MEIER TYPE THEOREMS FOR GENERAL BOUNDARY APPROACH AND σ-POROUS EXCEPTIONAL SETS

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In this paper we are concerned with determining under what conditions equality is obtained between two different cluster sets of a function f at a point on the boundary of its domain. Specifically for functions defined in the unit disc D in the complex plane taking values in the extended plane we show that the generalized angle cluster set equals the generalized outer angular cluster set at all points of the boundary of D except possibly for a σ-porous set. The definition of both generalized cluster sets includes the usual Stolz angle definition but this result generalizes the known results. In addition the proof is shorter than proofs of less general results.

If f is required to be meromorphic in D then an application of the principal result gives a decomposition of the boundary of D into a set of generalized normal points of f, a set of generalized Picard points of f, and a σ-porous set. The third result gives a different decomposition into generalized Plessner points; generalized pre-Meier points and a σ-porous set. Again these results generalize known results.

The notion of porosity was introduced in 1967 by E. P. Dolženko. While he defined porous sets in higher dimensions as well we shall limit our considerations to $C: |z| = 1$. Porous sets have zero Lebesgue measure and are of first Baire category and in addition isolate properties essential for certain cluster set considerations. In turn this allows generalizations of results of Meier [10] and others. In a series of papers Yoshida [14-20] extended results of Meier and others by using the notion of porosity. Yoshida's fundamental lemma [15, Lemma 1] has a very complicated statement and proof. In this paper we expand this lemma; offer a succinct proof which hopefully illuminates better the character of the results; and apply it to obtain generalizations of some of Yoshida's results as well as those of Dragosh ([6], [7]), Colwell [4] and Yanigahara [13].

2. **Definitions and notation.** Let D be the open unit disc and $P \subseteq C$. For each $e^{i\theta} \in C$, let $\gamma(\theta, \varepsilon, P)$ be the length of the largest subarc of the arc $(e^{i(\theta-\varepsilon)}, e^{i(\theta+\varepsilon)})$ which does not meet P. If no such arc exists define $\gamma(\theta, \varepsilon, P) = 0$. According to Dolženko [5], P is *porous* at $e^{i\theta}$ if

(2.0)
$$\varlimsup_{\varepsilon \to 0} \frac{\gamma(\theta, \varepsilon, P)}{\varepsilon} > 0 \; .$$

DEFINITION 1. A subset $P \subseteq C$ is a porous set if it is porous at each $p \in P$; P is σ-porous if it is the finite or countable union of porous sets.

Condition (2.0) implies that the density of P at $e^{i\theta}$ is not unity and so a porous set must have zero Lebesgue measure as does a σ-porous set. Because a porous set is nowhere dense, a σ-porous set is also of first Baire category. Thus porosity contains both the metric and topological features found in Meier-type theorems. It is known [3, p. 75] there exists a residual set on C with zero Lebesgue measure and so zero Lebesgue measure does not imply σ-porosity. The set of rationals is an example of a set of first category and measure 0 which is not porous. The existence of a set which is of first category and measure 0 but not σ-porous was asserted by Dolženko [5, pg. 3] and was constructed by Zajíček [21, Prop. 5.2].

Zero capacity and σ-porous sets do not have any obvious connections. If E is the Cantor middle thirds set on C and $e^{i\theta} \in E$, it is easy to see that $\gamma(\theta, 1/6^n, E) \geqq (1/6^n)(2/3)$, $n = 1, 2, \cdots$, and so E is porous. On the other hand E is known to have positive capacity. To complete the analysis we construct a residual set K (thus K is not σ-porous) which has zero capacity. It will be the intersection of complements of Cantor-type set. For convenience we construct the set on $[0, 1]$. Begin by letting H_1 be the closed interval of length $1/4$ symmetric about $1/2$; then let $\{F_i^{(2)}\}$, $i = 1, 2$, be the closed intervals symmetric about $1/2^2$ and $3/2^2$ respectively, with each interval of equal length and chosen so that, setting $H_2 = F_1^{(2)} \cup F_2^{(2)} \cup H_1$,

$$\operatorname{cap} H_2 \leqq \operatorname{cap} H_1 + \frac{1}{2^2} \; .$$

This is clearly possible. Proceed inductively defining H_n by adjoining to H_{n-1}, 2^{n-1} closed intervals of equal length symmetric about the points $i/2^n$, $i = 1, 3, \cdots, 2^n - 1$ and chosen so that

$$\operatorname{cap} H_n \leqq \operatorname{cap} H_{n-1} + \frac{1}{2^n} \; .$$

The H_n's are closed sets with $H_{n-1} \subseteq H_n$ and thus setting $K_1 = \bigcup_{n=1}^{\infty} H_n$,

$$\operatorname{cap} K_1 = \lim_{n \to \infty} \operatorname{cap} H_n \leqq \lim_{n \to \infty} \left[(\operatorname{cap} H_1) + \sum_{k=2}^{n} \frac{1}{2^k} \right] = \frac{1}{16} + \frac{1}{2} = \frac{9}{16} \; .$$

