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THE NON-ORIENTABLE GENUS OF THE n-CUBE

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For the purposes of embedding theory, a graph consists of a collection of points, called vertices, certain pairs of which are joined by homeomorphs of the unit interval, called edges. Edges may intersect only at vertices, and no vertex is contained in the interior of an edge. The graph thus becomes a topological space as a subspace of R^3. An embedding of a graph G in a compact 2-manifold (surface) S is then just an embedding of G in S as a topological space. The genus, $\gamma(G)$, of G is the minimum genus among all orientable surfaces into which G may be embedded. The non-orientable genus, $\tilde{\gamma}(G)$, is defined analogously. The n-cube, Q_n, is a well known graph wnich generalizes the square and the standard cube. In this paper the following formula is proven:

THEOREM.

$$\tilde{\gamma}(Q_n) = \begin{cases} 2 + 2^{n-2}(n-4) & n \geqq 6 \\ 3 + 2^{n-2}(n-4) & n = 4,5 \\ 1 & n \leqq 3 . \end{cases}$$

Introduction. In 1955, Ringel [3] showed that the orientable genus of the n-cube is given by

$$(1) \qquad \gamma(Q_n) = 1 + 2^{n-3}(n-4) .$$

This result was also obtained independently by Beinecke and Harary [1]. Since then, genus formulae, both orientable and non-orientable, have been obtained for several classes of graphs, including the complete bipartite graph, the octahedral graphs and many of the other multipartite graphs (see, for example [2]). As a result, the n-cube is perhaps the best known graph for which a genus question has remained open. The present paper fills this gap.

Preliminaries. Let $\boldsymbol{Z}_2^n$ be the elementary abelian 2-group of rank n, $\boldsymbol{Z}_2 X \cdots X \boldsymbol{Z}_2$. If $x \in \boldsymbol{Z}_2^n$, let $[x]_k$ be the kth ordinate of x, so that $x = ([x]_1, \cdots, [x]_n)$. Let $1_k \in \boldsymbol{Z}_2^n$ be the element such that $[1_k]_m = 1$ iff $m = k$, and let $\Delta_n = \{1_k \in \boldsymbol{Z}_2^n \mid 1 \leqq k \leqq n\}$. Then the n-cube, Q_n, is the Cayley graph $(\boldsymbol{Z}_2^n, \Delta_n)$; that is, $\boldsymbol{Z}_2^n$ is the vertex set of Q_n, and for $x, y \in \boldsymbol{Z}_2^n$, $\{x, y\}$ is an edge iff $x + y \in \Delta_n$. If $\{x, y\}$ is an edge, it is said to have the *color* $x + y$, and is called a $(x + y)$-edge.

Let $(x; c_1, \cdots, c_m)$, where $x \in \boldsymbol{Z}_2^n$ and $c_i \in \Delta_n$, denote the walk $x, x+c_1, x + c_1 + c_2, \cdots, x + c_1 + \cdots + c_m$ in Q_n. Note that any walk $x_0, \cdots, x_m$

may be represented in this manner by $(x_0; x_1 + x_0, x_2 + x_1, \cdots, x_m + x_{m-1})$. The walk $(x; c_1, \cdots, c_m)$ is a circuit iff $c_1 + \cdots + c_m = 0$. Thus any circuit in Q_n is of even length. A circuit of length m will be called an m-circuit. An edge will be said to occur in a circuit once for each time its incident vertices appear consecutively in the circuit.

If ε is a 2-cell embedding of a graph in a surface, the boundary of each face is a circuit in the graph. If B is the set containing the boundary circuit of each face in ε, the manifold structure of the surface implies that B satisfies the properties:

P1: Each edge occurs exactly twice among the circuits of B.

P2: If x is any vertex and B_x is any subset of B such that no edge incident to x occurs exactly once among the circuits in B_x, then one of $B - B_x$ or B_x contains all occurrences in B of edges incident to x.

If the surface is oriented and the induced orientation is given to every circuit in B, then B satisfies

P3: Each directed edge occurs once in B.

