PACIFIC JOURNAL OF MATHEMATICS
Vol. 74, No. 1, 1978

ON GROUPS WITH SPECIFIED LOWER CENTRAL SERIES QUOTIENTS

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Intrinsic necessary and sufficient conditions are established for a tower of groups to be the tower of lower central series quotients $\{G/\Gamma_s G\}$ of some group G, in the case in which $G/\Gamma_2 G$ is finitely generated and the case in which G is free. A process for constructing a large number of groups with the same lower central series quotient tower is also described.

1. **Introduction.** Given a group G, one can form nilpotent approximations $G/\Gamma_s G$ to G, where $\Gamma_s G$ is the normal subgroup of G generated by all simple s-fold commutators ($s = 1, 2, \cdots$). The *tower* formed by these lower central series quotients and the natural projections $G/\Gamma_{s+1}G \rightarrow G/\Gamma_s G$ deserves the title *nilpotent completion tower*, or simply *completion*, of G. We do *not* take the inverse limit of the tower, but rather view the tower either as a diagram or, preferably, as a *pro-group*. A. K. Bousfield [3] has studied the properties of a transfinite extension of this tower (generalized to incorporate a ring of "coefficients") with application to homological properties of topological spaces. G. Baumslag [2] has investigated groups which have the same completion as a free group. In this paper we study the following problems: Under what conditions is a tower of groups $\{G_s\}$ the completion of some group? Under these conditions, find (all) groups G such that $G_s \cong G/\Gamma_s G$.

Our principal results are as follows. Call a tower of groups $\{G_s\}$ a Γ-tower if, for each $s \geqq 1$, the sequence $1 \rightarrow \Gamma_s G_{s+1} \rightarrow G_{s+1} \rightarrow G_s \rightarrow 1$ is exact. If $\{G_s\}$ is a Γ-tower and G_2 is finitely generated, then $\{G_s\}$ is the completion of its inverse limit and, more generally, of each of a transfinite sequence of subgroups of its inverse limit. In particular, we obtain a large number of examples of parafree groups [2]. If $\{G_s\}$ is a Γ-tower, G_2 is free abelian, and $\{H_2 G_s\}$ has trivial projections, then $\{G_s\}$ is the completion of a free group. We do not yet know if every Γ-tower is the completion of a group.

In §2 we review pro-groups and establish the basic properties of the completion functor. In §3 we derive the properties of Γ-towers. A "decompletion" process in described in §4, which enables us to construct groups of small cardinality with a given completion, once one group with the given completion is known. We treat the finitely generated case in §5 and the free case in §6.

2. **Pro-groups and the completion functor.** Let $\mathscr{C}$ be any

category. The category tow-$\mathscr{C}$ has as objects *towers* in $\mathscr{C}$,

$$\cdots \longrightarrow X_{s+1} \longrightarrow X_s \longrightarrow \cdots \longrightarrow X_1 ,$$

written $\{X_s\}$ and called *pro-objects* over $\mathscr{C}$ (*pro-groups* in case $\mathscr{C}$ is the category of groups). The morphisms $X_{s+1} \to X_s$ within the tower (and their compositions) are called *projections*. Morphisms in tow-$\mathscr{C}$ are given by

$$\mathrm{Hom}_{\text{tow-}\mathscr{C}}(\{X_s\}, \{Y_s\}) = \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathscr{C}}(X_i, Y_j) .$$

For our purposes it is enough to note that a sequence of morphisms $\{X_s \to Y_s\}$ commuting with the projections in the towers $\{X_s\}$ and $\{Y_s\}$, that is, a morphism in the diagram category, represents a morphism from $\{X_s\}$ to $\{Y_s\}$ in tow-$\mathscr{C}$ and that cofinal towers are isomorphic. See [1], [4], or [7] for a fuller discussion of pro-objects. Although the category of pro-groups is, as we shall see (2.2), the "right" setting in which to study completions, the reader may view the towers in this paper simply as diagrams.