Let K_2 be the set obtained by the same method as used for K_1 except

that we begin by selecting the initial interval about 1/2 with capacity 1/32, and all subsequent closed intervals symmetric about the same midpoints but chosen so that the new H_n's satisfy

$$\operatorname{cap} H_n \leqq \operatorname{cap} H_{n-1} + \frac{1}{2^{n+1}} .$$

Thus $K_2 \subseteqq K_1$, and $\operatorname{cap} K_2 \leqq 9/2^3$. Defining K_n, $n > 2$, in the same fashion and setting $K = \bigcap_{n=1}^{\infty} K_n$, we have

$$\operatorname{cap} K = 0 .$$

Because each K_n is a residual set so also is K. (I wish to thank Professor W. Schneider for suggesting this construction.)

For future use we note that if P is not σ-porous then the subset P^* of P of all points at which P is not porous is uncountable. Otherwise since $P - P^*$ is porous—as is easily seen—and a singleton set is obviously porous we would have that P is σ-porous.

DEFINITION 2. If a property $\mathscr{P}$ holds for all points of $E \subseteqq C$ except possibly on a σ-porous subset of E, we say that $\mathscr{P}$ holds strongly on E, or that strongly on E points satisfy property $\mathscr{P}$.

Substs of C which are not σ-porous have the happy property of exerting strong collective influence on the boundary behavior of functions f at certain points of C. This is best illustrated by Theorem 3, although out first lemma indicates the basic connection between porosity and boundary behavior. This lemma was proved in the angular case by Dolženko [5, Eng. translation, pg. 5], and in a more general setting by Yoshida [15, Lemma 1], although Dragosh [6, Lemmata 4, 5] proved a horocyclic version using metric density instead of porosity. We present it in a still more general setting and a result of Satyanarayana and Weiss [12, Theorem 5] both simplifies the proof and extends Yoshida's result. (We thank Professor L. Garbanati for this last reference.)

We wish to define approach to a point of C inside a reasonably nice subdomain of D and so let $h(\varphi)$ be a real valued function defined in some open interval I_h on the real line symmetric about 0. On I_h we require that

(2.1)
 (i) h be continuous;
 (ii) $h(\varphi) = h(-\varphi)$;
 (iii) $h(0) = 0$, $h(1) = 1$, $h(\varphi_1) < h(\varphi_2) < 1$, $0 \leqq \varphi_1 < \varphi_2$.

Call such an h an *approach function*. In addition if $h''(\varphi)$ exists for $\varphi \in I_h$, $\varphi \neq 0$, and is strictly positive, then h is said to be a *convex*

approach function. An approach function h defines a boundary curve at $e^{i\theta} \in C$ as follows. Let $\varphi - \theta \in I_h$ and set

(2.2) $$z(\varphi; \theta, h) = (1 - h(\varphi - \theta))e^{i\varphi} .$$

Then $z(\varphi; \theta, h)$ is a symmetric curve with respect to the radius at $e^{i\theta}$ and determines a boundary domain at $e^{i\theta}$, according to the following

(2.3) $$\Omega(\theta, h) = \{re^{i\varphi} \mid r \leqq 1 - h(\varphi - \theta)\} .$$

Note that when $|I_h| < \pi$, Ω is convex if h is. For $0 \leqq r < 1$, a truncated boundary domain is

(2.4) $$\Omega_r(\theta, h) = \Omega(\theta, h) \cap \{|z| \geqq r\} .$$

Henceforth if $A \subseteq D$, A_r will mean $A \cap \{|z| \geqq r\}$. If $h(\varphi) = \varphi^{q+1}$, $q \geqq 0$, the case $q = 0$, defines the usual nontangential approach; for $q = 1$, we have horocyclic approach; and for arbitrary $q \geqq 0$, the situation considered by Yoshida. For $q > 0$ it is necessary to define right and left h-angles at $e^{i\theta}$. To this end for $c > 0$ put $h^c(\varphi) = h(\varphi/c)$, and for $0 < a < b$, define the right angle