Conversely, if a set B of circuits satisfies P1 and P2, it is the set of boundary circuits for some 2-cell embedding $\varepsilon(B)$. The embedding is orientable iff orientations may be assigned in B so that P3 holds. A set B satisfying P1 and P2 will be called a *boundary* set for the embedding $\varepsilon(B)$. If B contains only 4-circuits, $\varepsilon(B)$ is called *quadrilateral*. For convenience we refer to non-orientable quadrilateral 2-cell embeddings as *NQ-embeddings*.

Tube adding and covering sets. We now describe the tube adding construction of Beinecke and Harary (see also White [4]), and a kind of inverse to it.

A set A of circuits in Q_n is called a *covering* set if every vertex in Q_n occurs exactly once among the circuits in A. We will be particularly concerned with covering sets contained in boundary sets for Q_n, where $n \geqq 3$. In such a case, P2 implies that the covering set consists of disjoint cycles. Suppose B is a boundary set in Q_{n-1}, $n > 3$, containing a covering set A. We define a boundary set $B^*(B, A)$ for Q_n containing a covering set $D^*(B, A)$ as follows. Take two copies of the embedding $\varepsilon(B)$. Let F be a face in one copy of $\varepsilon(B)$ whose boundary circuit is in A, and let F^1 be the corresponding face in the other copy. Delete a 2-cell from the interiors of both F and F^1. Identify the boundaries of the deleted 2-cells so that the union of the remaining portions of F and F^1 forms a cylinder or tube. Perform the identifications in such a way that it is possible to draw edges in the interior of the tube joining each pair of corresponding vertices on the boundaries of F and F^1, so that there are no crossings. These new edges subdivide the tube into a necklace of quadrilaterals, the boundary circuits of which we denote by $b(F)$. As F has even length, we may choose a subset $d(F)$ of $b(F)$ containing pairwise disjoint 4-circuits such that every vertex on the boundary of F or F^1 occurs once in $d(F)$. Repeat this process for each face whose boundary circuit is in A. The result is an embedding of Q_n, with a boundary set containing the circuits from the two copies of $B-A$ and the $b(F)$'s for each F whose boundary is in A. Call is boundary set $B^*(B, A)$. Taking all $d(F)$'s produces a covering set $D^*(B, A)$ contained in $B^*(B, A)$. If $\varepsilon(B^*(B, A))$ is orientable, an orientation is induced on $\varepsilon(B)$. If $\varepsilon(B)$ is orientable, choosing opposite orientations on the two copies of $\varepsilon(B)$ at the beginning of the construction results in an orientation for $\varepsilon(B^*(B, A))$. Thus we have

PROPOSITION 1. *Let B be a boundary set for Q_{n-1}, $n > 3$, and let $A \subset B$ be a covering set. Then there are sets $B^*(B, A)$ and $D^*(B, A)$ such that*

(a) *$D^*(B, A) \subset B^*(B, A)$. $B^*(B, A)$ is a boundary set, and $D^*(B, A)$ is a covering set, for Q_n.*

(b) *$\varepsilon(B^*(B, A))$ is orientable iff $\varepsilon(B)$ is orientable.*

Note that if $B-A$ contains only 4-circuits, $B^*(B, A)$ will be quadrilateral. In turn, $B^*(B^*(B, A))$ and $D^*(B, A)$ will be quadrilateral. This may be repeated indefinitely giving

PROPOSITION 2. *If there is a non-orientable embedding $\varepsilon(B)$ of Q_{n-1}, $n > 3$, where B contains a covering set A such that $B-A$ contains only quadrilaterals, then there is an NQ-embedding of Q_m for each $m \geqq n$.*

Below this proposition will be used in the case where $n = 6$. Note also that if $X \subset B$ is a covering set disjoint from A, then the two copies of X in $B^*(B, A)$ form a covering set in $B^*(B, A)$. This fact will be used in constructing the embedding of Q_5 to be used in the application of Prop. 2.

