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THE SHEAF OF OUTER FUNCTIONS IN THE POLYDISC

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Let U^n be the unit polydisc in C^n. Define a presheaf by assigning to each relatively open subset W of $\bar{U}^n$ the multiplicative group of outer functions in the intersection $W \cap U^n$. If $\mathscr{O}$ denotes the associated sheaf, we prove that $H^q(\bar{U}^n, \mathscr{O})=0$ for all integers $q \geqq 1$.

1. **Introduction.** Classically, the outer functions in the open unit disc U are functions of the form

$$\lambda \exp \int_T \frac{w+z}{w-z} k(w) dm(w) ,$$

where m is the Haar measure on the unit circle T, k is an absolutely integrable real-valued function on T, and λ is a complex number of modulus one. Closely related to the class of outer functions is the Smirnov class $N^*(U)$, which consists of all functions that are holomorphic in U and admit an inner-outer factorization. The class $N^*(U)$ is an algebra, and the outer functions are precisely the invertible elements of this algebra. An alternative characterization of $N^*(U)$, considered by Rudin in [5], where it was extended to the polydisc U^n, is that a holomorphic function f in U belongs to $N^*(U)$ if and only if there exists a strongly convex function ϕ (depending on f) for which $\phi(\mathrm{Log}^+ |f|)$ has a harmonic majorant. This definition can be extended naturally to arbitrary polydomains $W_1 \times W_2 \times \cdots \times W_n$, the requirement now being that $\phi(\mathrm{Log}^+ |f|)$ have an n-harmonic majorant in $W_1 \times W_2 \times \cdots \times W_n$. We define the outer functions in $W_1 \times W_2 \times \cdots \times W_n$ to be the invertible elements of the algebra $N^*(W_1 \times W_2 \times \cdots \times W_n)$. (For the polydisc U^n, this definition can easily be seen to agree with the one given by Rudin in [5, Def. 4.4.3, p. 72].)

The correspondence that assigns to each polydomain W in C^n the group $O(W \cap U^n)$ of outer functions in the intersection $W \cap U^n$, defines a sheaf $\mathscr{O}$ on the closure $\bar{U}^n$ of U^n, which is locally determined in the sense that $\Gamma(\bar{U}^n, \mathscr{O})$ is canonically isomorphic to the group of outer functions in U^n. Our aim, in this article, is to show that the cohomology groups $H^q(\bar{U}^n, \mathscr{O})$ are trivial for all integers $q \geqq 1$.

Sheaves of a similar type (sheaves of germs of holomorphic functions satisfying boundary conditions on polydomains) have been studied by Nagel in [4], where a unified approach to many types of boundary behavior was given. Nagel's methods, however, do not appear to be applicable in our case. Instead, we use the methods developed by Stout in [7], which we also used in [8]. Indeed, the proof of Lemma 3.1 closely follows that of Lemma 1.2 of [7], and part of our conclusion is that the multiplicative Cousin problem with N^*-data can be solved in the polydisc.

II. Preliminaries. We denote the open unit disc $\{z \in \boldsymbol{C}: |z| < 1\}$ by U, and its boundary, the unit circle, by T. The cartesian product of n copies of U will be denoted by U^n. More generally, a *polydomain* in $\boldsymbol{C}^n$ will be a cartesian product $W_1 \times W_2 \times \cdots \times W_n$ of n domains (open connected sets) in $\boldsymbol{C}$. Similarly, T^n will be the cartesian product of n copies of T.

Let W be an open set in $\boldsymbol{C}^n$, a continuous function $h: W \to (-\infty, +\infty)$ is *n-harmonic* if it is harmonic in each complex variable separately; an upper semicontinuous function s: $W \to [-\infty, +\infty)$ is *n-subharmonic* if it is subharmonic in each complex variable separately. If h and s are as above, and if $s(z) \leqq h(z)$ for all $z \in W$, we say that h is an *n-harmonic majorant* of s in W.

The following proposition ([8, Th. 2.10, p. 301], see also [2]) shows that having an n-harmonic majorant is a local property under certain conditions.

PROPOSITION 2.1. *Let $W_1, W_2, \cdots, W_n$ be bounded domains in $\boldsymbol{C}$ such that the boundary of each W_j consists of finitely many mutually disjoint Jordan curves. Let $W = W_1 \times W_2 \times \cdots \times W_n$, and let $\{U_\alpha\}$ be a relatively open covering of the closure $\bar{W}$ of W. If s is a positive n-subharmonic function in W with "local" n-harmonic majorants h_α in each intersection $U_\alpha \cap W$, then s must have an n-harmonic majorant in all of W.*

Let W be a polydomain in $\boldsymbol{C}^n$. We define $N^*(W)$ to be the class of all holomorphic functions f in W such that $\phi(\mathrm{Log}^+|f|)$ has an n-harmonic majorant for some strongly convex function ϕ. We recall that a function $\phi: (-\infty, +\infty) \to [0, +\infty)$ is *strongly convex* if it is convex, nondecreasing, and if $\lim_{t\to\infty} \phi(t)/t = +\infty$. Given two (or finitely many) strongly convex functions ϕ_α, it is always possible to find a strongly convex ϕ such that $\phi \leqq \phi_\alpha$ for all α. This, together with the arithmetic properties of Log^+, shows that $N^*(W)$ is closed under pointwise addition and multiplication, and is therefore an algebra. The class $O(W)$ of *outer functions in W* is defined to be the group of all invertible elements of the algebra $N^*(W)$. If W is simply connected, then $f \in O(W)$ if and only if $f = \exp g$, where $g = u + iv$ is holomorphic and where $\phi(|u|)$ has an n-harmonic majorant in W for some strongly convex function ϕ. The additive group formed by such functions g will be denoted $P(W)$.

