NUMERICAL ALGORITHMS FOR OSCILLATION VECTORS OF SECOND ORDER DIFFERENTIAL EQUATIONS INCLUDING THE EULER-LAGRANGE EQUATION FOR SYMMETRIC TRIDIAGONAL MATRICES

John Gregory

We give numerical algorithms for second order differential equations. More specifically we consider the problem of numerically determining oscillation points and vectors for numerical solutions of the equation $(r(t)x'(t))' + p(t)x(t) = 0$ and focal points and vectors for the quadratic form $J(x) = \int_a^b (rx'^2 - px^2)dt$.
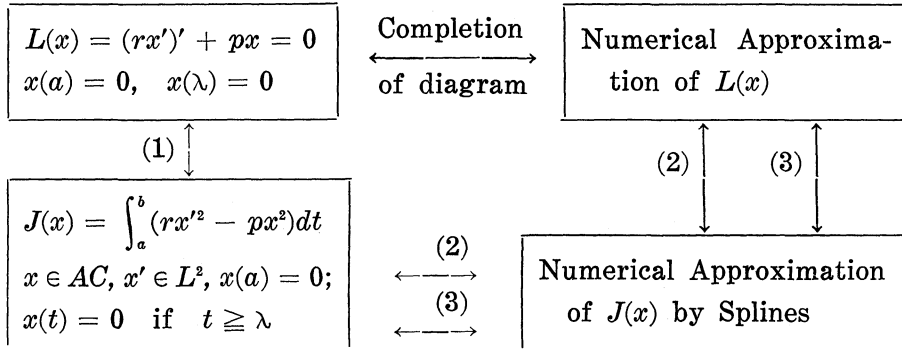
As a biproduct of our work we obtain some new theoretical and numerical results for symmetric tridiagonal matrices. Many of our results may be extended to eigenvalue and eigenvector problems, integral and partial differential equations, and higher order problems described in § 4. Many of our matrix results may be extended to more general banded symmetric matrices.

In a broad sense this work is a numerical application of an approximation theory of quadratic forms on Hilbert spaces given by the author. Our ideas are based on generalizations of a basic idea of Hestenes; namely on the consideration of the "negative signature of quadratic forms [5]." Our ideas are similar to finding roots of polynomials by looking at sign changes as opposed to the more difficult problem of solving equations.

For convenience of presentation we now describe the basic procedure: (i) Let $(r(t)x'(t))' + p(t)x(t) = 0$ be the differential equation which is the Euler-Lagrange equation of the quadratic form $J(x) = \int_a^b (rx'^2 - px^2)dt$. Let $x_0(t)$ be "a" solution of the differential equation satisfying $x_0(a) = 0$. (ii) Approximate the vectors $x(t)$ on $[a, b]$ by Spline functions of degree 1 (order 2) so that the approximating finite dimensional quadratic form is $J(x_\mu; \mu) = x_\mu^T D_\mu x_\mu$; where x_μ is a "piecewise linear function," D_μ is a symmetric tridiagonal matrix, and μ is a parameter denoting the distance between "knot points." (iii) Obtain the "Euler-Lagrange equations" for D_μ; call the solution $c(\mu)$. (iv) Show that $c(\mu)$ converges to $x_0(t)$ as $\mu \to 0$ in the strong L^2 derivative norm sense. We remark that we have previously shown that the negative signature (negative eigenvalues) of the matrix D_μ "agrees" with the negative signature of the quadratic

form $J(x)$.

To better visualize our ideas we think of the "commutative diagram" pictured below.

$$L(x) = (rx')' + px = 0$$
$$x(a) = 0, \quad x(\lambda) = 0$$

Completion
of diagram

Numerical Approxima-
tion of $L(x)$

(1)

$$J(x) = \int_a^b (rx'^2 - px^2)dt$$
$$x \in AC, \, x' \in L^2, \, x(a) = 0;$$
$$x(t) = 0 \quad \text{if} \quad t \geqq \lambda$$

(2)

(3)

(2)   (3)

Numerical Approximation
of $J(x)$ by Splines

Arrow (1) denotes ideas originally given by Hestenes which were redeveloped by the author to "fit" into the overall picture. Arrow (2) denotes approximation ideas previously given by the author [3], [4]. Arrow (3) denotes the new ideas (algorithms) in this paper which include the oscillation vector (eigenvector) results and the "Euler-Lagrange equations" for tridiagonal matrices.