We now describe an inverse to the construction of $B^*(B, A)$. Suppose $\varepsilon(B)$ is a quadrilateral embedding of Q_n, $n \geqq 3$. Let S be the set of edges not colored 1_n but lying on a face containing a 1_n-edge, and let x be a vertex. There is exactly one 1_n-edge incident to x, so x lies on at most two faces containing 1_n-edges. As $n \geqq 3$, the faces containing a given 1_n-edge are distinct with distinct boundaries. It follows that x is incident to exactly two distinct edges in S, and that each edge in S lies on one face which does not contain a 1_n-edge as well as on one which does. Thus S spans a set A of disjoint cycles, which is a covering set for Q_n. Let $B_i = \{C \in B \mid x \in$

$C \Rightarrow [x]_n = i\}$ and define A_i analogously. Note that if $\{x, y\} \in S$, then $\{x, y\}$ lies on $(x; 1_n, x + y, 1_n, x + y)$ so that $\{x + 1_n, y + 1_n\} \in S$. Thus if $x_0, \cdots x_{m-1}, x_0 \in A_i$ then $x_0 + 1_n, \cdots, x_{m-1} + 1_n, x_0 + 1_n \in A_{1-i}$. Also, as no 1_n-edge lies in S, $A = A_0 \cup A_1$. Now delete every 1_n-edge, the interior of every face containing a 1_n-edge, and all nth ordinates from $\varepsilon(B)$. The result is disjoint embeddings ε_0 and ε_1 of Q_{n-1} in compact 2-manifolds with boundary. The boundaries of the faces in ε_i comprise B_i and the manifold boundary in ε_i is the union of the cycles in A_i. Thus $B_i \cup A_i$ is a boundary set for Q_{n-1}. Since A was a covering set and $A = A_0 \cup A_1$, A_i is a covering set in $B_i \cup A_i$. Moreover since nth ordinates have been deleted $A_0 = A_1$. If $B_0 = B_1$, then $B = B^*(B_0, A_0)$. Thus we have

PROPOSITION 3. *Let B be a boundary set for a quadrilateral embedding $\varepsilon(B)$ of Q_n and let A_i and B_i be defined as above. Then for, $i = 0, 1$*

(a) *B_i is a boundary set containing the covering set A_i.*

(b) *$A_0 = A_1$.*

(c) *If $\varepsilon(B)$ is non-orientable and both $\varepsilon(B_0)$ and $\varepsilon(B_1)$ are orientable then $B_0 \neq B_1$.*

Proof of the theorem. Since Q_1, Q_2 and Q_3 are planar, they trivially embed in the projective plane, as claimed.

For $n \geqq 2$, Q_n has girth four. The standard Euler formula argument then gives

$$\tilde{\gamma}(Q_n) \geqq 2 + 2^{n-2}(n - 4) \qquad n \geqq 2$$

where equality holds iff there is an NQ-embedding of Q_n. For any graph G, $\tilde{\gamma}(G) \leqq 2\gamma(G) + 1$, since a crosscap may be added to any orientable embedding producing a non-orientable embedding with Euler characteristic lowered by one. Using (1) it follows that

$$\tilde{\gamma}(Q_n) \leqq 3 + 2^{n-2}(n - 4) .$$

Thus in order to complete the proof, it suffices to exhibit NQ-embeddings for $n \geqq 6$, and to show that NQ-embeddings do not exist for Q_4 and Q_5.

NQ-Embeddings of Q_n, $n \geqq 6$. Figure 1 depicts a 2-cell embedding ε of Q_4 in the nonorientable surface of genus four. The sides α and β of the rectangle are to be identified in the standard way to produce a torus. The labelled edges are to be identified so that labels and directions coincide. The regions inside the rectangles of labelled edges are deleted from the torus so that the identification of the edges results in a manifold.