Let Ω be the family of all cartesian products $W_1 \times W_2 \times \cdots \times W_n$, where each W_j is connected and relatively open in $\bar{U}^n$. The pressheaves $W \to P(W \cap U^n)$, $W \to O(W \cap U^n)$, defined for W in Ω, induce sheaves $\mathscr{P}$ and $\mathscr{O}$ on $\bar{U}^n$. There is a canonical map $P(U^n) \to \Gamma(\bar{U}^n, \mathscr{P})$ which is clearly one-one and a group homomorphism. To see that it is also onto, suppose $\{W_\alpha\}$ is a finite covering of $\bar{U}^n$ (by members of Ω) and suppose that $f = u + iv$ is a holomorphic function in U^n whose restriction to each intersection $W_\alpha \cap U^n$ is in $P(W_\alpha \cap U^n)$. For each α let ϕ_α be a strongly convex function such that $\phi_\alpha(|u|)$ has an n-harmonic majorant in $W_\alpha \cap U^n$. Choose a strongly convex ϕ such that $\phi \leqq \phi_\alpha$ for all α. The n-subharmonic function $\phi(|u|)$ has n-harmonic majorants in the intersections $W_\alpha \cap U^n$. Consequently, by (2.1), it has an n-harmonic majorant in U^n. The function f then belongs to $P(U^n)$, and the canonical map $P(U^n) \to \Gamma(\bar{U}^n, \mathscr{P})$ is therefore an isomorphism. In a similar way we show that $O(U^n)$ and the group $\Gamma(\bar{U}^n, \mathscr{O})$ of global sections of $\mathscr{O}$, are canonically isomorphic. More generally, if W is a member of Ω, $\Gamma(W, \mathscr{P})$ and $\Gamma(W, \mathscr{O})$ can be naturally identified with the class of holomorphic functions in W whose restriction to any $V \in \Omega$ such that $\bar{V} \subset W$, is in $P(V \cap U^n)$ and in $O(V \cap U^n)$ respectively.

In § IV we prove that $H^q(\bar{U}^n, \mathscr{O}) = 0$ for all integers $q \geqq 1$. First we need some technical results.

III. A generalized Cartan lemma. The following lemma is the crux of our work. It is a modified version of [7, Lemma 1.2, p. 380].

Let λ_1 and λ_2 be disjoint closed arcs on the circle T, and let S^2 be the extended complex plane. For $j = 1, 2$, define V_j to be the union of the disc U, its exterior $S^2 - \bar{U}$, and the interior (relative to T) of λ_j.

LEMMA 3.1. *If $f \in P(U^n)$, there exist functions f_j which are holomorphic in $V_j \times U^{n-1}$, and such that:*

(a) $f = f_1 + f_2$ *on* U^n,

(b) $f_j \in P(U^n)$,

(c) $f_j \in P((S^2 - \bar{U}^n) \times U^{n-1})$,

(d) $f_j \in P(D_j \times U^{n-1})$, *for some open disc D_j containing λ_j.*

Proof. We use the notation and terminology of [5]. In particular, m_n will be the Haar measure on T^n, Z^n will be the set of all n-tuples of integers, Z^n_+ the set of all $\alpha \in Z^n$ such that $\alpha_1 \geqq 0$, $\alpha_2 \geqq 0, \cdots, \alpha_n \geqq 0$, and $Y_n = Z^n_+ \cup (-Z^n_+)$. For $z = (z_1, z_2, \cdots, z_n) \in U^n$ and

$w = (w_1, w_2, \cdots, w_n) \in T^n$, $P_n(z_1, w)$ will be the n-dimensional Poisson kernel; i.e.,

$$(1) \qquad P_n(z, w) = \sum_{\alpha \in Z^n} |z|^{|\alpha|} \left(\frac{z\bar{w}}{|z|} \right)^{\alpha},$$

where

$$|z|^{|\alpha|} = |z_1|^{|\alpha_1|} \cdots |z_n|^{|\alpha_n|}, \quad \text{and} \quad \left(\frac{z\bar{w}}{|z|} \right)^{\alpha} = \left(\frac{z_1\bar{w}_1}{|z_1|} \right)^{\alpha_1} \cdots \left(\frac{z_n\bar{w}_n}{|z_n|} \right)^{\alpha_n}.$$

We define $K_n(z, w)$ to be the summation in (1) restricted to the lattice points of Y_n. It can be verified that $K_n(z, w)$ is the real part of

$$H_n(z, w) = \frac{2}{(1 - \bar{w}_1 z_1) \cdots (1 - \bar{w}_n z_n)} - 1.$$

In what follows, we will use an alternative characterization of $P(U^n)$. It is a consequence of [5, Th. 3.1.2, p. 37] and [5, Th. 3.2.4, p. 41] that a holomorphic function f belongs to $P(U^n)$ if and only if its real part u is the Poisson integral of some function $u^* \in L^1(T^n)$; if this is the case, then $u^*(w) = \lim_{r \to 1^-} u(rw_1, rw_2, \cdots, rw_n)$ for almost all $w \in T^n$ (with respect to the measure m_n).

Suppose now that $f \in P(U^n)$, and write $f = f' + f''$, where $f'(z_1, z_2, \cdots, z_n) = f(0, z_2, \cdots, z_n)$. The function f' is clearly in $P(S^2 \times U^{n-1})$; therefore it suffices to prove the lemma for f'' instead of f. Let u be the real part of f'' and let u^* be the radial boundary values of u. Since the Fourier coefficients

$$\hat{u}^*(\alpha) = \int_{T^n} \bar{w}^{\alpha} u^*(w) dm_n(w)$$

vanish for all $\alpha \notin Y_n$, we can write:

$$u(z) = \int_{T^n} P_n(z, w) u^*(w) dm_n(w) = \int_{T^n} K_n(z, w) u^*(w) dm_n(w).$$

The kernel $H_n(z, w)$ is holomorphic in z, $H_n(0, w) \equiv 1$, and $K_n(z, w) = \operatorname{Re} H_n(z, w)$. Therefore, since $f''(0) = 0$, we have

$$f''(z) = \int_{T^n} H_n(z, w) u^*(w) dm_n(w).$$

Choose an infinitely differentiable real-valued function χ on the circle T, such that χ is identically zero on an open connected subset T_1 of T which contains λ_1, and identically one on a similar neighborhood T_2 of λ_2 in T. Define

$$f_1''(z) = \int_{T^n} H_n(z, w)\chi(w)u^*(w)dm_n(w) ,$$
$$f_2''(z) = \int_{T^n} H_n(z, w)[1 - \chi(w)]u^*(w)dm_n(w) .$$

It is clear that f_j'' is not only holomorphic in $V_j \times U^{n-1}$ but at all points in $T_j \times U^{n-1}$ as well, and that $f_1''(z) + f_2''(z) = f''(z)$ for all $z \in U^n$.