(2.5) $R\mathscr{A}(\theta, a, b, h) = \{re^{i\varphi} \mid 1 - h^a(\varphi - \theta) < r < 1 - h^b(\varphi - \theta); \varphi \geqq \theta\} ,$

while a left h-angle $L\mathscr{A}(\theta, a, b, h)$ is defined as in (2.5) except $\varphi \geqq \theta$ is replaced by $\varphi \leqq \theta$. A reduced right h-angle $R\alpha(\theta, a, b, h)$ (or reduced left h-angle) is defined by replacing h^a and h^b in (2.5) by h/a and h/b respectively. Any choice of $0 < a < b$ defines a right and left h-angle and we omit any of the parameters in the symbol if we can safely ignore them.

The reduced h-angle α is so called because for a convex approach function h, and any $0 < a < b$,

(2.6) $$R\alpha\left(\theta, \frac{2a}{a+b}, \frac{2b}{a+b}, h^{(a+b)/2}\right) \subseteq R\mathscr{A}(\theta, a, b, h) ;$$

with left h-angles also satisfying (2.6). To see this we combine the definition of angles in (2.5) and the fact that for convex approach functions h' is increasing for $\varphi \neq 0$ to argue via the generalized mean value theorem that

(2.7) $$\frac{\dfrac{a+b}{2a} h\left(\dfrac{(\varphi - \theta)2}{a+b}\right)}{h\left(\dfrac{\varphi - \theta}{a}\right)} = \frac{\left(\dfrac{a+b}{2a}\right)\left(\dfrac{2}{a+b}\right)h'\left(\overline{\dfrac{(\varphi - \theta)2}{a+b}}\right)}{\dfrac{1}{a}h'\left(\overline{\dfrac{\varphi - \theta}{a}}\right)} > 1 .$$

A similar result holds for the other pair of boundary curves. Because $h^{(a+b)/2}$ is a convex approach function the result of Satyanaraya and

Weiss [12, Theorem 5] implies that the hyperbolic Hausdorff distance at $e^{i\theta}$ between the boundary curves of the reduced h-angle in (2.6) is asymptotic to $1/2 \log (1 + 2|a - b|/1 - 2|a - b|)$ as $|z| \to 1$, and so not only does a right h-angle defined by a convex h contain a reduced h-angle but one of a positive hyperbolic width.

Let $\rho(a, b)$ denote the hyperbolic distance in D and $N(a, s) = \{b \in D \,|\, \rho(a, b) < s\}$. The above result reinterpreted says that a (reduced) h-angle at $e^{i\theta}$—provided h is convex—contains (close to $e^{i\theta}$) discs of a fixed hyperbolic radius. Finally note that if $h(\theta) = k\theta^{q+1}$, $q \geqq 0$, $k > 0$, then h-angles and reduced h-angles are equivalent.

3. **Fundamental lemma.** The lemma first is given in a set-theoretic fashion using approach functions, and then in a hyperbolic geometric setting using convex approach functions.

LEMMA 1 (Spotlight Lemma). *Let $P \subseteqq C$, such that P is not porous at $e^{i\theta}$. Given an approach function h and positive numbers $a < b$, r', c, there is a value $r > 0$, such that*

$$(3.0) \qquad \Omega_r(\theta; h^c) \subseteqq \bigcup_{e^{i\varphi} \in P} R\mathscr{A}_{r'}(\varphi, a, b, h) \,;$$

and the same result holds for left h-angles.

REMARK. The right h-angles are congruent under rotation and can be thought of as beams of light with a positive width emanating from each point of P. The beams are hyperbolically wide if h is convex. The lemma says that if all the lights are turned on there is an $r > 0$ such that the set $\Omega_r(\theta, h^c)$ is fully illuminated.