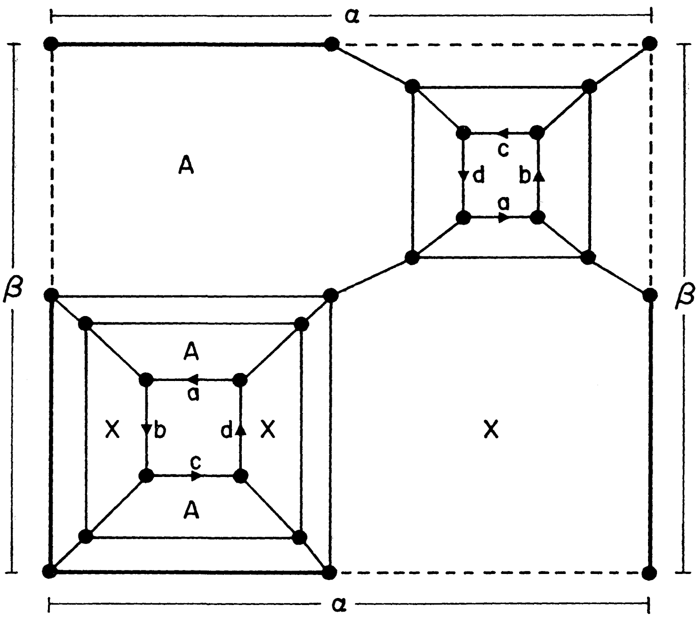

FIGURE 1

Let B be the boundary set for ε, and A the covering set in B comprising the boundaries of those faces labelled A in the figure. The faces marked X give a second covering set in B disjoint from A. All circuits in $B - X - A$ have length four. Thus $B^*(B, A)$ is a boundary set for Q_5 containing a covering set X^* which arises from the two copies of X in the construction of $B^*(B, A)$. $B(B, A) - X^*$ contains only 4-circuits. $\varepsilon(B^*(B, A))$ is non-orientable. Applying Prop. 2 we get the desired NQ-embedding of Q_m, $m \geqq 6$.

Non-existence of NQ-embeddings for Q_4 and Q_5. Suppose there is an NQ-embedding of Q_n, $n \geqq 3$. By Prop. 3 there are two boundary sets of Q_{n-1} both containing the same covering set which contains all non-quadrilateral faces. Moreover, if both of these boundary sets give orientable embeddings, they must be distinct. We prove that this cannot occur for Q_4 or Q_5 by showing that there is at most one such boundary set for Q_3 or Q_4 containing a given covering set, and that all the resulting embeddings are orientable.

Suppose B is a boundary set for Q_n, $n = 3, 4$, containing a covering set A such that $B - A$ contains only 4-circuits. Certain arguments regarding this situation will be used repeatedly below. We therefore represent them symbolically, as now described.

A1: If $\{x, y\}$ is known to occur in A, $T_1, \cdots, T_{n-2}$ are distinct 4-circuits containing $\{x, y\}$ known not to be in $B - A$, and R is the remaining 4-circuit containg $\{x, y\}$, then $R \in B - A$. This follows from the fact that $B - A$ contains only 4-circuits, that every edge occurs in $B - A$, and that there are precisely $n - 1$ 4-circuits in Q_n containing any given edge. We denote this argument by A1($\{x, y\}$, $T_1, \cdots, T_{n-2}$) $\Rightarrow R \in B - A$.

A2: If $\{x, y\}$ is known not to occur in A, $T_1, \cdots, T_{n-3}$ are distinct 4-circuits containing $\{x, y\}$ not in $B - A$ and R is either of the remaining two 4-circuits containing $\{x, y\}$, then $R \in B - A$. This follows from the fact that if $\{x, y\}$ does not occur in A, it occurs twice in $B - A$. The notation is A2($\{x, y\}$, $T_1, \cdots, T_{n-3}$) $\Rightarrow R \in B - A$.

A3: If $\{x, y_1\}$ and $\{x, y_2\}$ are distinct edges incident to x occurring in A and $\{x, z\}$ is a third edge incident to x, then $\{x, z\}$ does not occur in A. This is denoted A3($\{x, y_1\}$, $\{x, y_2\}$) $\Rightarrow \{x, z\} \notin A$. By abuse of notation we let $\{x, z\} \notin A$ mean $\{x, z\}$ does not occur in A.

A4: If $\{x_1, y_1\}, \cdots, \{x, y_{n-2}\}$ are distinct edges not occurring in A and $\{x, z\}$ is either of the remaining two edges incident to x, then $\{x, z\}$ occurs in A. This is denoted A4($\{x_1, y_1\}, \cdots, \{x, y_{n-2}\}$) $\Rightarrow \{x, z\} \in A$.