In § 2 we present the preliminaries necessary for the remainder of our results. In particular we discuss the connection between splines of degree 1 and the approximation theory of quadratic forms previously given by the author. In § 3 we show how oscillation points and the oscillation vector $c(\sigma)$ are obtained. In particular we show that if $c(\sigma)$ and $x_0(t)$ are as described in (iv), then $\int_a^b [c'(\sigma) - x_0'(t)]^2 dt \to 0$ as the spline parameter $\sigma \to 0$, where the prime denotes differentiation. In § 4 we note how more general problems can be reduced to those of this paper. We will not give any numerical results in this paper. However, our algorithm is very efficient giving faster and more accurate results than the standard computer routines for comparison cases (I.B.M.-S.S.P. routines).

II. Preliminaries. In this section we give the Spline approximation setting associated with the quadratic form in (1) and the differential equation in (2). These results are given in [4] and are included for ease of exposition.

Our fundamental quadratic form is $J(x) = J(x, x)$ where

$$(1) \qquad J(x, y) = \int_a^b [r(t)x'(t)y'(t) - p(t)x(t)y(t)]dt \, .$$

$J(x)$ is the quadratic form whose Euler-Lagrange equation is the second order equation

(2) $(r(t)x'(t))' + p(t)x(t) = 0$.

For convenience we assume $r(t)$ and $p(t)$ are continuous functions on $a \leqq t \leqq b$, although these restrictions may be relaxed, and $r(t) > 0$.

Let $\mathscr{A}$ denote the arcs $x(t)$ which are absolutely continuous on $\varLambda = [a, b]$ and have square integrable derivates $x'(t)$. The norm, $\| x \| = (x, x)^{1/2}$ on the Hilbert space $\mathscr{A}$ is defined from

(3) $(x, y) = x(a)y(a) + \int_a^b x'(t)y'(t)dt$.

Let $\varSigma$ denote the set of real numbers of the form $\sigma = 1/n (n = 1, 2, \cdots)$ and zero. The metric on $\varSigma$ is the absolute value function of the difference. Let $\sigma = 1/n$, define the partition

$$\pi(\sigma) = (a = a < a_1 < a_2 \cdots < a_n = b) ,$$

where

(4) $a_k = k\dfrac{b - a}{n} + a \quad (k = 0, \cdots, n)$.

The space $\mathscr{A}(\sigma)$ is the set of continuous broken linear functions with vertices at $\pi(\sigma)$. Let $\mathscr{A}(0)$ denote the subset of $\mathscr{A}$ satisfying $x(a) = 0$ and $x(b) = 0$.

For each λ in $\varLambda$ let $\mathscr{H}(\lambda)$ denote the arcs $x(t)$ in $\mathscr{A}$ satisfying $x(a) = 0$ and $x(t) \equiv 0$ on $[\lambda, b]$. Finally if $\mu = (\lambda, \sigma)$ is in the metric space $\mathscr{M} = \varLambda \times \varSigma$ with metric $d(\mu_1, \mu_2) = |\lambda_2 - \lambda_1| + |\sigma_2 - \sigma_1|$, let $\mathscr{B}(\mu) = \mathscr{A}(\sigma) \cap \mathscr{H}(\lambda)$. Thus an arc $x(t)$ in $\mathscr{B}(\lambda, \sigma)$ is a spline of degree 2 on $[a, a_k]$ where $a_k \leqq \lambda < a_{k+1}$, such that $x(a) = 0$ and $x(t) \equiv 0$ on $[a_k, b]$.

To construct $J(x: \mu)$: Let $r_\sigma(t) = r(a_k)$, $p_\sigma(t) = p(a_k)$ if t is in $[a_k, a_{k+1})$; $r_\sigma(b) = r(b)$, $p_\sigma(b) = p(b)$. Finally for $\mu = (\lambda, \sigma)$ let $J(x; \mu) = J(x, x; \mu)$, where

(5) $J(x, y; \mu) = \int_a^b [r_\sigma(t)x'(t)y'(t) - p_\sigma(t)x(t)y(t)]dt$

is defined for arcs $x(t)$, $y(t)$ in $\mathscr{B}(\mu)$.