Proof. It is easy to show that it is sufficient to prove the lemma for the case $c = 1$, and $\theta = 0$. We first consider the right angle situation. Suppose the lemma is false. Then for some choice of $0 < a < b$ and $r' > 0$, there is a sequence $\{z_n\}$ tending to 1 inside $\Omega(0, h)$ such that

$$(3.1) \qquad z_n \notin \bigcup_{e^{i\theta} \in P} R\mathscr{A}_{r'}(\theta, a, b, h) \,,$$

that is, the z_n lie in the umbra of all right h-angle lights at P. Let $z(\varphi; \theta_n^a, h^a)$ and $z(\varphi; \theta_n^b, h^b)$ be the unique h-angle boundary curves through z_n. See Figure 1 for a description of these and other pertinent sets. Assume first that $\theta_n^b < \theta_n^a < 0$. The arc $(e^{i\theta_n^b}, e^{i\theta_n^a})$ then lies in the complement of P. We will show that the ratio

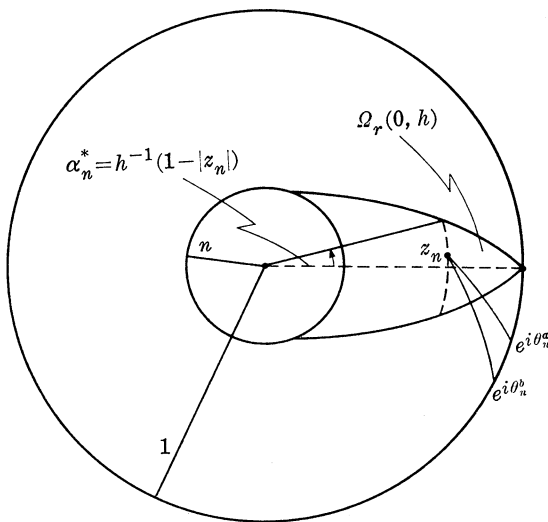$$(3.2) \qquad \frac{\theta_n^b - \theta_n^a}{\theta_n^b}$$

FIGURE 1

is bounded away from zero which contradicts the porosity hypothesis. The two curves $z(\varphi; \theta_n^a, h^a)$ and $z(\varphi; \theta_n^b, h^b)$ were chosen to meet at $z_n = |z_n| e^{i\varphi_n}$. Thus

$$(3.3) \qquad |z_n| = 1 - h\left(\frac{\varphi_n - \theta_n^a}{a}\right) = 1 - h\left(\frac{\varphi_n - \theta_n^b}{b}\right).$$

On the boundaries of right h-angles h^{-1} exists and is positive. From (3.3)

$$(3.4) \qquad \frac{\varphi_n - \theta_n^a}{a} = h^{-1}(1 - |z_n|) = \frac{\varphi_n - \theta_n^b}{b}.$$

Since $z_n \in \Omega(0, h)$ we have that $|z_n| < 1 - h(\varphi_n)$ and after applying h^{-1} to this inequality

$$(3.5) \qquad \frac{|\varphi_n|}{h^{-1}(1 - |z_n|)} < 1.$$

Using (3.4) and (3.5), the ratio in (3.2) equals

$$\frac{b - a}{b - \dfrac{\varphi_n}{h^{-1}(1 - |z_n|)}} > \frac{b - a}{b + 1} > 0.$$

Thus a contradiction is reached. If $0 \le \theta_n^b < \theta_n^a$ a lower bound for $(\theta_n^a - \theta_n^b)/\theta_n^a$ is found to be $b - a/a + 1$, while if $\theta_n^b < 0 < \theta_n^a$, P is porous at 1 by definition. The proof for left angles is done by an entirely similar analysis.

LEMMA 2. *Let $P \subseteqq C$, and suppose P is not porous at $e^{i\theta}$. Let h be a convex approach function and $0 < a < b$, $n' > 0$ and $c > 0$ be given. Then there exists $r > 0$, $s > 0$, such that for each sequence $\{z_n\}$ tending to $e^{i\theta}$ in $\Omega_r(\theta, h^c)$, there is a corresponding sequence $\{e^{i\varphi_n}\}$ in P, with the property that*

(3.6) $$N(z_n, s) \subseteqq R\mathscr{A}_{r'}(\varphi_n, a, b, h) , \qquad all \quad n .$$

A corresponding statement holds for left h-angles as well.

Proof. Choose $0 < a < a' < b' < b$. Then by Lemma 1 the collection of rays $R\mathscr{A}_{r'}(\varphi, a', b', h)$, $e^{i\varphi} \in P$, cover $\Omega_r(\theta, h^c)$ for some $r > 0$. Select a point $e^{i\theta_n} \in P$ for which $z_n \in R\mathscr{A}_{r'}(\varphi_n, a', b', h)$. Of course z_n is also in the larger ray $R\mathscr{A}_{r'}(\varphi_n, a, b, h)$—larger by a fixed hyperbolic distance according to the remarks following (2.7) and the observation that each of the two sets between the larger and smaller rays is itself a ray. Consequently $R\mathscr{A}_{r'}(\varphi_n, a, b, h)$ contains $N(z_n, s)$ from some $s > 0$, and all n, and the right h-angle statement of the lemma is proved. A similar argment validates the left angle case.