A5: Since $n \geqq 3$, P2 implies that distinct circuits in B cannot share consecutive edges. Thus if $R \in B$ shares consecutive edges with T, $T \notin B$. This is denoted A5(R) $\Rightarrow T \notin B$.

A6: Suppose $n = 4$, R_1 and R_2 are circuits in B and T is a third circuit such that every edge incident to x occurs either twice or not at all in $\{R_1, R_2, T\}$. Then by P2, $T \notin B$. This is denoted A10(x, R_1, R_2) $\Rightarrow T \notin B$.

PROPOSITION 4. *Suppose B is a boundary set for Q_n. $A \subset B$ is a covering set, and $B - A$ contains only 4-circuits. Then*

(1) *If $n = 3$, B is of the form $\{(x; b, c, b, c), (x; c, a, c, a), (x + a + b; b, c, b, c)(x + a + b; c, a, c, a)\} \cup A$, where $A = \{(x; a, b, a, b), (x + c; b, a, b, a)\}$, $x \in \boldsymbol{Z}_2^3$ and $\{a, b, c\} = \Delta_3$.*

(2) *Suppose $n = 4$. For $x \in \boldsymbol{Z}_2^4$ and $\Delta_4 = \{a, b, c, d\}$, define the following circuits in Q_4:*

$$R_1 = (x; d, a, d, a) \quad R_2 = (x; b, c, b, c) \quad R_3 = (x; c, d, c, d)$$
$$R_4 = (x + a + c; d, a, d, a) \quad R_5 = (x + a + c; b, c, b, c)$$
$$R_6 = (x + a + c; c, d, c, d) \quad R_7 = (x + b + d; d, a, d, a)$$
$$R_8 = (x + b + d; b, c, b, c) \quad R_9 = (x + b + d; c, d, c, d)$$
$$R_{10} = (x + a + b + c + d; d, a, d, a)$$
$$R_{11} = (x + a + b + c + d; b, c, b, c)$$
$$R_{12} = (x + a + b + c + d; c, d, c, d) \quad R_{13} = (x + a; d, b, d, b)$$
$$R_{14} = (x + a + c; d, b, d, b)$$
$$C_1 = (x; a, b, a, b) \quad C_2 = (x + c; a, b, a, b)$$
$$C_3 = (x + d; a, b, a, b) \quad C_4 = (x + c + d; a, b, a, b)$$

$$C_5 = (x;\ a,\ c,\ a,\ b,\ a,\ c,\ a,\ b)$$
$$C_6 = (x + b + d;\ a,\ c,\ a,\ b,\ a,\ c,\ a,\ b)\ .$$

Then B has one of the following three forms, for some choice of x and a, b, c, d:

(a) $B = \{R_1, R_2, \cdots, R_{12}\} \cup A$, *where* $A = \{C_1, C_2, C_3, C_4\}$.

(b) $B = \{R_1, \cdots, R_7, R_9, R_{10}, R_{12}, C_3, C_4\} \cup A$, *where* $A = \{C_1, C_2, R_8, R_{11}\}$.

(c) $B = \{R_1, \cdots, R_5, R_7, \cdots, R_{11}, R_{13}, R_{14}\} \cup A$, *where* $A = \{C_5, C_6\}$.

Proof. (1) Suppose $C \in A$ contains a walk of the form $(x;\ a, b, c)$. $\mathrm{A3}(\{x + a, x\}, \{x + a, x + a + b\}) \Rightarrow \{x + a, x + a + c\} \notin A$. $\mathrm{A2}(\{x + a, x + a + c\}) \Rightarrow R = (x + a;\ b, c, b, c) \in B - A$. However $\mathrm{A5}(C) \Rightarrow R \notin B$. Thus there is no such walk in a circuit in A, so that A contains only 4-circuits. Thus B must contain all six 4-circuits in Q_3. The result follows easily by proper choice of a, b, c.

(2) We first show that if A contains circuits of length greater than four, then (c) holds.

