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ON MASSEY PRODUCTS

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We present two examples: one showing the necessity of the condition that the two triple products must vanish *simultaneously* in order for the quadruple product to be defined; and one showing that the higher order products of a path connected, simply connected space are not completely determined by the differentials in the Eilenberg-Moore spectral sequence of its path-loop fibration.

1. Introduction. This paper is concerned with two examples concerning Massey products. Example I fills a gap in the literature since there are many references (see [2], [1], [3]) to the fact that the two triple products $\langle u, v, w\rangle$ and $\langle v, w, x\rangle$ must vanish *simultaneously* in order for the quadruple product $\langle u, v, w, x\rangle$ to be defined, yet there appears to be no example proving this. Example II gives a path connected, simply connected space which has higher order products which are not determined by the differentials in the Eilenberg-Moore spectral sequence of its path-loop fibration. This is to show that the higher products are a richer source of information about a space than the above-mentioned spectral sequence, and this should be contrasted with J. Peter May's result that matrix Massey products completely determine the differentials in the Eilenberg-Moore spectral sequence (see [4]).

2. Definitions. Because of different conventions in the literature used to define Massey products and to state what it means for the two triple products to vanish simultaneously, we present the following definitions:

Let X be a topological space and R a commutative ring with identity. $H^*(X; R)$ will denote the singular cohomology ring and $C^*(X; R)$ the singular cochain complex. (We could use in these definitions any cochain complex which has an associative product.) If $a \in H^p(X; R)$ or $C^p(X; R)$, we will write $\bar{a} = (-1)^p a$. We first define the triple product.

DEFINITION 1. Let u, v, and w be homogeneous entries from $H^*(X; R)$ of degrees p, q, and r respectively. Choose representative cocycles u', v', and w' for these classes respectively, and assume that $uv = 0$ and $vw = 0$. We may select cochains a^{12} and a^{23} such that

$$\delta(a^{12}) = \overline{u'}v'$$
$$\delta(a^{23}) = \overline{v'}w' .$$

One can show that the cochain

$$z' = \overline{u'}a^{23} + \overline{a^{12}}w'$$

is actually a cocycle of degree $p + q + r - 1$.

We define the triple product $\langle u, v, w \rangle$ to be the collection of all cohomology classes $[z'] \in H^{p+q+r-1}(X; R)$ that we can obtain by the above procedure. This definition differs by the sign (-1) from the definition in [3] and by the sign $(-1)^{q+1}$ from Massey's original definition in [6]. One can show that the indeterminacy of the triple product is

$$u \cdot H^{q+r-1}(X; R) + H^{p+q-1}(X; R) \cdot w .$$

In order to give the definition of the quadruple product $\langle u, v, w, x \rangle$, we must indicate what it means for the two triple products $\langle u, v, w \rangle$ and $\langle v, w, x \rangle$ to vanish simultaneously. Select representative cocycles u', v', w', x' respectively, for u, v, w, x and assume that $uv = 0$, $vw = 0$, and $wx = 0$. Then we may select cochains a^{12}, a^{23}, and a^{34} such that

$$\delta(a^{12}) = \overline{u'}v' , \quad \delta(a^{23}) = \overline{v'}w' , \quad \delta(a^{34}) = \overline{w'}x' .$$

If we let

$$y' = \overline{u'}a^{23} + \overline{a^{12}}w' \quad \text{and} \quad z' = \overline{v'}a^{34} + \overline{a^{23}}x' ,$$

these are cocycle representatives of $\langle u, v, w \rangle$ and $\langle v, w, x \rangle$ respectively. We say that $\langle u, v, w \rangle$ and $\langle v, w, x \rangle$ vanish *simultaneously* if it is possible to make the choices above so that both y' and z' are coboundaries. The purpose of Example I is to show that it is possible for both $\langle u, v, w \rangle$ and $\langle v, w, x \rangle$ to vanish, but not to vanish simultaneously.

DEFINITION 2. Let u, v, w and x be homogeneous entries from $H^*(X; R)$ of degrees p, q, r and s respectively. Assume that

(1) $uv = 0$, $vw = 0$, $wx = 0$

(2) $\langle u, v, w \rangle$ and $\langle v, w, x \rangle$ vanish simultaneously.