We consider $\mathscr{C}$ as a full subcategory of tow-$\mathscr{C}$ by identifying an object X in $\mathscr{C}$ with the tower $\{X_s\}$ in which each X_s is X and each projection the identity. A pro-object isomorphic to an element of $\mathscr{C}$ is called *constant*. The inclusion functor $\mathscr{C} \to$ tow-$\mathscr{C}$ is left adjoint to the inverse limit functor $\varprojlim$: tow-$\mathscr{C} \to \mathscr{C}$, if the latter exists. In that case, $\{X_s\}$ is constant if and only if $\{X_s\} \cong \varprojlim X_s$.

We next define the completion functor. Recall [9, Chapter 5] that if A and B are subgroups of a group G, then $[A, B]$ denotes the subgroup of G generated by all *commutators* $[a, b] = a^{-1}b^{-1}ab$ for $a \in A$, $b \in B$. Inductively define the *lower central series* of G by $\Gamma_1 G = G$, $\Gamma_{s+1}G = [\Gamma_s G, G]$. Thus $\Gamma_s G$ is generated by all *simple s-fold commutators* $[g_1, g_2, \cdots, g_s] = [[\cdots [g_1, g_2], g_3] \cdots, g_s]$ of elements of G. Let $\Gamma_\omega G = \bigcap_{s=1}^\infty \Gamma_s G$. A group is *nilpotent* if $\Gamma_s G = 0$ for some $s < \omega$ and *residually nilpotent* if $\Gamma_\omega G = 0$. Each $\Gamma_s G$ is normal in G; $G/\Gamma_s G$ is nilpotent for $s < \omega$ and $G/\Gamma_\omega G$ is residually nilpotent. The inclusions $\Gamma_{s+1}G \subset \Gamma_s G$ give rise to epimorphisms $G/\Gamma_{s+1}G \to G/\Gamma_s G$, and we call the pro-group $\{G/\Gamma_s G\}$ the *completion* of G. Denoting the category of groups [resp. nilpotent groups] by $\mathscr{G}$ [resp. $\mathscr{N}$], we more generally define the completion functor $\boldsymbol{C}$: tow-$\mathscr{G} \to$ tow-$\mathscr{N}$.

DEFINITION 2.1. Let $\{G_s\} \in$ tow-$\mathscr{G}$. Then $\boldsymbol{C}\{G_s\}$ is the pro-group $\{G_s/\Gamma_s G_s\}$, called the *completion* of $\{G_s\}$. There is a canonical morphism $\{G_s\} \to \boldsymbol{C}\{G_s\}$ induced by the identity.

The proofs of the following propositions are fairly straightforward and hence omitted.

PROPOSITION 2.2. *$\boldsymbol{C}$ is left adjoint to the inclusion functor* tow-$\mathscr{N} \to$ tow-$\mathscr{G}$, *and $\boldsymbol{C}$ restricted to $\mathscr{G}$ is left adjoint to the inverse limit functor from* tow-$\mathscr{N}$ *to $\mathscr{G}$. Furthermore $\{G_s\} \to \boldsymbol{C}\{G_s\}$ is an isomorphism if and only if $\{G_s\}$ is isomorphic to a tower of nilpotent groups.*

PROPOSITION 2.3. *For any group G,*

(i) $1 \to \Gamma_s(G/\Gamma_{s+1}G) \to G/\Gamma_{s+1}G \to G/\Gamma_s G \to 1$ *is exact for each* $s < \omega$;

(ii) $\Gamma_i(G/\Gamma_s G) \cong \Gamma_i G/\Gamma_s G$ *for* $i \leqq s \leqq \omega$;

(iii) $(G/\Gamma_s G)/\Gamma_i(G/\Gamma_s G) \cong G/\Gamma_i G$ *for* $i \leqq s \leqq \omega$.