If h_1 and h_2 are two approach functions with $h_2 < h_1$, and if some collection of h_2 rays cover $\Omega_r(\theta, h_2)$, the same collection also cover $\Omega_r(\theta, h_1)$. For example if a collection of horocyclic rays cover an oricyclic domain they also cover *any* angular domain.

4. Consequences of the spotlight lemma. The first consequence of Lemma 1 is a statement about cluster sets of arbitrary functions and generalizes the results of Dragosh [6, Lemma 6], Doležnko [5, Theorem 1], Yoshida [15, Theorem 2] and Yanagihara [13]. If $E \subseteqq D$ such that $e^{i\theta} \in \bar{E} \cap C$, then the cluster set of a function f along E at $e^{i\theta}$ will be denoted by $C(f, \theta, E)$, with again any parameter omitted if safely done. And let W be the extended plane.

THEOREM 1. *Let f be defined in D taking values in W, and h a given approach function. Then strongly on C, for any choice of $0 < a < b$, and $c > 0$,*

(4.0) $$C(f, \mathscr{A}(\theta, a, b, h)) = C(f, \Omega(\theta, h^c)) .$$

Proof. There is no loss of generality in assuming all angles are right h-angles. Further by a simple argument it is easy to show that we need prove the result only in the case a, b, and c are rational with $b < c$, and so we suppose the result is false in this case. Then for some rational numbers $0 < a < b < c$ there exists a non-σ-porous set $E \subseteqq C$ such that for each $e^{i\theta} \in E$ the set on the left in (4.0) is a

proper subset of the set on the right. Select any countably dense subset P of W and let R be the positive rationals. Then $e^{i\theta} \in E$ if and only if for some choice of $p_0 \in P$ and $r_0 \in R$ the closed disk $D(p_0, r_0)$ with center p_0 and radius r_0 is entirely contained in the complement of $f(\mathscr{A}_{r'}(\theta, a, b, h))$ for some rational $r' \in R$, and such that $D(p_0, r_0/2) \cap C(f, \Omega(\theta, h^{r_1})) \neq \varnothing$. The set of all $e^{i\theta} \in E$ which satisfies the above situation relative to the parameters (a, b, c, r', p_0, r_0) we denote by $E(a, b, c, r', p_0, r_0)$. Thus E is the countable union of all such $E(a, b, c, r', p_0, r_0)$:

$$(4.1) \qquad\qquad E = UE(a, b, c, r', p_0, r_0) \, .$$

Then at least one set in the union (4.1) is not porous. Let $E(a, b, c, r', p_0, r_0) = \widetilde{E}$ be such a nonporous set and select a point $e^{i\theta} \in \widetilde{E}$ at which $\widetilde{E}$ is not porous. From Lemma 1 there is some value $r > 0$ for which

$$(4.2) \qquad\qquad \bigcup_{e^{i\varphi} \in \widetilde{E}} \mathscr{A}_{r'}(\varphi, a, b, h) \supseteq \Omega_r(\theta, h^c) \, .$$

From our definition of $\widetilde{E}$ we know that if $e^{i\varphi} \in \widetilde{E}$ then $f(\mathscr{A}_{r'}(\varphi, a, b, h))$ omits the closed disk with center p_0 and radius r_0 and so (4.2) implies that $f(\Omega_r(\theta, h^c))$ also omits this same closed disk. But then $C(f, \Omega(\theta, h^c))$ does not meet $D(p_0, r_0/2)$ contrary to the fact that $e^{i\theta} \in \widetilde{E}$. Indeed E is σ-porous.

To state some immediate consequences of the lemma in familiar form we need some more definitions. Let h be an approach function and f a function in D with values in W. A point $e^{i\theta}$ is called an *h-Plessner* (*h-Fatou*) point of f if $C(f, \mathscr{A}(\theta, h)) = W(C(f, \mathscr{A}(\theta, h))) = \{w_0\})$, for all h-angles at $e^{i\theta}$. For $h(\theta) = \theta$ we obtain the standard Plessner (Fatou) point. The *h-Fatou value* w_0 is to be independent of the particular h-angle used. If we use reduced h-angles in place of standard h-angles in the above we say $e^{i\theta}$ is a *reduced h-Plessner* or *reduced h-Fatou point*.