Select representative cocycles u', v', w' and x' for u, v, w and x respectively, and because of condition (1) we may select cochains a^{12}, a^{23}, and a^{34} such that

$$\delta(a^{12}) = \overline{u'}v' , \quad \delta(a^{23}) = \overline{v'}w' , \quad \delta(a^{34}) = \overline{w'}x' .$$

Making use of condition (2) we may assume that a^{12}, a^{23}, and a^{34} were selected so that there exist cochains a^{13} and a^{24} such that

$$\delta(a^{13}) = \overline{u'}a^{23} + \overline{a^{12}}w' \quad \text{and} \quad \delta(a^{24}) = \overline{v'}a^{34} + \overline{a^{23}}x' .$$

One can show that the cochain

$$z' = \overline{u'}a^{24} + \overline{a^{12}}a^{34} + \overline{a^{13}}x'$$

is actually a cocycle of degree $p + q + r + s - 2$.

We define the quadruple product $\langle u, v, w, x\rangle$ to be the collection of all cohomology classes $[z'] \in H^{p+q+r+s-2}(X; R)$ that we can obtain by the above procedure. This definition differs by the sign (-1) from the definition in [3] and by the sign $(-1)^{p+r+1}$ from the definition in [2]. An element in the indeterminacy of the quadruple product can be written as an element of a marix triple product which we can write as

$$\left\langle (u, z_1), \begin{pmatrix} v & z_2 \\ 0 & w \end{pmatrix}, \begin{pmatrix} z_3 \\ x \end{pmatrix} \right\rangle$$

where $z_i \in H^*(X; R)$, $1 \leqq i \leqq 3$, by a result due to Kraines (see [3]).

3. EXAMPLE I. We will give an example of a topological space whose cohomology contains four classes u_1, u_2, u_3, and u_4 such that $\langle u_1, u_2, u_3\rangle$ and $\langle u_2, u_3, u_4\rangle$ are both defined and vanish, but do not vanish simultaneously. Thus in the cohomology of this space the quadruple product $\langle u_1, u_2, u_3, u_4\rangle$ is not defined, even though $\langle u_1, u_2, u_3\rangle = 0$ and $\langle u_2, u_3, u_4\rangle = 0$.

Let $X_0 = (S_1 V S_2 V S_3 V S_4 V S_5) \bigcup_{f_0} e_0$ be the wedge of five oriented spheres S_i of dimension p_i, $1 \leqq i \leqq 5$, where $p_5 = p_2 + p_3 - 1$, with an oriented $(p_1 + p_2 + p_3 - 1)$-cell e_0 attached by the map f_0 which is a representative of the homotopy class $[i_1, [i_2, i_3]] + [i_1, i_5]$, where i_j denotes both the inclusion map $S_j \to S_1 V S_2 V S_3 V S_4 V S_5$ and its homotopy class, $1 \leqq j \leqq 5$. We will consider two spaces $X_1 = X_0 \bigcup_{f_1} e_1$ and $X_2 = X_0 \bigcup_{f_2} e_2$, where e_1 and e_2 are oriented $(p_2 + p_3 + p_4 - 1)$-cells, f_1 is a representative of the homotopy class $[i_2, [i_3, i_4]] + [i_5, i_4]$, and f_2 is a representative of $[i_2, [i_3, i_4]] - [i_5, i_4]$.

In $H^*(X_1; \boldsymbol{Z})$ and $H^*(X_2; \boldsymbol{Z})$ we will let u_i denote the cohomology class represented by the cocycle which assigns $+1$ to the cell S_i and 0 elsewhere, $1 \leqq i \leqq 5$, and z_i denote the class represented by the cocycle which assigns $+1$ to the cell e_i and 0 elsewhere, $0 \leqq i \leqq 2$. Then using Lemma 7 in [6], in $H^*(X_1; \boldsymbol{Z})$ we have

$$\langle u_1, u_2, u_3\rangle = (-1)^{p_1+p_2+1}z_0 = (-1)^{p_1+p_2+1}u_1u_5$$

and

$$\langle u_2, u_3, u_4\rangle = (-1)^{p_2+p_3+1}z_1 = (-1)^{p_2+p_3+1}u_5u_4 .$$

Similarly in $H^*(X_2; \boldsymbol{Z})$ we have $\langle u_1, u_2, u_3\rangle = (-1)^{p_1+p_2+1}z_0 = (-1)^{p_1+p_2+1}u_1u_5$ and $\langle u_2, u_3, u_4\rangle = (-1)^{p_2+p_3+1}z_2 = (-1)^{p_2+p_3}u_5u_4$. Since $H^*(X_i) \cdot u_3 = 0$ and $u_2 \cdot H^*(X_i) = 0$, the indeterminacy of $\langle u_1, u_2, u_3\rangle$ is $u_1 \cdot H^*(X_i)$

and the indeterminacy of $\langle u_2, u_3, u_4\rangle$ is $H^*(X_i)\cdot u_4$, $i = 1, 2$. Thus for both X_1 and X_2, $\langle u_1, u_2, u_3\rangle$ and $\langle u_2, u_3, u_4\rangle$ both vanish. Depending on the parity of p_2, either X_1 or X_2 will be the space required.