3. Γ-towers. By 2.2 every tower of nilpotent groups is, up to isomorphism in tow-$\mathscr{G}$, its own completion. Our problem is to characterize those towers which are completions of groups.

DEFINITION 3.1. *A Γ-tower* is a tower of groups $\{G_s\}$ such that, for each $s \geqq 1$, the sequence

$$1 \longrightarrow \Gamma_s G_{s+1} \longrightarrow G_{s+1} \longrightarrow G_s \longrightarrow 1$$

is exact.

PROPOSITION 3.2. *Let $\{G_s\}$ be a Γ-tower. Then for each s,*

(i) $1 \to \Gamma_i G_s \to G_s \to G_i \to 1$ *is exact for all* $i < s$;

(ii) $G_s/\Gamma_i G_s \cong G_i$ *for all* $i < s$;

(iii) $\Gamma_s G_s \cong 1$;

(iv) *if* $\Gamma_s G_{s+1} = 1$, *then* $G_k \cong G_s$ *for all* $k > s$;

(v) *if P is a set of generators of G_2 and P' is a set of elements of G_s which maps onto P by the projection $G_s \to G_2$, then P' generates G_s.*

Proof. We prove (i) by induction on $s - i$. The statement is true by definition when $s - i = 1$. Denote the projection $G_m \to G_n$ by $p_{m,n}$ for $m > n$. Clearly $p_{s,i}$ is surjective; we must show that $\Gamma_i G_s = \ker p_{s,i}$. Let $x \in \Gamma_i G_s$. Then $p_{s,s-1}(x) \in \Gamma_i G_{s-1}$, so by induction $p_{s,s-1}(x) \in \ker p_{s-1,i}$, whence $x \in \ker p_{s,i}$. Conversely, suppose $x \in \ker p_{s,i}$. Then $p_{s,s-1}(x) \in \ker p_{s-1,i}$. By induction $p_{s,s-1}(x) \in \Gamma_i G_{s-1}$; thus we can write $p_{s,s-1}(x) = \prod_{j=1}^{N} [a_{j,1}, a_{j,2}, \cdots, a_{j,i}]$. Since $p_{s,s-1}$ is surjective, we can choose $b_{j,l} \in G_s$ such that $p_{s,s-1}(b_{j,l}) = a_{j,l}$ for $1 \leqq j \leqq N, 1 \leqq l \leqq i$. Let $y = \prod_{j=1}^{N} [b_{j,1}, b_{j,2}, \cdots, b_{j,i}]$. Then $xy^{-1} \in \ker p_{s,s-1} = \Gamma_{s-1}G_s \subset \Gamma_i G_s$. But $y \in \Gamma_i G_s$, so $x \in \Gamma_i G_s$. Clearly (i) implies (ii), and (iii) is immediate from the definition. To prove (iv), note that the natural surjection $G_k/\Gamma_{s+1}G_k \to G_k/\Gamma_s G_k$ induces an isomorphism $G_{s+1} \to G_s$ by (ii) and the hypothesis; hence $\Gamma_{s+1}G_k \cong \Gamma_s G_k$. But then the definition of the lower central series and (iii) imply that $\Gamma_s G_k \cong \Gamma_k G_k \cong 1$. Hence by (ii), $G_s \cong G_k/\Gamma_s G_k \cong G_k$. Finally (v) follows from [9, Lemma 5.9].

By 2.3 (i) CG is a Γ-tower for every group G. We conjecture the converse: Given a Γ-tower $\{G_s\}$, there exists a group G such that $G/\Gamma_s G \cong G_s$.

In §5 we prove this conjecture in case G_2 is finitely generated, and in §6 we prove it in case G_2 is free abelian and $\{H_2 G_s\} \cong 0$.

4. Constructing small decompletions. If $CG = \{G_s\}$, then the natural map $G \to \varprojlim G_s$ has kernel $\Gamma_\omega G$. By 2.3 (iii) the residually nilpotent group $G/\Gamma_\omega G$ has the same completion as G. We therefore make the following definition.