We now give some important definitions: The *signature (index)* of a quadratic form $Q(x)$ on a subspace $\mathscr{B}$ of $\mathscr{A}$ is the dimension of a maximal, linear subclass $\mathscr{C}$ of $\mathscr{B}$ such that $x \neq 0$ in $\mathscr{C}$ implies $Q(x) < 0$. The *nullity* of $Q(x)$ on $\mathscr{B}$ is the dimension of the set $\mathscr{B}_0 = \{x$ in $\mathscr{B} \mid Q(x, y) = 0$ for all y in $\mathscr{B}\}$. A vector x in B_0 is said to be a Q *null vector* of $\mathscr{B}$. The vector z is Q orthogonal to $\mathscr{B}$ if z satisfies $Q(z, y) = 0$ for all y in $\mathscr{B}$. For each $\mu = (\lambda, \sigma)$ in M let $s(\mu) = s(\lambda, \sigma)$ and $n(\mu) = n(\lambda, \sigma)$ denote the index and nullity of the quadratic form $J(x; \mu)$ given in (5) if $\sigma \neq 0$. Let $s(\lambda, 0)$ and

$n(\lambda, 0)$ denote the index and nullity of the quadratic form $J(x)$ given by (1) on $\mathscr{H}(\lambda)$. Let $\sigma = 0$ in Σ. A point λ at which $s(\lambda, 0)$ is discontinuous is an *oscillation point* of $J(x; 0)$ relative to $\{\mathscr{H}(\lambda):$ λ in $\varLambda = [a, b]\}$. Let $\sigma \neq 0$ in Σ. A point $\lambda(\sigma)$ at which $s(\lambda, \sigma)$ is discontinuous is an *oscillation point* of $J(x; \sigma)$ relative to $B(\mu)$. The oscillation points are denoted by $\lambda_m(\sigma)$, $m = 1, 2, 3, \cdots$.

In Reference [3] we show that the mth oscillation point $\lambda_m(\sigma)$ is a continuous function of σ if $\lambda_m(\sigma) < b$. Furthermore if $\sigma = 1/n$; $a_k = a + k(b - a)/n$ for $k = 0, 1, 2, \cdots, n$;

$$z_k(t) = \begin{cases} 1 - |t - a_k|/\sigma & \text{if } t \text{ in } [a_{k-1}, a_{k+1}] \\ 0 & \text{otherwise} \end{cases}$$

for $k = 1, 2, 3, \cdots$; and $x(t) = b_\alpha z_\alpha(t)$ is in $\mathscr{B}(\mu)$ for $\mu = (\lambda, \sigma)$, $\sigma \neq 0$. A straightforward calculation shows that $J(x; \mu) = b_\alpha b_\alpha e_{\alpha\beta}(\mu) = x^T D(\mu)x$ where $x = (b_1, b_2, \cdots)^T = \Sigma b_\alpha z_\alpha(t)$, $e_{\alpha\beta}(\mu) = J(z_\alpha, z_\beta; \mu)$, and $D(\mu)$ is a symmetric, tridiagonal matrix "increasing" in λ so that the upper $k \times k$ submatrix of $D(a_{k+1}, \sigma)$ is $D(a_k, \sigma)$.

III. **The approximating oscillation vector.** In this section we define the elements of $D(\sigma)$, construct the finite dimension spline function $c(\sigma)$, and show that $c(\sigma)$ approximates "the" solution $x_0(t)$ of (2) subject to $x_0(a) = 0$ in a strong (derivative) sense. We note that $c(\sigma)$ is the Euler-Lagrange equation for the matrix $D(\sigma)$ in the sense that the signature $s(\mu)$ or number of negative eigenvalues of $D(\sigma)$ counts the number of oscillations of $c(\mu)$. Since b is not relevant to our discussion for technical reasons we will modify the definition of $\{a_k\}$. Thus let $\sigma = 1/n$ and define $a_k = a + k\sigma$ for $k = 0, 1, 2, \cdots$. The reader may assume $b - a = m\sigma$ for m sufficiently large.