Suppse $C \in A$ has length greater than four. Then C contains a walk of the form $(x + b;\ b, a, c)$. $\mathrm{A3}(\{x, x + b\}, \{x, x + a\}) \Rightarrow \{x, x + c\} \notin A$. $\mathrm{A5}(C) \Rightarrow T_1 = (x;\ a, c, a, c) \notin B$. $\mathrm{A2}(\{x, x + c\}, T_1) \Rightarrow R_2, R_3 \in B - A$. Similarly, $C_1 \notin B$ and $R_5, R_{13} \in B - A$. Since $\{x + a, x + a + c\}$ occurs twice in $\{C, R_5\} \subset B$, P1 implies that $R_6 \notin B$. Suppose $\{x + a + c, x + c\} \notin A$. Then $\mathrm{A2}(\{x + a + c, x + c\}, T_1) \Rightarrow R_4, C_2 \in B - A$. Then every edge incident to $x + c$ occurs twice in $B - A$, implying that $x + c$ does not occur in A. This is not possible, as A is a covering set. So $\{x + a + c, x + c\} \in A$. $\mathrm{A3}(\{x + a + c, x + c\}, \{x + a + c, x + a\}) \Rightarrow \{x + a + c, x + a + c + d\} \notin A$. $\mathrm{A2}(\{x + a + c, x + a + c + d\}, R_6) \Rightarrow R_4, R_{14} \in B - A$. $\{x + c, x + c + d\}$ occurs twice in $\{R_3, R_4\} \subset B - A$, so P1 implies that $\{x + c, x + c + d\}$ does not occur in A. $\mathrm{A4}(\{x + c, x\}, \{x + c, x + c + d\}) \Rightarrow \{x + c, x + b + c\} \in A$. Thus we have shown that if a walk of the form $(x + b;\ b, a, c)$ is contained in a circuit $C \in A$, then $(x + b;\ b, a, c, a, b)$ is contained in C. It follows that since $(x + a;\ c, a, b)$ is in C $(x + a;\ c, a, b, a, c)$ is in C. Continuing in this fashion, we get $C = C_5$. Then $\mathrm{A5}(C_5) \Rightarrow T_2 = (x + b;\ a, c, a, c) \notin B$. $\mathrm{A3}(\{x + b, x\}, \{x + b, x + a + b\}) \Rightarrow \{x + b, x + b + c\} \notin A$. $\mathrm{A2}(\{x + b, x + b + c\}, T_2) \Rightarrow R_9 \in B - A$. $\mathrm{A1}(\{x, x + a\}, T_1, C_5) \Rightarrow R_1 \in B - A$. $\mathrm{A1}(\{x + b, x + a + b\}, T_2, C_5) \Rightarrow R_7 \in B - A$. $\mathrm{A5}(C_5) \Rightarrow C_2 \notin B$. $\mathrm{A1}(\{x + b + c, x + a + b + c\}, C_2, T_2) \Rightarrow R_{10} \in B - A$. $\mathrm{A6}(R_{10}, R_{14} \Rightarrow C_4 \notin B$. $\mathrm{A6}(R_{13}, R_7) \Rightarrow C_3 \notin B$. $\mathrm{A6}(R_7, R_9) \Rightarrow T_3 = (x + b + d;\ a, c, a, c) \notin B$. $\mathrm{A6}(R_1, R_3) \Rightarrow T_4 = (x + d;\ a, c, a, c) \notin B$. Two of three 4-circuits containing $\{x + b + d, x + a + b + d\}$, namely T_3 and C_3 are not in B. Thus $\{x + b + d, x + a + b + d\}$ occurs at most once among the 4-circuits in B. It follows that $\{x + b + d, x + a + b + d\}$ occurs in A. Similarly, $\{x+d, x+a+d\}$, $\{x+b+c+d, x+a+b+c+d\}$ and $\{x+c+d, x+a+c+d\}$ each occur in A. Let C^1 be the circuit in A containing $\{x+d, x+a+d\}$. Suppose $\{x+d, x+c+d\} \in A$. Then $\{x+d, x+c+d\} \in C^1$ and $A6(R_1, R_3) \Rightarrow C^1 \notin B$. So $\{x+d, x+c+d\} \notin A$. $A4(\{x+d, x+c+d\}, \{x+d, x\}) \Rightarrow \{x+d, x+b+d\} \in A$. $A4(\{x+c+d, x+d\}, \{x+c+d, x+c\}) \Rightarrow \{x+c+d, x+b+c+d\} \in A$. Thus C^1 contains $(x+a+d; a, b, a)$. As $C_3 \notin B$, $C^1 \neq C_3$ so $\{x+a+d, x+a+b+d\} \notin A$. $A4(\{x+a+d, x+a+b+d\}, \{x+a+d, x+a\}) \Rightarrow \{x+a+d, x+a+c+d\} \in A$. $A4(\{x+a+b+d, x+a+d\}, \{x+a+b+d, x+a+b\} \Rightarrow \{x+a+b+d, x+a+b+c+d\} \in A$. Thus $C^1 = C_6$. Finally, $A2(\{x+d, x+c+d\}, T_4) \Rightarrow R_8 \in B - A$ and $A2(\{x+a+d, x+a+b+d\}, C_3) \Rightarrow R_{11} \in B - A$. Thus (c) holds.