DEFINITION 4.1. Let $\{G_s\}$ be a Γ-tower. A subgroup G of $\varprojlim G_s$ is a *proper decompletion* of $\{G_s\}$ if the natural maps $G \to G_s$ induce isomorphisms $G/\Gamma_s G \cong G_s$ for all s.

Aside from the case in which a Γ-tower $\{G_s\}$ is constant (and hence itself its only proper decompletion), $\varprojlim G_s$ is uncountable because each surjection $G_{s+1} \to G_s$ has nontrivial kernel by 3.2 (iv). We shall see in the next section that $\varprojlim G_s$ is a proper decompletion of $\{G_s\}$ if G_2 is finitely generated, but we now describe a process for obtaining decompletions with small cardinality.

PROPOSITION 4.2. *Let H be a proper decompletion of a nonconstant Γ-tower $\{G_s\}$. Let K be a subset of H. Let $\mathfrak{m}$ be the maximum of the cardinality of K, the cardinality of G_2, and $\aleph_0$. Then there exists a proper decompletion of $\{G_s\}$ containing K, contained in H, and of cardinality $\mathfrak{m}$.*

Proof. We shall construct an increasing sequence of subgroups, $A_1 \subset A_2 \subset \cdots$, of H, each of which is obtained from the preceding one by adjoining at most $\mathfrak{m}$ elements of H, and whose union is the desired decompletion. For each element g in a generating set for G_2, let $x_g \in H$ map to g under the natural surjection $H \to G_2$. Let A_1 be the subgroup of H generated by K and all the x_g's. Since $A_1 \to G_2$ is surjective, $A_1 \to G_s$ is surjective for all s by 3.2 (v), and the cardinality of A_1 is $\mathfrak{m}$. Assume by induction that we have defined $A_n \subset H$ such that A_n has cardinality $\mathfrak{m}$ and $A_n \to G_s$ is surjective for all s. Consider the groups $K_s = \ker(A_n \to G_s)$. Clearly $\Gamma_s A_n \subset K_s$, since $\Gamma_s G_s = 1$ by 3.2 (iii), but it might happen that there are elements in K_s which are not in $\Gamma_s A_n$. Such elements are in $\Gamma_s H$, however, since H is a proper decompletion of $\{G_s\}$. Form A_{n+1} as the subgroup of H generated by A_n and a collection of at most m elements of H needed to express all the elements of K_s as products of simple s-fold commutators, for all s. Clearly A_{n+1} satisfies the inductive hypotheses. Then $A = \bigcup_{n=1}^{\infty} A_n$ is perforce the desired decompletion.

PROPOSITION 4.3. *The union of a nested family of proper decompletions of a Γ-tower is again a proper decompletion.*

The proof is clear.

5. The finitely generated case. In this section we use a lemma of Bousfield [3] to show that Γ-towers with finitely generated G_2 are actually completion towers, and we construct many decompletions of them. In view of 3.2(v), it makes sense to call such a tower a *finitely generated* Γ-tower.

THEOREM 5.1. *Let $\{G_s\}$ be a finitely generated Γ-tower, and let $\hat{G} = \varprojlim G_s$. Then $\hat{G}$ is a proper decompletion of $\{G_s\}$.*

The proof involves the notion of N-series [3], [9, p. 391].

DEFINITION 5.2. An *N-series* in a group G is a descending series of subgroups (indexed by positive integers)

$$G = K_1 \supset K_2 \supset K_3 \supset \cdots$$

such that $[K_r, K_s] \subset K_{r+s}$ for all r, s. There is an associated Lie ring $\bigoplus_{r \geqq 1} K_r/K_{r+1}$ with Lie product

$$[\ ,\ ]\colon K_r/K_{r+1} \otimes K_s/K_{s+1} \longrightarrow K_{r+s}/K_{r+s+1}$$

induced by the commutator.