We first prove that $f_1'' \in P(U^n)$ (a symmetric argument will show that $f_2'' \in P(U^n)$).

The function $\chi(w_1)$ is the sum of its Fourier series $\sum_{-\infty}^{+\infty} c_k w_1^k$, which converges uniformly and absolutely in T. Since χ is real-valued, we have $c_{-k} = \bar{c}_k$; also $|c_k| = O(k^{-q})$ for all integers $q \geqq 1$.

If u_1 is the real part of f_1'', we have

$$u_1(z) = \int_{T^n} K_n(z, w)\chi(w_1)u^*(w)dm_n(w) .$$

To show that $f_1'' \in P(U^n)$ it suffices to find a function $A \in L^1(T^n)$ such that

$$\int_{T^n} K_n(z, w)\chi(w_1)u^*(w)dm_n(w) = \int_{T^n} P_n(z, w)A(w)dm_n(w) . \tag{3.1.1}$$

This is trivially verified, with $A(w) = \chi(w_1)u^*(w)$, if $u(z_1, z_2, \cdots, z_n)$ depends only on z_1; for instance, if the radial boundary values of $u(z_1, 0, \cdots, 0)$ take the place of u^*. It therefore suffices to establish (3.1.1) with u^* replaced by the radial boundary values of $u(z_1, z_2, \cdots, z_n) - u(z_1, 0, \cdots, 0)$. We assume then, without loss of generality, that $u(z_1, 0, \cdots, 0)$ is identically zero, or equivalently, that $\hat{u}^*(\alpha_1, 0, \cdots, 0) = 0$ for all integers α_1. Write

$$\begin{aligned} u_1(z) &= c_0 \int_{T^n} K_n(z, w)u^*(w)dm_n(w) \\ &\quad + \sum_{k=1}^{\infty} \int_{T^n} K_n(z, w)[c_k w_1^k + \bar{c}_k \bar{w}_1^k]u^*(w)dm_n(w) . \end{aligned} \tag{3.1.2}$$

Let μ_1 and ν_1 be identically zero on T^n, and define, for $k = 2, 3, \cdots$, and for almost all $w \in T^n$,

$$\mu_k(w) = \int_T u^*(w_1\bar{y}, w_2, \cdots, w_n) \sum_{j=1}^{k-1} \bar{y}^j dm_1(y) ,$$
$$\nu_k(w) = \int_T u^*(w_1\bar{y}, w_2, \cdots, w_n) \sum_{j=1}^{k-1} \bar{y}^j dm_1(y) ,$$

where m_1 is the Haar measure on the circle T. The functions μ_k, ν_k belong to $L^1(T^n)$, and have L^1-norms no greater than $(k-1)\|u^*\|_1$. A simple calculation shows that the Fourier coefficients $\hat{\mu}_k(\alpha_1, \alpha_2, \cdots, \alpha_n)$

are zero unless $1 - k \leqq \alpha_1 \leqq -1$, in which case they agree with $\hat{u}^*(\alpha_1, \alpha_2, \cdots, \alpha_n)$. We now recall that $f''(0, z_2, \cdots z_n) \equiv 0$, and consequently that the Fourier coefficients $\hat{u}^*(0, \alpha_2, \cdots, \alpha_n)$ are all zero. This, together with the assumption that $\hat{u}^*(\alpha_1, 0, \cdots, 0) = 0$ for all integers α_1, and the series expansion of $K_n(z, w)$, shows (as in [7, p. 384]) that

$$\int_{T^n} K_n(z, w) w_1^k u^*(w) dm_n(w) = \int_{T^n} P_n(z, w) w_1^k [u^*(w) - \mu_k(w)] dm_n(w) . \tag{3.1.3}$$

Similarly

$$\int_{T^n} K_n(z, w) \bar{w}_1^k u^*(w) dm_n(w) = \int_{T^n} P_n(z, w) \bar{w}_1^k [u^*(w) - \nu_k(w)] dm_n(w) . \tag{3.1.4}$$

If we define $A_k(w) = c_k w_1^k [u^*(w) - \mu_k(w)] + \bar{c}_k \bar{w}_1^k [u^*(w) - \nu_k(w)]$, and combine (3.1.3) and (3.1.4), we have

$$\int_{T^n} K_n(z, w)[c_k w_1^k + \bar{c}_k \bar{w}_1^k] u^*(w) dm_n(w) = \int_{T^n} P_n(z, w) A_k(w) dm_n(w) . \tag{3.1.5}$$

The estimates $\|\mu_k\|_1 \leqq (k-1)\|u^*\|_1$, $\|\nu_k\|_1 \leqq (k-1)\|u^*\|_1$, show that $\|A_k\|_1 \leqq 2\|u^*\|_1 |c_k| k$. Since $\sum_{k=1}^{\infty} |c_k| k$ converges, the series $\sum_{k=1}^{\infty} A_k$ converges absolutely in $L^1(T^n)$. If $A = c_0 u^* + \sum_{k=1}^{\infty} A_k$, (3.1.2) and (3.1.5) show that (3.1.1) is verified. Consequently $f_1'' \in P(U^n)$.

Next we prove that f_1'' and f_2'' are in $P((S^2 - \bar{U}^n) \times U^{n-1})$.

A direct calculation yields

$$f_1''(z_1, z_2, \cdots, z_n) - f_1''\left(\frac{1}{\bar{z}_1}, z_2, \cdots, z_n\right) = \int_{T^n} P_1(z_1, w_1) \frac{2}{(1 - z_2\bar{w}_2) \cdots (1 - z_n \bar{w}_n)} \chi(w_1) u^*(w) dm_n(w) \tag{3.1.6}$$

for all $z \in U^n$. (Here, $P_1(z_1, w_1)$ is the one-dimensional Poisson kernel.) Taking real parts in (3.1.6), we get

$$\begin{aligned} u_1(z_1, z') - u_1\left(\frac{1}{\bar{z}_1}, z'\right) &= \int_{T^n} P_1(z_1, w_1)[1 + K_{n-1}(z', w')]\chi(w_1) u^*(w) dm_n(w) \\ &= \int_{T^n} P_1(z_1, w_1)\chi(w_1) u^*(w) dm_n(w) \\ &\quad + \int_{T^n} P_1(z_1, w_1) K_{n-1}(z', w')\chi(w_1) u^*(w) dm_n(w) , \end{aligned} \tag{3.1.7}$$

where $z' = (z_2, \cdots, z_n)$ and $w' = (w_2, \cdots, w_n)$.