Henceforth we shall always assume a sequence $\{h_n\}$ of approach functions has the two additional properties that $h_n \geqq h_{n+1}$, and $h_1(\theta) = \theta$. If $e^{i\theta}$ is an h_n-Plessner (h_n-Fatou) point of f for every $h_n \in \mathscr{H}$, we say $e^{i\theta}$ is an *$\mathscr{H}$-Plessner* (*$\mathscr{H}$-Fatou*) point of f. In the Fatou case we do not assume that the various h_n-Fatou values are equal. The sequence of polynomial approach functions $h_n(\theta) = \theta^n$, $n = 1, 2, \cdots$, we call $\mathscr{P}$. Finally let $K_h(f)$ be the set of all points $e^{i\theta}$ at which the cluster sets of f along any two h-angles at $e^{i\theta}$ are equal, and put $K_{\mathscr{H}}(f) = \bigcap_{h \in \mathscr{H}} K_{\mathscr{H}}(f)$. Because a countable union of σ-porous sets is again σ-porous Theorem 1 can be rephrased as

THEOREM 1′.   *Let $\mathscr{H} = \{h_n\}$ be a sequence of approach functions and f a function in D.   Then strongly* (i) *Plessner points of f are $\mathscr{H}$-Plessner points of f;* (ii) *for any fixed n the h_n-Fatou points of f are also h_k-Fatou points for any $k \leq n$ (with the same h_n-Fatou value);* (iii) *$K_*(f)$ is all of C.*

To arrive at Meier-type theorems we need to limit ourselves to meromorphic functions and state a few more definitions.   For $e^{i\theta} \in C$, let $\varLambda(f, \theta, h)$ be the set of all values $w \in W$ such that w is assumed infinitely often by f in *each* (right or left) h-angle at $e^{i\theta}$, h an approach function.   If we use only right (left) h-angles in the above we write $R\varLambda(f, \theta, h)$ $(L\varLambda(f, \theta, h))$.   If $W - \varLambda(f, \theta, h)$ contains at most two values then $e^{i\theta}$ is called an *h-Picard point of f*.   Let χ be the chordal distance on W.   A point $e^{i\theta}$ is an *h-normal point of f* provided for each $c > 0$, and for any two sequences $\{z_n\}$ and $\{z_n'\}$ with $z_n \in \varOmega(\theta, h^c)$ all n, the property $\rho(z_n, z_n') \to 0$, $n \to \infty$ implies $\chi(f(z_n), f(z_n')) \to 0$, $n \to \infty$.   (In the angular case this is a definition of J. M. Anderson [1, p. 103].)   If $\mathscr{H} = \{h_n\}$ is a sequence of approach functions and $e^{i\theta}$ is an h_n-Picard (h_n-normal) point of f for each $h_n \in \mathscr{H}$ then $e^{i\theta}$ is said to be an *$\mathscr{H}$-Picard ($\mathscr{H}$-normal)* point of f.

THEOREM 2.   *Let f be meromorphic in D and h a convex approach function.   Let $Q \subseteq C$ be a set of non-h-Picard points of f. If Q is not σ-porous there exists a nondenumerable set $Q^* \subseteq Q$ of h-normal points of f.*

Proof.   In the fashion of Theorem 1 — and following Meier [10] — we write Q as a countable union of fairly nice sets.   Let R be the positive rationals.   In addition let $\mathscr{T}$ be the (countable) collection of triples $\{D_1, D_2, D_3\}$, where   each D_i, $i = 1, 2, 3$, is a closed disc in W with rational center and radius and the triple of sets are pairwise disjoint.   Thus there exists a $d_T > 0$ such that if $T = (D_1, D_2, D_3)$, then $\chi(D_i, D_j) \geq d_T$, $i \neq j$.   Distinguish the set of all points $e^{i\varphi} \in Q$ at which f omits some three values $w_i^{(\varphi)}$, $i = 1, 2, 3$, (all distinct), respectively in the three truncated h-angles at $e^{i\theta}$, $\mathscr{A}_{r_{1i}}(\varphi, r_{2i}, r_{3i}, h)$ each $r_{hi} \in R$ and $w_i^{(\varphi)} \in D_i$, $i = 1, 2, 3$, for some $T = (D_1, D_2, D_3) \in \mathscr{T}$.   Thus

$$(4.3) \qquad \chi(w_i^{(\varphi)}, w_j^{(\varphi)}) \geq d_T , \qquad i \neq j ; \qquad i, j = 1, 2, 3 .$$