LEMMA 5.3 (Bousfield [3]). *Let $\{K_s\}$ be an N-series in a group G such that*

(i) *the natural map $G \to \varprojlim G/K_s$ is an isomorphism;*

(ii) *the Lie product*

$$[\ ,\ ]\colon G/K_2 \otimes K_s/K_{s+1} \longrightarrow K_{s+1}/K_{s+2}$$

is surjective for all s; and

(iii) *G/K_2 is finitely generated. Then $K_s = \Gamma_s G$ for all $s \geqq 1$.*

Proof of 5.1. Let $K_s = \ker(\hat{G} \to G_s)$. It suffices to show that $\{K_s\}$ is an N-series in $\hat{G}$ satisfying the conditions of 5.3. Express elements of $\hat{G}$ as sequences $(g_1, g_2, \cdots)$ such that $g_i \in G_i$ and g_{i+1} projects to g_i for all i. Then $K_s = \{(g_1, g_2, \cdots) \in \hat{G}: g_i = 0 \text{ for } i \leqq s\} = \{(g_1, g_2, \cdots) \in \hat{G}: g_i \in \Gamma_s G_i \text{ for all } i\}$ by 3.2 (i) and 3.2 (iii). Since $[\Gamma_r G_i, \Gamma_s G_i] \subset \Gamma_{r+s} G_i$ for all i [9, p. 293], $[K_r, K_s] \subset K_{r+s}$. Conditions (i) and (iii) of 5.3 are given. To verify condition (ii), let $\bar{g} = (g_1, g_2, \cdots) \in K_{s+1}$. Then $g_{s+2} \in \Gamma_{s+1} G_{s+2}$, so $g_{s+2} = \prod_{j=1}^N [y_{j,s+2}, z_{j,s+2}]$ for some elements $y_{j,s+2} \in \Gamma_s G_{s+2}$ and $z_{j,s+2} \in G_{s+2}$. Since $\{G_s\}$ is a tower of surjections, we may extend to $\bar{y}_j = (y_{j,1}, y_{j,2}, \cdots) \in K_s$ and $\bar{z}_j = (z_{j,1}, z_{j,2}, \cdots) \in \hat{G}$. Then $\bar{g}$ and $\prod_{j=1}^N [\bar{y}_j, \bar{z}_j]$ differ only by an element of K_{s+2}, so the Lie product is onto K_{s+1}/K_{s+2}.

Combining 5.1 with 4.2 and 4.3 we can construct inductively a transfinite sequence of decompletions as follows. Let $\{G_s\}$ and $\hat{G}$ be as in 5.1, with $\{G_s\}$ not constant. Apply 4.2 to the empty subset of $\hat{G}$ to obtain a countable proper decompletion G^1. Given the proper decompletion G^α, for an ordinal α, if $G^\alpha \neq \hat{G}$, let $x \in \hat{G} - G^\alpha$ and apply 4.2 to $G^\alpha \cup \{x\}$ to obtain a proper decompletion $G^{\alpha+1}$, containing, but of the same cardinality as, G^α. For limit ordinals λ, let $G^\lambda = \bigcup_{\alpha<\lambda} G^\alpha$, which is a proper decompletion by 4.3. Note that G^α is countable for $\alpha < \omega$ and has cardinality equal to the cardinality of α for $\alpha \geqq \omega$. This process terminates at $\hat{G}$, which has the cardinality of the continuum, $\mathfrak{C}$. Although there is no guarantee that the G^α's are not isomorphic, any two with different cardinality will be nonisomorphic, and every cardinality between $\aleph_0$ and $\mathfrak{C}$, inclusive, is represented. Since it is consistent to assume [5] that $\mathfrak{C}$ is an arbitrarily large cardinal, we have proved the following existence theorem.