Since the Fourier coefficients of $\chi(w_1)u^*(w) = \sum_{-\infty}^{+\infty} c_k w_1^k u^*(w)$ are zero for all lattice points not in $Z \times Y_{n-1}$, and since

$$P_1(z_1, w_1)K_{n-1}(z', w') = \sum_{\alpha \in Z \times Y_{n-1}} |z|^{|\alpha|}\left(\frac{z\bar{w}}{|z|}\right)^{\alpha},$$

we can write

$$\begin{aligned}(3.1.8)\qquad &\int_{T^n} P_1(z_1, w_1)K_{n-1}(z', w')\chi(w_1)u^*(w)dm_n(w) \\ &= \int_{T^n} P_n(z, w)\chi(w_1)u^*(w)dm_n(w) .\end{aligned}$$

On the other hand, if we define

$$v^*(w_1) = \int_{T^{n-1}} u^*(w_1, w')dm_{n-1}(w') ,$$

then

$$\begin{aligned}&\int_{T^n} P_1(z_1, w_1)\chi(w_1)u^*(w)dm_n(w) \\ (3.1.9)\qquad &= \int_{T} P_1(z_1, w_1)\chi(w_1)v^*(w_1)dm_1(w_1) \\ &= \int_{T^n} P_n(z, w)\chi(w_1)v^*(w_1)dm_n(w) .\end{aligned}$$

Substituting (3.1.8) and (3.1.9) in (3.1.7) yields

$$u_1(z_1, z') - u_1\left(\frac{1}{\bar{z}_1}, z'\right) = \int_{T^n} P_n(z, w)[\chi(w_1)v^*(w_1) + \chi(w_1)u^*(w)]dm_n(w) ,$$

which allows us to write

$$u_1\left(\frac{1}{\bar{z}_1}, z'\right) = \int_{T^n} P_n(z, w)[u_1^*(w) - \chi(w_1)v^*(w_1) - \chi(w_1)u^*(w)]dm_n(w) .$$

The above exhibits $u_1(1/\bar{z}_1, z')$ as the Poisson integral of a function in $L^1(T^n)$. This implies ([5, Th. 3.2.4, p. 41]) that there exists a strongly convex ϕ and an n-harmonic function h in U^n such that

$$\phi\left(\left|u_1\left(\frac{1}{\bar{z}_1}, z'\right)\right|\right) \leqq h(z_1, z') ,$$

for all $(z_1, z') \in U^n$. Consequently

$$\phi(|u_1(z)|) \leqq h\left(\frac{1}{\bar{z}_1}, z'\right) ,$$

for $z \in (S^2 - \bar{U}) \times U^{n-1}$. Since $h(1/\bar{z}_1, z')$ is n-harmonic in $(S^2 - \bar{U}) \times U^{n-1}$, f_1'' must belong to $P((S^2 - \bar{U}) \times U^{n-1})$. Similarly, we show that $f_2'' \in P((S^2 - \bar{U}) \times U^{n-1})$.

Finally, we prove part (d) *of the lemma.*

Denote by h_U and $h_{S^2-\bar{U}}$ the least n-harmonic majorants of $|u_1|$ in U^n and $(S^2 - \bar{U}) \times U^{n-1}$ respectively. (That h_U and $h_{S^2-\bar{U}}$ exist is a direct consequence of parts (b) and (c) of the lemma.) As functions of the single variable z_1, $h_U(z_1, 0, \cdots, 0)$ and $h_{S^2-\bar{U}}(z_1, 0, \cdots, 0)$ are positive harmonic functions (in U, and in $S^2 - \bar{U}$). Therefore, as is well known, they must have nontangential boundary values at almost all points of T. Choose in each of the two connected components of $T_1 - \lambda_1$ a point where both $h_U(z_1, 0, \cdots, 0)$ and $h_{S^2-\bar{U}}(z_1, 0, \cdots, 0)$ simultaneously have a nontangential boundary value. Call these points ζ' and ζ'', and let C be a circle which intersects the circle T precisely at ζ' and ζ''. If C has center a and radius ρ we write $C = a + \rho T$. Let D_1 be the disc bounded by $a + \rho T$, and let $W_1 = U \cup T_1 \cup (S^2 - \bar{U})$. As we mentioned earlier, f_1'' is holomorphic in $W_1 \times U^{n-1}$. Thus, for each $z' = (z_2, \cdots, z_n) \in U^{n-1}$, the function $z_1 \rightarrow f_1''(z_1, z')$ is holomorphic in W_1. Since the closure of D_1 is contained in W_1, the function $u_1(z_1, z')$ can be represented there as the Poisson integral of its values on the circle $a + \rho T$, i.e.,

$$u_1(z_1, z') = \int_T u_1(a + \rho w_1, z') P_1\left(\frac{z_1 - a}{\rho}, w_1\right) dm_1(w_1)\,, \tag{3.1.10}$$

for all $z = (z_1, z') \in D_1 \times U^{n-1}$.

Similarly, for each z_1 in U or in $S^2 - \bar{U}$, the function $z' \rightarrow f_1''(z_1, z')$ is holomorphic in U^{n-1}, and belongs to $P(U^{n-1})$ by parts (b) and (c) of the lemma. Thus $z' \rightarrow u_1(z_1, z')$ has radial boundary values $u_1(z_1, w')$ in $L^1(T^{n-1})$, and

$$u_1(z_1, z') = \int_{T^{n-1}} u_1(z_1, w') P_{n-1}(z', w') dm_{n-1}(w')\,, \tag{3.1.11}$$

for all $z = (z_1, z')$ either in U^n or in $(S^2 - \bar{U}) \times U^{n-1}$.

