

REMARK. It should be noted that if $m_1 = 0$ and $\sum_{n=2}^{\infty}1/m_n = \infty$, then $A(S)$ does not have a reproducing kernel. This is because of the Müntz-Szasz theorem, which shows that in this case $A(S)$ will contain the restrictions to $[0, 1)$ of all functions in $C([0, 1])$. Hence, if $x \in [0, 1)$, the linear functional on $A(S)$ given by $f \to f(x)$ is not bounded, so that $A(S)$ does not have a reproducing kernel ([2], p. 317).

## 3. Moment sequences.

Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of complex numbers and let $f \in L^2([0, 1))$. Consider the condition

$$\alpha_n = \int_0^1 x^{m_n}f(x)dx\,, \qquad \text{for} \quad n = 1, 2, 3, \cdots\,. \tag{3.1}$$

THEOREM 3.1. *Let $(\alpha_n)_{n=1}^{\infty}$ be a given sequence of complex numbers. Then if $S$ is lacunary, a sufficient condition for* (3.1) *to hold for some $f$ in $L^2$ is that*

$$\sum_{n=1}^{\infty} m_n |\alpha_n| < \infty . \tag{3.2}$$

*This condition is not necessary.*

*If $S$ satisfies the condition*

$$\sum_{n=2}^{\infty} \frac{\left(\sum_{j=1}^{n-1} m_j\right)}{m_n} < \infty , \tag{3.3}$$

*then a sufficient condition for* (3.1) *to hold for some $f \in L^2$ is that*

$$\sum_{n=1}^{\infty} m_n |\alpha_n|^2 < \infty . \tag{3.4}$$

*If $S$ satisfies the condition*

$$\sum_{n=2}^{\infty} \frac{\left(\sum_{j=1}^{n-1} \sqrt{m_j}\right)^2}{m_n} < \infty , \tag{3.5}$$

*then* (3.4) *is a necessary and sufficient condition for* (3.1) *to hold for some $f \in L^2$.*

*Proof.* By virtue of [2], pp. 226–227, (3.1) holds for some $f \in L^2$ if and only if

$$\sum_{n=1}^{\infty} \left| \sum_{j=1}^{n} a_{nj}\alpha_j \right|^2 < \infty , \tag{3.6}$$

in which case $\sum_{n=1}^{\infty} (\sum_{j=1}^{n} a_{nj}\alpha_j) p_n$ will do for $f$.

If $S$ satisfies (3.3) or (3.5) then $S$ is lacunary so in any case we may choose $\alpha > 1$ so that (2.8) holds when $i > j$. Also, choose $C$ so that (2.7) holds and use (2.3), (2.4), and (2.5) to obtain for $n \geqq 2$,

$$\left| \sum_{j=1}^{n} a_{nj}\alpha_j \right| \leqq C\sqrt{2m_n + 1}\left( |\alpha_n| + \sum_{j=1}^{n-1} \frac{2m_j + 1}{m_n - m_j} |\alpha_j| \right) ,$$

so that

$$\left| \sum_{j=1}^{n} a_{nj}\alpha_j \right| \leqq \frac{C\sqrt{2m_n + 1}}{m_n}\left( m_n |\alpha_n| + \frac{\alpha}{\alpha - 1} \sum_{j=1}^{n-1} (2m_j + 1)|\alpha_j| \right) . \tag{3.7}$$

If (3.2) holds this shows that there is $F$ so that

$$\left| \sum_{j=1}^{n} a_{nj}\alpha_j \right| \leqq \frac{F}{\sqrt{m_n}} ,$$

for all $n \geqq 2$, so that (3.6) and hence (3.1) hold. (3.2) is not necessary since

$$1/(m_n + 1) = \int_0^1 x^{m_n} dx \ , \qquad \text{but} \ \sum_{n=1}^{\infty} m_n/(m_n + 1) = \infty \ .$$

Now assume that (3.3) holds. Because of (2.5) and (2.7), (3.4) is equivalent to the condition

$$\sum_{n=1}^{\infty} |a_{nn}\alpha_n|^2 < \infty \ . \tag{3.8}$$

Also the approach used to derive (3.7) shows that there is a constant $G$, depending only on $S$, so that

$$\left|\sum_{j=1}^{n-1} a_{nj}\alpha_j\right| \leqq \frac{G}{\sqrt{m_n}}\left(\sum_{j=1}^{n-1}(2m_j + 1)|\alpha_j|\right) , \quad \text{for } n \geqq 2 \ . \tag{3.9}$$

Now let (3.4) hold. An application of Schwartz's inequality shows that

$$\left|\sum_{j=1}^{n-1} m_j\alpha_j\right|^2 \leqq \left(\sum_{j=1}^{n-1} m_j\right)\left(\sum_{j=1}^{n-1} m_j|\alpha_j|^2\right) \leqq \left(\sum_{j=1}^{n-1} m_j\right)\left(\sum_{j=1}^{\infty} m_j|\alpha_j|^2\right) .$$

Since $S$ is lacunary and (3.3) holds we have, for some $J$, $\sum_{n=2}^{\infty}|\alpha_n| \leqq J\sum_{n=2}^{\infty} 1/\sqrt{m_j} < \infty$. These facts, together with (3.3), (3.8), and (3.9) imply that (3.6), and hence (3.1), hold (the latter for some $f \in L^2$).