The three h-angles associated with the triad $(w_1^{(\varphi)}, w_2^{(\varphi)}, w_3^{(\varphi)})$ are one of eight possible combinations of h-angles according as to whether they are left or right angles.   We use $\kappa = 1, 2, \cdots, 8$ to designate the combinations.

Label the set of all $e^{i\varphi} \in Q$ satisfying the above conditions as $Q(\{r_{ki}\}_{i,i=1}^{3}, T, \kappa)$. Then Q is the countable union of all such sets. Consequently at least one set in the union, say $\tilde{Q}$ is not porous, and let Q^* be the nondenumerable subset of $\tilde{Q}$ at which $\tilde{Q}$ is not porous. To show any $e^{i\theta} \in Q^*$ is a h-normal point of f fix a domain of the form $\Omega(\theta, h^c)$ and consider two sequences $\{z_n\}$, $\{z'_n\}$, with $z_n \in \Omega(\theta, h^c)$ and $\rho(z_n, z'_n) \to 0$, $n \to \infty$. Notice that for each value $i = 1, 2, 3$, the corresponding collection of h-angles emanating from points of $\tilde{Q}$ is composed of all right or left h-angles and so by Lemma 2 where are three sequences in $\tilde{Q}$, $e^{\varphi_n^{(i)}}$, $i = 1, 2, 3$, such that for all $n > N_i$, and some $\rho_i > 0$

$$N(z_n, \rho_i) \subseteqq \mathscr{A}_{r_{1i}}(\varphi_n^{(i)}, r_{2i}, r_{3i}, h), \qquad i = 1, 2, 3.$$

Thus f omits *all* three values $\{w_i^{\varphi_n^{(i)}}\}$, $i = 1, 2, 3$, in $N(z_n, \rho_0)$, $\rho_0 = \min \rho_i$, $i = 1, 2, 3$, for n sufficiently large, and the triad of points $\{w_i^{\varphi_n^{(i)}}\}$ satisfies (4.3) for fixed d_T. According to Gauthier [8, p. 281], this implies that $\chi(f(z_n), f(z'_n)) \to 0$, $n \to \infty$, that is, $e^{i\theta}$ is an h-normal point. The result of Gauthier cited here involves the ρ-sequences of Lange [9].

This result which generalizes Dragosh [7, Theorem 9] and Yoshida [17, Theorem 4] can be phrased as a strong decomposition of C.

THEOREM 2′. *Let f be meromorphic in D and $\mathscr{H}$ a sequence of convex approach functions. Then strongly on C points are either $\mathscr{H}$-Picard points of f or $\mathscr{H}$-normal points of f.*

Proof. For any $h \in \mathscr{H}$ the set of points of C which are neither h-Picard points nor h-normal points of f must be σ-porous else by Theorem 2 that set contains h-normal points.

The theorems of Meier, Yoshida, and to some extent Dragosh previously mentioned arise by exploiting the behavior of f at an h-normal point. To generalize and strengthen these theorems a few more standard cluster sets need to be defined. Because we rely on the usual normal function arguments we sometimes have to use both convex approach functions and reduced angles in order to insure that these angles have a positive hyperbolic distance from boundary to boundary. Again we mention that for approach functions of the form $h(\theta) = \theta^{q+1}$, $q \geqq 0$, h-angles and reduced h-angles are equivalent.

Let h be a given approach function and f a complex function in D. For $e^{i\theta} \in C$, let

$$\pi(f, \theta, h) = \bigcap_{\gamma} C(f, \theta, \gamma),$$

where the intersection is taken over all curves γ which tend to $e^{i\theta}$ and which have the property that each γ is contained in its own reduced h-angle at $e^{i\theta}$. A curve tending to $e^{i\theta}$ is said to be an h-admissible curve if there is a nested sequence of reduced angles at $e^{i\theta}$ $\{\alpha(\theta, a_n, b_n, h)\}$ with $b_n - a_n \to 0$, such that each angle contains a truncated subarc γ_{r_n} of γ. Let

$$\pi^*(f, \theta, h) = \bigcap_{\gamma} C(f, \theta, h) ,$$

where the intersection is over all h-admissible curves γ at $e^{i\theta}$. If we restrict the reduced angles to be either all right or left reduced h-angles in these two definitions we write $R\pi$, $R\pi^*$ or $L\pi$, $L\pi^*$ as the corresponding cluster sets. Suppose h is a convex approach function and $e^{i\theta}$ is both a reduced h-Plessner and an h-normal point of f. It is easy to see that

$$\pi^*(f, \theta, h) = W .$$

In addition if there is some value $w_0 \notin R\Lambda(f, \theta, h)$, then we can prove