A point $a + \rho w_1$ on the circle $a + \rho T$ will be contained in U or in $S^2 - \bar{U}$, or will be one of the two intersections ζ' and ζ'' of $a + \rho T$ with T. In the first two cases, by (3.1.11), we have

$$u_1(a + \rho w_1, z') = \int_{T^{n-1}} u_1(a + \rho w_1, w') P_{n-1}(z', w') dm_{n-1}(w')\,.$$

Substituting the above in (3.1.10), we obtain

$$\begin{aligned} u_1(z_1, z') = \int_T \Big\{ \int_{T^{n-1}} u_1(a + \rho w_1, w') P_{n-1}(z', w') dm_{n-1}(w') \Big\} \\ P_1\left(\frac{z_1 - a}{\rho}, w_1\right) dm_1(w_1)\,, \end{aligned} \tag{3.1.12}$$

for all $z = (z_1, z') \in D_1 \times U^{n-1}$.

The function $u_1(a + \rho w_1, w')$ is measurable on T^n, and for each $w_1 \in T$ belongs (as a function of w') to $L^1(T^n)$. We next show that $u_1(a + \rho w_1, w')$ is in $L^1(T^n)$.

For a fixed point $a + \rho w_1$ in $a + \rho T$, the function

$$I(a + \rho w_1, z') = \int_{T^{n-1}} |u_1(a + \rho w_1, w')| P_{n-1}(z', w') dm_{n-1}(w')$$

is the least $n - 1$-harmonic majorant of $z' \to |u_1(a + \rho w_1, z')|$ in U^{n-1}. Since $|u_1|$ has n-harmonic majorants h_U and $h_{S^2-\bar{U}}$ in U_n and $S^2 - \bar{U}$ respectively, we have the inequalities

$$(3.1.13) \quad I(a + \rho w_1, z') \leqq h_U(a + \rho w_1, z'), \quad \text{if} \quad a + \rho w_1 \in U,$$

and

$$(3.1.14) \quad I(a + \rho w_1, z') \leqq h_{S^2-\bar{U}}(a + \rho w_1, z'), \quad \text{if} \quad a + \rho w_1 \in S^2 - \bar{U}.$$

Recalling that $h_U(z_1, 0)$ has limits as z_1 approaches ζ' and ζ'' nontangentially, it follows that $h_U(z_1, 0)$ is bounded on the intersection $a + \rho T \cap U$ (since the circle $a + \rho T$ meets T nontangentially at ζ' and ζ''). Similarly, $h_{S^2-\bar{U}}(z_1, 0)$ is bounded on $a + \rho T \cap S^2 - \bar{U}$. Thus there exists a constant M such that $h_U(a + \rho w_1, 0) \leqq M$ if $a + \rho w_1 \in U$, and $h_{S^2-\bar{U}}(a + \rho w_1, 0) \leqq M$ if $a + \rho w_1 \in S^2 - \bar{U}$. Therefore, if we let $z' = 0$ in (3.1.13), we get

$$\int_{T^{n-1}} |u_1(a + \rho w_1) w')| dm_{n-1}(w') \leqq M,$$

for all $w_1 \in T$. Hence

$$\int_T \int_{T^{n-1}} |u_1(a + \rho w_1, w')| dm_{n-1}(w') dm_1(w_1) \leqq M,$$

which shows that $u_1(a + \rho w_1, w')$ is in $L^1(T^n)$. In conjunction with (3.1.12), we can now assert that $u_1(z_1, z')$ is the Poisson integral of $u_1(a + \rho w_1, w')$ in $D_1 \times U^{n-1}$. Consequently, $f_1'' \in P(D_1 \times U^{n-1})$. A parallel argument shows that there is a disc D_2 containing λ_2 such that $f_2'' \in P(D_2 \times U^{n-1})$.

For the next proposition consider the open intervals $J_1 = (-1, 1/2)$, $J_2 = (-1/2, 1)$, and $J = (-1, 1)$. Let K be an arbitrary bounded open interval. Define the rectangles $Q_1 = J_1 + iK$, $Q_2 = J_2 + iK$, $Q = J + iK$, and let $L = L_2 \times L_3 \times \cdots \times L_n$ be an arbitrary polyrectangle (open) in C^{n-1}.

PROPOSITION 3.2. (*Generalized Cartan lemma*). *If* $g \in P((Q_1 \cap Q_2) \times L)$, *there exist* $g_1 \in P(Q_1 \times L)$ *and* $g_2 \in P(Q_2 \times L)$ *such that*

$g = g_1 + g_2$ *on* $(Q_1 \cap Q_2) \times L$.

Proof. Without loss of generality, assume that the rectangles L_m are all equal. Let ϕ be a conformal mapping from the disc U onto L_m, and ψ be a conformal mapping from U to $Q_1 \cap Q_2$. Extend ψ to a homeomorphism between the closures $\bar{U}$ and $\overline{Q_1 \cap Q_2}$. Let $\Lambda_1 = \{-1/2 + iy: y \in K\}$, $\Lambda_2 = \{1/2 + iy: y \in K\}$, and λ_1, λ_2 be the pre-images of Λ_1, Λ_2 under ψ. Let V_1, V_2 be the domains constructed from λ_1, λ_2 as in (3.1). By the reflection principle, we can extend ψ to a conformal mapping ψ_1 from V_1 onto the rectangle $S_1 = (-1/2, 3/2) + iK$; i.e., given $|z| > 1$ define $\psi_1(z) = 1 - \bar{\psi}(1/\bar{z})$. Similarly, ψ can be extended to a conformal mapping ψ_2 from V_2 onto $S_2 = (-3/2, 1/2) + iK$.

Define Φ: $U^n \to (Q_1 \cap Q_2) \times L$ by $\Phi(z_1, z_2, \cdots, z_n) = (\psi(z_1), \phi(z_2), \cdots, \phi(z_n))$, and let Φ_j: $V_j \times U^{n-1} \to S_j \times L$ be the extension of Φ obtained replacing ψ by ψ_j, for $j = 1, 2$.

Suppose $g \in P((Q_1 \cap Q_2) \times L)$. Since the composition $f = g \circ \Phi$ is in $P(U^n)$, there exist functions f_1, f_2 satisfying the properties (a), (b), (c), and (d) of (3.1). If $g_j = f_j \circ \Phi_j$, the following can be verified:

(a′) $g = g_1 + g_2$ on $(Q_1 \cap Q_2) \times L$,

(b′) $g_j \in P((Q_1 \cap Q_2)) \times L$,

(c′) $g_j \in P((S_j - \overline{Q_1 \cap Q_2})) \times L$,

(d′) $g_j \in P(\Phi_j(D_j \times U^{n-1}))$, for $j = 1, 2$.