Condition (3.5) is stronger than (3.3), so that if (3.5) holds then (3.4) implies (3.1). Conversely let (3.5) and (3.1) hold, the latter for some $f \in L^2$. An application of the Schwartz inequality to (3.1) shows that there is a constant $H$ so that $|\alpha_n| \leqq H/\sqrt{m_n}$, for $n = 2, 3, \cdots$. Hence $\sum_{n=1}^{\infty}|\alpha_n| < \infty$ and also

$$\sum_{j=1}^{n-1} m_j|\alpha_j| \leqq H\left(\sum_{j=1}^{n-1}\sqrt{m_j}\right) .$$

Because of (3.5) we conclude from (3.9) that $\sum_{n=1}^{\infty}|\sum_{j=1}^{n-1} a_{nj}\alpha_j|^2 < \infty$. Since (3.1) implies (3.6) we deduce that (3.8) holds and this is equivalent to (3.4).

This result suggests introducing a sequence space $L_S$ as follows. A sequence $(\alpha_n)_{n=1}$ belongs to $L_S$ if and only if there is $f \in L^2$ such that $\alpha_n = \int_0^1 x^{m_n} f(x)dx$, for $n = 1, 2, \cdots$. It is shown in [5], p. 237 that if $S$ consists of integers, then $L_S \subseteq \ell^2$.

COROLLARY 3.2. *If $S$ satisfies* (3.5), *there is a subsequence $S_1$ of $S$ so that $L_{S_1} \subsetneqq L_S$. We also have $L_S \neq \ell^2$, regardless of whether or not $S$ satisfies* (3.5).

COROLLARY 3.3. *Let $S$ be a lacunary sequence of integers and let $\alpha = (\alpha_n)_{n=1}^{\infty}$ be a given sequence of complex numbers. Then if* *there are real* $\gamma$ *and* $\delta$, *with* $\delta < 1$, *so that* $|\alpha_n| \leqq \gamma\delta^{m_n}$ *for* $n = 1, 2, \cdots$, *then* $\alpha \in L_S$.

These results suggest that rather than using the functions $x^{m_n}(n = 1, 2, \cdots)$ in (3.1) it may be more appropriate to use the functions $m_n x^{m_n}$ or $\sqrt{m_n}\, x^{m_n}$. The following is essentially a rewording of Theorem 3.1.

THEOREM 3.4. *Let* $(\alpha_n)_{n=1}^{\infty}$ *be a sequence of complex numbers. If* $S$ *is lacunary and* $\sum_{n=1}^{\infty} |\alpha_n| < \infty$ *then there is* $f \in L^2$ *so that* $\alpha_n = m_n \int_0^1 x^{m_n} f(x)dx$ *for* $n = 1, 2, \cdots$.

*If* $S$ *satisfies* (3.5) *then* $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ *if and only if there is* $f \in L^2$ *so that* $\alpha_n = \sqrt{m_n} \int_0^1 x^{m_n} f(x)dx$, *for* $n = 1, 2, \cdots$.

REMARKS. If $\mu$ is a measure supported by $[0, \delta]$, where $\delta < 1$, and we let $\alpha_n = \int_0^1 x^{m_n} d\mu(x)$ for $n = 1, 2, \cdots$ then, assuming that $S$ is a lacunary sequence of integers, Corollary 3.3 applies to give a function $f \in L^2$ so that (3.1) holds. That is, the measure $\mu$ can be absolutely continuous with respect to Lebesgue measure.

As remarked in the introduction, if $\sum_{i=2}^{\infty} 1/m_i < \infty$ and $j$ is given, there is a measure $\mu$ on $[0, 1]$ so that $\int_0^1 x^{m_n} d\mu(x) = 0$ if $n \neq j$ and $\int_0^1 x^{m_j} d\mu(x) \neq 0$. If $\delta$ is given $(1 \geqq \delta > 0)$ and $S$ is a lacunary sequence of integers, the measure $\mu$ can be chosen to be supported by $[0, \delta]$ and be absolutely continuous with respect to Lebesgue measure. To see this, apply Corollary 3.3 to the sequence $(\varepsilon_n \delta^{m_n})_{n=1}^{\infty}$, where $\varepsilon_n = 0$ if n $\neq j$ and $\varepsilon_j = 1$. We obtain $f \in L^2$ so that $\varepsilon_n \delta^{2m_n} = \int_0^1 (\delta x)^{m_n} f(x)dx$, from which the result follows by a change of variable.

## REFERENCES


1. J. A. Clarkson and P. Erdös, *Approximation by polynomials*, Duke Math. J., **10** (1943), 5-11.
2. P. Davis, *Interpolation and Approximation*, Blaisdell, 1965.
3. R. E. Edwards, *Fourier Series: A Modern Introduction*, vols. I and II, Holt, Rinehart and Winston, 1967.
4. C. Goffman and G. Predrick, *First Course in Functional Analysis*, Prentice Hall, 1965.
5. G. Hardy, J. Littlewood, and G. Pólya, *Inequalities*, Cambridge, 1967.
6. G. G. Jonson, *Moment sequences in Hilbert space*, Pacific J. Math., **46** (1973), 201-207.
7. Y. Katznelson, *An Introduction to Abstract Harmonic Analysis*, Wiley, 1968.
8. W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1966.


Received March 18, 1977 and in revised form June 15, 1977.

THE UNIVERSITY OF WOLLONGONG
NEW SOUTH WALES, AUSTRALIA