Now suppose A contains only 4-circuits. We show that either (a) or (b) holds.

Since there are 4-circuits in A, let x, a, b, c, and d be chosen so that $C_1 \in A$ and $R_1 \in B - A$. By A3, no edge incident to a vertex in C_1 but not itself in C_1 occurs in A. $A6(R_1, C_1) \Rightarrow T_1 = (x; b, d, b, d)$, $T_2 = (x+a; b, d, b, d) \notin B$. By A2, it follows that $R_3, R_6, R_7, R_9, R_{12} \in B-A$. By A1, $R_2R_5, \in B-A$. At this point, the set of circuits known to be in $B-A$ and the circuit in A are fixed under the permutation $(ab)(cd)$. Thus if either $\{x+c, x+b+c\}$ or $\{x+d, x+a+d\}$ occurs in A, we may assume without loss that $\{x+c, x+b+c\} \in A$. Suppose neither does occur in A. Then, $A4(\{x+d, x\}, \{x+d, x+a+d\}) \Rightarrow \{x+d, x+b+d\}, \{x+d, x+c+d\} \in A$ and $(\{x+c, x\}, \{x+c, x+b+c\}) \Rightarrow \{x+c, x+c+d\} \in A$. It follows that walk $(x+b+d; b, c, d)$ is contained in some circuit in A. But then this circuit is not of length four, contrary to hypothesis. We conclude that, in fact, $\{x+c, x+b+c\} \in A$. Then $A6(R_2, R_3) \Rightarrow T_3 = (x+c; b, d, b, d) \notin B$. As $R_2 \in B-A$, the only remaining 4-circuit containing $\{x+c, x+b+c\}$ which may be in A is C_2. Thus $C_2 \in A$. $A2(\{x+c, x+c+d\}, T_3) \Rightarrow R_4 \in B-A$ and $A2(\{x+b+c, x+b+c+d\}, T_3) \Rightarrow R_{10} \in B-A$. A6 shows that $T_4 = (x+a+c; b, d, b, d)$, $T_5 = (x+b+d; a, c, a, c)$, $T_7 = (x; a_5, c_5a_3c)$, $T_8 = (x+b; a, c, a, c)$ and $T_6 = (x+d; a, c, a, c) \notin B$. Thus the only remaining 4-circuits which may be in B are C_3, C_4, R_8, R_{11}. All four must be in B in order to satisfy P1. A must contain a disjoint pair of them in order to be a covering set. The two possible choices for such a pair yield (a) and (b). This completes the proof of the proposition.

It is easily checked that each of the boundary sets B in Prop. 4 represent orientable embeddings. Moreover, if two such B's are distinct, so are the covering sets A contained in them. It follows from Prop. 3 that none of the sets B in Prop. 4 may arise as B_i^1, where $\varepsilon(B^1)$ is an NQ-embedding. Thus there are no NQ-embeddings of Q_4 or Q_5.

This completes the proof of the theorem.

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