We claim that $g_j \in P(Q_j \times L)$.

The set $\psi_1(D_1)$ is the intersection of an open subset of $\boldsymbol{C}$, that contains λ_1, with S_1, and $\Phi_1(D_1 \times U^{n-1}) = \psi_1(D_1) \times L$. Consequently, we can find a relatively open polydomain W_δ in the closure $\overline{S_1 \times L}$ such that $W_\delta \cap (S_1 \times L) = \Phi_1(D_1 \times U^{n-1})$. It is also clear that there are relatively open polydomains W_β and W_γ in $\overline{Q_1 \times L}$ such that $W_\beta \cap (Q_1 \times L) = (Q_1 \cap Q_2) \times L$, and $W_\gamma \cap (Q_1 \times L) = (Q_1 - \overline{Q_1 \cap Q_2}) \times L$. Thus we have a covering $W_\beta, W_\gamma, W_\delta$ of $\overline{Q_1 \times L}$ with the properties:

(b″) $g_1 \in P(W_\beta \cap (Q_1 \times L))$,

(c″) $g_1 \in P(W_\gamma \cap (Q_1 \times L))$,

(d″) $g_1 \in P(W_\delta \cap (Q_1 \times L))$.

The hypotheses of (2.1) are satisfied, so $g_1 \in P(Q_1 \times L)$. A parallel argument shows that $g_2 \in P(Q_2 \times L)$.

IV. The Čech cohomology of $\bar{U}^n$ with coefficient in $\mathscr{Q}$. Our goal is that $H^q(\bar{U}^n, \mathscr{Q}) = 0$ for all integers $q \geqq 1$. The standard exact sequence $0 \to Z \to \mathscr{P} \xrightarrow{\exp 2\pi i} \mathscr{Q} \to 0$ reduces this to proving $H^q(\bar{U}^n, \mathscr{P}) = 0$. If X is the cartesian product of n bounded open rectangles in $\boldsymbol{C}$, we have analogous sheaves $\mathscr{P}$ and $\mathscr{Q}$ on $\bar{X}$, and the vanishing of the cohomology of $\bar{U}^n$ with coefficients in $\mathscr{P}$

is entirely equivalent to the corresponding result for $\bar{X}$. In the sequel, we work with X instead of U^n. The reason for this preference is that it allows for the systematic partitioning into smaller polyrectangles used in (4.4).

Let $I = (-1, 1)$. Define $R = I + iI$, and set $X = R^n$. Let $\mathscr{W}$ be the family of all cartesian products of open rectangles whose edges are parallel to the real and imaginary axes of $\boldsymbol{C}$.

Fix an open covering $\mathscr{U} \subset \mathscr{W}$ of $\bar{X}$. *A q-simplex* σ of $\mathscr{U}$ is a $q+1$-tuple $(U_0, U_1, \cdots, U_q)$ of sets in $\mathscr{U}$; its *support* $|\sigma|$ is the intersection $U_0 \cap U_1 \cap \cdots \cap U_q$. If $W \in \mathscr{W}$ is contained in X, define $C^q(\mathscr{U}(W), P)$ to be the group of all alternating functionals γ (*q-cochains*) that assign to each q-simplex σ of $\mathscr{U}$ a function $\gamma(\sigma)$ in $P(|\sigma| \cap W)$ (the zero function if the intersection is empty), and let $\delta: C^q(\mathscr{U}(W), P) \to C^{q+1}(\mathscr{U}(W), P)$ be the standard coboundary operator. The groups $C^q(\mathscr{U}(W), P)$ together with the homomorphisms δ form a cochain complex with *cocycles* $Z^q(\mathscr{U}(W), P)$, *coboundaries* $B^q(\mathscr{U}(W), P)$, and *cohomology groups* $H^q(\mathscr{U}(W), P)$. It is an immediate consequence of (2.1) that $H^0(\mathscr{U}(W), P)$ equals $P(W)$. We define $H^q(\mathscr{U}(W), P) = 0$ if $q < 0$.

If V is a polyrectangle in $\mathscr{W}$ such that $V \subset W$, we have restriction homomorphisms $\rho_{VW}: C^q(\mathscr{U}(W), P) \to C^q(\mathscr{U}(V), P)$ which can easily be seen to commute with the coboundary operators. (If $\gamma \in C^q(\mathscr{U}(W), P)$ and σ is a q-simplex of $\mathscr{U}$, $\rho_{VW}\gamma(\sigma)$ is the restriction of the function $\gamma(\sigma)$ to $|\sigma| \cap V$.) When clear in the context, we shall denote $\rho_{VW}\gamma$ also by γ, and refer to it as the *restriction of* γ *to* $\mathscr{U}(V)$.

For (4.1), (4.2), (4.3), let $I_1 = (-1, 1/2)$, $I_2 = (-1/2, 1)$, $R_1 = I_1 + iI$, $R_2 = I_2 + iI$, and set $X_1^1 = R_1 \times R^{n-1}$, $X_1^2 = R_2 \times R^{n-1}$.

LEMMA 4.1. *If $q \geqq 0$ and if $\gamma \in C^q(\mathscr{U}(X_1^1 \cap X_1^2), P)$, there exist $\gamma_1 \in C^q(\mathscr{U}(X_1^1), P)$ and $\gamma_2 \in C^q(\mathscr{U}(X_1^2), P)$ such that $\gamma = \gamma_1 - \gamma_2$, with the appropriate restrictions to $\mathscr{U}(X_1^1 \cap X_1^2)$.*

Proof. We first observe that (3.2) remains valid if J_1, J_2, J are arbitrary open intervals such that $J_1 \cup J_2 = J$, and such that either $J_1 \subset J_2$, or $J_2 \subset J_1$, or length $J_1 \geqq 1/2$ length J, and length $J_2 \geqq 1/2$ length J. If σ is a q-simplex of $\mathscr{U}$, the polyrectangles $|\sigma| \cap X_1^1$, $|\sigma| \cap X_1^2$, $|\sigma| \cap X_1^1 \cap X_1^2$, will satisfy the modified hypotheses of (3.2); they can be taken as the polyrectangles $Q_1 \times L$, $Q_2 \times L$, and $(Q_1 \cap Q_2) \times L$ of (3.2).