THEOREM 5.4. *Let $\{G_s\}$ be a nonconstant finitely generated Γ-tower, and let $\aleph_\alpha$ be the* αth *infinite cardinal number. Then it is consistent with* ZFC (set theory plus the axiom of choice) *that there exist $\aleph_\alpha$ nonisomorphic, residually nilpotent groups with completion $\{G_s\}$.*

Letting $\{G_s\}$ be the completion of a finitely generated free group, we obtain a "large number" of examples of parafree groups [2].

6. Completions of free groups. In this section we completely characterize those towers which are completions of (not necessarily finitely generated) free groups. We first need two basic results relating group homology and completion. (These propositions lead Bousfield [3] to call a certain transfinite extension of $\{G/\Gamma_s G\}$ the *homological localization tower* for G.) Given a pro-group $\{G_s\}$ and an integer $n \geqq 1$, define $H_n\{G_s\}$ to be the pro-abelian-group $\{H_n G_s\}$, where $H_n G_s$ is the ordinary homology of the group G_s with trivial integer coefficients [8, p. 290]. In particular $H_1\{G_s\} \cong \{G_s/\Gamma_2 G_s\}$.

PROPOSITION 6.1 (W. G. Dwyer). *If $\{G_s\} \to \{G'_s\}$ is a morphism of pro-groups which induces an isomorphism $H_1\{G_s\} \to H_1\{G'_s\}$ and an epimorphism $H_2\{G_s\} \to H_2\{G'_s\}$, then $C\{G_s\} \to C\{G'_s\}$ is an isomorphism.*

The proof [6] is similar to the proof of the classical version of the theorem due to J. Stallings [10].

PROPOSITION 6.2. *Let $\{G_s\}$ be a pro-group. Then the natural morphism $\{G_s\} \to C\{G_s\}$ induces an isomorphism $H_1\{G_s\} \to H_1C\{G_s\}$ and an epimorphism $H_2\{G_s\} \to H_2C\{G_s\}$.*

Proof. $H_1G_s \cong H_1(G_s/\Gamma_s G_s)$ by 2.3 (iii). By [10], for each s the short exact sequence

$$1 \longrightarrow \Gamma_s G_s \longrightarrow G_s \longrightarrow G_s/\Gamma_s G_s \longrightarrow 1$$

gives rise to a natural exact sequence

$$H_2G_s \longrightarrow H_2(G_s/\Gamma_s G_s) \longrightarrow \Gamma_s G_s/\Gamma_{s+1} G_s\ .$$

That $H_2\{G_s\} \to H_2C\{G_s\}$ is an epimorphism now follows by forming the corresponding exact sequence of towers and noting that $\{\Gamma_s G_s/\Gamma_{s+1}G_s\} \cong 0$ because each projection is the trivial homomorphism.

THEOREM 6.3. *Let $\{G_s\}$ be a nonconstant Γ-tower. Then $\{G_s\}$ has a free group as a proper decompletion if and only if G_2 is free abelian and $H_2\{G_s\} \cong 0$.*

Proof. The first condition is clearly necessary, and the second follows from 6.2 since $H_2F = 0$ for F free. To show sufficiency, let F be the free group on a set of free abelian generators for G_2, and let $\varphi_2 \colon F \to G_2$ be induced by the identity. Lift φ_2 to a morphism $\varphi \colon F \to \{G_s\}$. By 3.2 (ii) and the hypothesis that $H_2\{G_s\} \cong 0$, $H_1\varphi$ is an isomorphism and $H_2\varphi$ is an epimorphism. Hence $C\varphi$ is an isomorphism by 6.1. In fact a diagram chase, using the characterization of isomorphism in tow-$\mathscr{G}$ in [4], shows that each level $F/\Gamma_s F \to G_s$ of $C\varphi$ is an isomorphism. Finally since free groups are residually nilpotent [2], the image of F in $\varprojlim G_s$ is free and a proper decompletion of $\{G_s\}$.

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Received November 29, 1976 and in revised form August 4, 1977.

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