As in [2] on p. 148, we can define in our situation an operation denoted by $(\langle u_1, u_2, u_3\rangle, \langle u_2, u_3, u_4\rangle)$ which consists of all ordered pairs $(\overline{u_1'}a^{23} + \overline{a^{12}}u_3', \overline{u_2'}a^{34} + \overline{a^{23}}u_4')$ that one can obtain, where previous notation has been used here. One can see that in our situation the operation $(\langle u_1, u_2, u_3\rangle, \langle u_2, u_3, u_4\rangle)$ is a coset of $H^{p_1+p_2+p_3-1}(X_i) \times H^{p_2+p_3+p_4-1}(X_i)$ modulo the subgroup Image (Φ), where $\Phi\colon H^{p_2+p_3-1}(X_i) \to H^{p_1+p_2+p_3-1}(X_i) \times H^{p_2+p_3+p_4-1}(X_i)$ is defined by $\Phi(z) = (\bar{u}_1 z, \bar{z}u_4)$. For our example we desire $(\langle u_1, u_2, u_3\rangle, \langle u_2, u_3, u_4\rangle) \notin$ Image (Φ).

But if p_2 is even, then in $H^*(X_1)$ we have

$$\begin{aligned}(\langle u_1, u_2, u_3\rangle, \langle u_2, u_3, u_4\rangle) &= ((-1)^{p_1+p_2+1}u_1u_5, (-1)^{p_2+p_3+1}u_5u_4) \\ &= (-\bar{u}_1u_5, \bar{u}_5u_4) \notin \text{Image}\,(\Phi)\,.\end{aligned}$$

And if p_2 is odd, then in $H^*(X_2)$ we have

$$\begin{aligned}(\langle u_1, u_2, u_3\rangle, \langle u_2, u_3, u_4\rangle) &= ((-1)^{p_1+p_2+1}u_1u_5, (-1)^{p_2+p_3}u_5u_4) \\ &= (\bar{u}_1u_5, -\bar{u}_5u_4) \notin \text{Image}\,(\Phi)\,, \quad \text{as desired.}\end{aligned}$$

Note also that if p_2 is even, in $H^*(X_2)$, or if p_2 is odd, in $H^*(X_1)$, we have examples of spaces where the quadruple product is defined, but is not strictly defined in the sense of May (see [3], p. 538).

4. EXAMPLE II. The differentials in the Eilenberg-Moore spectral sequence associated with the path-loop fibration of a path connected simply connected space are completely determined by higher order Massey products (see [3]). This might lead one to conjecture that the differentials in the spectral sequence contain all the information about the higher order products that our definition contains. But to see that this is not the case, we will consider the Eilenberg-Moore spectral sequence of the path-loop fibration of the following space.

Let $X_0 = S_1 V S_2 V S_3 V S_4$ be the wedge of four oriented spheres, where $\dim S_i = p_i > 1$, $p_4 = p_1 + p_3 - 1$, and we let $r = p_1 + p_2 + p_3 - 2$. Let i_j denote the homotopy class of the inclusion $S_j \to X_0$ and let f be a map from an oriented r-sphere into X_0 which represents $[i_1, [i_2, i_3]] - [i_2, i_4]$. If e is an oriented $(r+1)$-cell, we may attach e to X_0 via the map f and call the resulting space X_1. In $H^*(X_1; \boldsymbol{Z})$ let u_i be the cohomology class which is represented by the cocycle which assigns $+1$ to S_i and 0 elsewhere, and let z be the cohomology class represented by the cocycle which assigns $+1$ to e and 0 elsewhere.

By using Lemma 7 in [6], we can easily show that the triple product $\langle u_1, u_2, u_3\rangle = (-1)^{p_1+p_2}u_2u_4 = (-1)^{p_1+p_2+1}z$ and thus is nontrivial; however in the Eilenberg-Moore spectral sequence associated with the path-loop fibration of X_1, this triple product, being decomposable, would be trivial.

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