Let $\gamma \in C^q(\mathscr{U}(X_1^1 \cap X_1^2), P)$ and let σ be a q-simplex of $\mathscr{U}$. Since $\gamma(\sigma) \in P(|\sigma| \cap X_1^1 \cap X_1^2)$, we can decompose it as a difference $\gamma_1(\sigma) - \gamma_2(\sigma)$ of functions $\gamma_j(\sigma) \in P(|\sigma| \cap X_1^j)$. Repeating this for each q-simplex we construct $\gamma_1 \in C^q(\mathscr{U}(X_1^1), P)$, $\gamma_2 \in C^q(\mathscr{U}(X_1^2), P)$ such that

$\gamma_1 - \gamma_2 = \gamma$.

LEMMA 4.2. *Let* $q \geqq 0$. *If* $\gamma_1 \in C^q(\mathscr{U}(X_1^1), P)$ *and* $\gamma_2 \in C^q(\mathscr{U}(X_1^2), P)$ *have identical restrictions to* $\mathscr{U}(X_1^1 \cap X_1^2)$, *then* γ_1 *and* γ_2 *must be the restrictions to* $\mathscr{U}(X_1^1)$ *and* $\mathscr{U}(X_1^2)$, *respectively, of some* $\gamma \in C^q(\mathscr{U}(X), P)$.

Proof. Let σ be a q-simplex of $\mathscr{U}$. Then $\gamma_1(\sigma) \in P(|\sigma| \cap X_1^1)$, $\gamma_2(\sigma) \in P(|\sigma| \cap X_1^2)$, and $\gamma_1(\sigma)$ agrees with $\gamma_2(\sigma)$ on the intersection $|\sigma| \cap X_1^1 \cap X_1^2$. Let $\gamma(\sigma)$ be the analytic continuation of $\gamma_1(\sigma)$ given by $\gamma_2(\sigma)$. It follows from (2.1) that $\gamma(\sigma) \in P(|\sigma| \cap X)$. Repeating this procedure for each σ, we define $\gamma \in C^q(\mathscr{U}(X), P)$ with the requirements of the lemma.

DEFINITION 4.3. For each integer $q \geqq 0$, we construct the sequence of homomorphisms $0 \to C^q(\mathscr{U}(X_1^1 \cup X_1^2), P) \xrightarrow{\phi} C^q(\mathscr{U}(X_1^1, P) \oplus C^q(\mathscr{U}(X_1^2), P) \xrightarrow{\psi} C^q(\mathscr{U}(X_1^1 \cap X_1^2), P) \to 0$, where $\phi(\gamma) = (\gamma, \gamma)$ and $\psi(\gamma_1, \gamma_2)) = \gamma_1 - \gamma_2$ (with obvious restrictions). Lemmas (4.1), (4.2) assert that it is an exact sequence. It can be verified that the homomorphisms ϕ, ψ commute with the coboundary operator δ. Consequently, the above is a short exact sequence of the cochain complexes $\{C^q(\mathscr{U}(X_1^1 \cup X_1^2), P), \delta\}$, $\{C^q(\mathscr{U}(X_1^1), P) \oplus C^q(\mathscr{U}(X_1^2), P), \delta \oplus \delta\}$, $\{C^q(\mathscr{U}(X_1^1 \cap X_1^2), P), \delta\}$. As is well known ([1, Th. 3.7, p. 128]), there is an associated long exact sequence

$$
\begin{aligned}
(4.3.1) \qquad & 0 \longrightarrow \cdots \longrightarrow H^{q-1}(\mathscr{U}(X_1^1 \cap X_1^2), P) \\
& \xrightarrow{\delta^*} H^q(\mathscr{U}(X_1^1 \cup X_1^2), P) \\
& \xrightarrow{\phi^*} H^q(\mathscr{U}(X_1^1), P) \oplus H^q(\mathscr{U}(X_1^2), P) \\
& \xrightarrow{\psi^*} H^q(\mathscr{U}(X_1^1 \cap X_1^2), P) \xrightarrow{\delta^*} \cdots .
\end{aligned}
$$

Since, by (3.2), $0 \to P(X_1^1 \cup X_1^2) \xrightarrow{\phi} P(X_1^1) \oplus P(X_1^2) \xrightarrow{\psi} P(X_1^1 \cap X_1^2) \to 0$ is exact, we can assume that in (4.3.1) the first term following zero is $H^1(\mathscr{U}(X_1^1 \cup X_1^2), P)$.

PROPOSITION 4.4. *For any polyrectangle* X *in* $\mathscr{W}$, *for any covering* $\mathscr{U} \subset \mathscr{W}$ *of* $\bar{X}$, *and for any integer* $q \geqq 1$, *the cohomology groups* $H^q(\mathscr{U}(X), P)$ *are trivial.*

Proof. We argue by induction. Suppose that either $q = 1$, or that $q > 1$ and the proposition is true for all positive integers $\leqq q - 1$. Let X be a member of $\mathscr{W}$; assume without loss of generality that $X = X_1^1 \cup X_1^2$ is the polyrectangle of (4.3.1).

If $q > 1$, the inductive hypothesis, applied to $X_1^1 \cap X_1^2$, implies $H^{q-1}(\mathscr{U}(X_1^1 \cap X_1^2), P) = 0$. Hence, the homomorphism

$$(4.4.1) \qquad H^q(\mathscr{U}(X_1^1 \cup X_1^2), P) \xrightarrow{\phi^*} H^q(\mathscr{U}(X_1^1), P) \oplus H^q(\mathscr{U}(X_1^2), P)$$

is one-one for all $q \geqq 1$ (the case $q = 1$ is trivial).