(4.4) $w_0 \in R\pi(f, \theta, h)$;

with a similar result holding for left h-angles. To show (4.4) consider the reduced (truncated) right h-angle in which f omits w_0. Because $e^{i\theta}$ is a reduced h-Plessner point of f we can find a sequence $\{z_n\}$ in a smaller h-angle inside of which $f(z_n) \to w_0$ and so there is a $\rho_0 > 0$ such that f omits w_0 in $N(z_n, \rho_0)$ for n sufficiently large. Because $e^{i\theta}$ is an h-normal point this implies that in any $\Omega(\theta, h^c)$ containing the reduced right angle f tends to w_0 in $\bigcup_{n=1} N(z_n, \rho)$, for *any* choice of $\rho > 0$. Each curve tending to $e^{i\theta}$ in any reduced right h-angle meets each of the sets $N(z_n, \rho)$ for some value ρ and n large. Thus w_0 is in the cluster set of f along all curves in the reduced right angle and (4.4) holds. These remarks together with Theorem 2 give

THEOREM 3. *Let f be meromorphic in D and h a convex approach function. Then in a strong sense the reduced h-Plessner points of f are either h-Picard points or points $e^{i\theta}$ at which both*
  (i) *$\pi^*(f, \theta, h) = W$;*
  (ii) *$R\Lambda(f, \theta, h) \cup R\pi(f, \theta, h) = W = L\Lambda(f, \theta, h) \cup L\pi(f, \theta, h)$.*

This theorem strengthens results of Colwell [4, Theorem 3] and Dragosh [7, Theorem 10 (i), (ii), (iii)] in the case $h(\theta) = \theta$; and Yoshida's result [18, Theorem 1], if $h(\theta) = \theta^{q+1}$, $q \geqq 0$.

We could restate Theorem 3 for a sequence of approach functions but do so only for the polynomial sequence $\mathscr{P}$ for then reduced

angles and angles are equivalent and a more familiar appearing theorem results.

THEOREM 3'. *Let f be meromorphic in D. Then in a strong sense the Plessner points of f are either $\mathscr{P}$-Picard points or points $e^{i\theta}$ at which both*

(i) $\pi^*(f, \theta, h_n) = W$, *and*

(ii) $RA(f, \theta, h_n) \cup R\pi(f, \theta, h_n) = W = LA(f, \theta, h_n) \cup L\pi(f, \theta, h_n)$, *for each $h_n \in \mathscr{P}$.*

We close by showing how our results contain result of Yoshida on pre-Meier points previously referred to. Theorem 1' tells us that strongly points of C are $K_h(f)$ points while Theorem 2' say that strongly points are either *h-Picard* or *h*-normal point. If we restrict $h(\theta) = \theta^{q+1}$, $q \geqq 0$, and suppose $e^{i\theta}$ is an *h*-normal point contained in $K_h(f)$ but is not an *h*-Plessner point, then for any angle $\alpha(\theta, h)$

(4.5) $\pi^*(f, \theta, h) = C(f, \alpha(\theta, h)) \subsetneqq W$.

A point satisfying (4.5) is called an *h*-pre-Meier point. Of course any *h*-Picard point of f is already an *h*-Plessner point of f so we have shown the result of Yoshida in [19, Theorem 1].

THEOREM 4. *Let f be meromorphic in D and $h(\theta) = \theta^{q+1}$, $q \geqq 0$, then strongly on C points are either h-Plessner points or h-pre-Meier points of f.*

5. Although we did not use the idea of order of curves, for any approach function h the two boundary curves of reduced *h*-angles have the same order of contact. According to an earlier result of Seidel and Walsh [11, p. 200] the non-Euclidean Frechet distance (and hence the non-Euclidean Hausdorff distance) between the two boundary curves is finite.

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