Suppose that $H^q(\mathscr{U}(X), P) \neq 0$, and let ζ be a cocycle in $Z^q(\mathscr{U}(X), P)$ that does not cobound. Since ϕ^* in (4.4.1) is one-one, the restrictions of ζ to $\mathscr{U}(X_1^1)$, and to $\mathscr{U}(X_1^2)$, cannot both cobound. Let $X_1^{k_1}$ be the polyrectangle on which ζ fails to cobound. The procedure that led to (4.3.1) can be repeated for $X_1^{k_1}$, a subdivision $X_1^{k_1} = X_2^1 \cup X_2^2$, and the same covering $\mathscr{U}$. As before, if we apply the inductive hypothesis to $X_2^1 \cap X_2^2$, the homomorphism

$$H^q(\mathscr{U}(X_2^1 \cup X_2^2), P) \xrightarrow{\phi^*} H^q(\mathscr{U}(X_2^1), P) \oplus H^q(\mathscr{U}(X_2^2), P)$$

will also be one-one. Iterating this procedure, proceeding cyclicly through the real and imaginary coordinates of $\boldsymbol{C}^n$, we obtain a nested sequence $X_1^{k_1} \supset X_2^{k_2} \supset \cdots \supset X_m^{k_m} \supset \cdots$ of polyrectangles with diameters eventually decreasing to zero, on none of which the cocycle induced by ζ cobounds. This leads to a contradiction: $\mathscr{U} = \{U_\alpha\}$ is an open covering of $\bar{X}$, so for some integer m and some U_α in the covering, we will have $X_m^{k_m} \subset U_\alpha$; if m is so chosen, the restriction of ζ to $\mathscr{U}(X_m^{k_m})$ trivially cobounds, i.e., if γ is defined by $\gamma(U_0, \cdots, U_{q-1}) = \zeta(U_\alpha, U_0, \cdots, U_{q-1})$, then $\delta\gamma = \zeta$.

COROLLARY 4.5. *Let X be a polyrectangle in $\boldsymbol{C}^n$ and U^n be the unit polydisc in $\boldsymbol{C}^n$. Then, for all integers $q \geqq 1$,*

(a) $H^q(\bar{X}, \mathscr{P}) = 0$,

(b) $H^q(\bar{U}^n, \mathscr{P}) = 0$,

(c) $H^q(\bar{U}^n, \mathscr{Q}) = 0$.

Proof. As was noted earlier, (b) and (c) are direct consequences of (a).

To prove (a) it suffices to show that $H^q(\mathscr{V}, \mathscr{P}) = 0$ for any covering $\mathscr{V} \subset \mathscr{W}$ of $\bar{X}$ (since such coverings are cofinal in the class of all open coverings of $\bar{X}$). Choose such a covering $\mathscr{V}$, and let $\mathscr{U} \subset \mathscr{W}$ be a refinement of $\mathscr{V}$ such that the closure of each member U of $\mathscr{U}$ is contained in some polyrectangle μU of $\mathscr{V}$. Let $\sigma = (U_0, U_1, \cdots, U_q)$ be a simplex of $\mathscr{U}$ and let $\mu\sigma = (\mu U_0, \mu U_1, \cdots, \mu U_q)$ be the corresponding simplex of $\mathscr{V}$. Recall that a section $\gamma \in \Gamma(|\mu\sigma|, \mathscr{P})$ can be naturally identified with a holomorphic function f_γ in $|\mu\sigma| \cap X$, and that the restriction of f_γ to $|\sigma| \cap X$ will be in $P(|\sigma| \cap X)$. With this in mind, we construct, for each integer $q \geqq 0$, a one-one homomorphism

$$C^q(\mathscr{V}, \mathscr{P}) \xrightarrow{\mu^*} C^q(\mathscr{U}(X), P)$$

defined by letting $\mu^*(\gamma)(\sigma)$ be the function f_γ restricted to $|\sigma| \cap X$. A straightforward calculation shows that μ^* commutes with the coboundary operators; consequently it induces a one-one homomorphism

$$H^q(\mathscr{V}, \mathscr{P}) \xrightarrow{\mu^*} H^q(\mathscr{U}(X), P) .$$

It follows, by (4.4), that $H^q(\mathscr{V}, \mathscr{P}) = 0$, which completes the proof.

V. Remarks. In [7] Stout proved that the multiplicative Cousin problem with bounded data can be solved in the polydisc U^n. If, as in [6], we let $\mathscr{H}$ be the sheaf of germs of locally bounded holomorphic functions, and $\mathscr{E}$ be the sheaf of multiplicative groups of invertible elements of $\mathscr{H}$, this is equivalent to the assertion that $H^1(\bar{U}^n, \mathscr{E})$ is trivial. If we apply the methods of § IV to the sheaf $\mathscr{P}_b$ (defined in [6]) of locally bounded pluriharmonic functions on $\bar{U}^n$, it follows that $H^q(\bar{U}^n, \mathscr{E}) = 0$ for all $q > 1$, as well.

The methods used for the study of the sheaf $\mathscr{P}$ can be also applied to obtain similar results for the sheaves $\mathscr{H}^p$, induced by assigning to each relatively open polydomain $W \subset \bar{U}^n$ the Hardy space $\mathscr{H}^p(W \cap U^n)$. If $p > 1$, Lemma 3.1 holds word for word if everywhere we replace the letter P by the symbol $\mathscr{H}^p$; it then can be proven, as was done for the sheaf $\mathscr{P}$, that $H^q(U^n, \mathscr{H}^p) = 0$ for all $q \geqq 1$. A simpler procedure, however, is to show that the sheaves $\mathscr{H}^p$ correspond to a particular case of the boundary conditions studied by Nagel in [4].

Finally, we mention that as a consequence of $H^1(\bar{U}^n, \mathscr{Q}) = 0$, it is possible to solve the multiplicative Cousin problem with N^*-data in U^n (in [8], the corresponding problem for the Nevanlinna class N was shown to be solvable). By standard arguments (such as in [3, Cor. 2, p. 47]) it can be shown that $H^1(\mathscr{U}, \mathscr{Q}) = 0$ for any covering $\mathscr{U} = \{U_\alpha\}$ of $\bar{U}^n$. If $\mathscr{U}$ consists of relatively open polydomains, and if for each α we are given $f_\alpha \in N^*(U_\alpha \cap U^n)$ such that $f_\alpha f_\beta^{-1}$ is an outer function in the intersection $U_\alpha \cap U_\beta \cap U^n$ (a cocycle in $Z^1(\mathscr{U}, \mathscr{Q})$), there must exist $F \in N^*(U^n)$ with the property that Ff_α^{-1} is an outer function in $U_\alpha \cap U^n$ for each α.

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