As indicated above the matrix element $e_{\alpha\beta}$ of $D(\sigma)$ is $J(z_\alpha, z_\beta; \sigma)$. A straightforward calculation shows that $e_{\alpha\beta} = 0$ if $|\alpha - \beta| \geq 2$, $e_{\alpha\beta} = e_{\beta\alpha}$,

(6a) $$e_{\alpha\alpha} = r(a_{\alpha-1}) + r(a_\alpha) - \sigma^2[p(a_{\alpha-1}) + p(a_\alpha)]/3 ,$$

and

(6b) $$e_{\alpha,\alpha+1} = - r(a_\alpha) - \sigma^2 p(a_\alpha)/6 .$$

We remark that similar expressions are obtained for the case of second order integral differential equations and for eigenvalue problems.

Our first computational lemma may be found (for example) in [2; p. 81] with $\lambda = 0$ in the reference. It involves a Sturm-sequence argument. We note that we will assume σ small enough so that

$e_{\alpha\alpha} > 0$ and $e_{\alpha,\alpha+1} < 0$. We define recursively: $p_0 = 1$, $p_1 = e_{11}$, $\cdots$

$$(7) \qquad p_r = e_{rr}p_{r-1} - e_{r,r-1}^2 p_{r-2} \quad (r = 2, 3, 4, \cdots)$$

and note that p_r is the determinant of the upper $r \times r$ submatrix of $D(\sigma)$ which we denote by $D^{(r)}(\sigma)$. In Theorem 2, result (d) is proven in Theorem 5 and included here for completeness.

LEMMA 1. *The number of agreements in sign of two successive members of the sequence* $\{p_r\}$ *is equal to the number of eigenvalues of* $D(\sigma)$ *which are greater than* 0.

THEOREM 2. *The following nonnegative integers are equal:*

(a) $s(a_{k+1}, \sigma) + n(a_{k+1}, \sigma)$,

(b) $k - l(k)$, *where* $l(k)$ *is the number of agreements in sign of* $\{p_0, p_1, p_2, \cdots, p_k\}$,

(c) *the number of nonpositive eigenvalues of* $D^{(k)}(\sigma)$,

and

(d) *the number of times the vector* $c(\sigma)$, *defined below, "crosses the axis" on the interval* $(a_0, a_{k+1}]$.

THEOREM 3. *There exists* $\delta > 0$ *such that if* $\sigma < \delta$ *and* a_{k+1} *is not an oscillation point of* (2), *i.e.,* $x_0(a_{k+1}) \neq 0$ *where* x_0 *is a solution of* (2) *such that* $x_0(a) = 0$, *then the nonnegative integers in Theorem 2 are equal to:*

(e) *the number of oscillation points of* (2) *on* (a, a_{k+1}) *and*

(f) $s(a_{k+1}, 0) = s(a_{k+1}, 0) + n(a_{k+1}, 0)$.

We remark that one method is complete. Namely the elementary calculations of $e_{\alpha\alpha}$ and $e_{\alpha,\alpha-1}$ in (6) and the number of sign changes of p_r in (7) allow us to determine the number of oscillations of (2).

We now wish to consider a second method for finding oscillation points. In this method we will actually construct the numerical oscillation vector $c(\sigma)$ which agrees with (2) in the sense that

$$\int_a^b [c'(\sigma) - x_0'(t)]^2 dt \longrightarrow 0 \quad \text{as} \quad \sigma \longrightarrow 0 .$$

Given the matrix $D(\sigma) = (e_{\alpha\beta})$ define a sequence $\{c_1, c_2, c_3, \cdots\}$ of real numbers as follows:

$$(8a) \qquad c_1 e_{11} + c_2 e_{12} = 0 ,$$

$$(8b) \qquad c_1 e_{21} + c_2 e_{22} + c_3 e_{32} = 0 ,$$

and

(8c) $\qquad c_{r-1}e_{r,r-1} + c_r e_{rr} + c_{r+1}e_{r,r+1} = 0 \quad (r = 3, 4, 5, \cdots)\,.$

As before we assume σ small enough so that $e_{rr} > 0$ and $e_{r,r+1} < 0$ in (6). Given the sequence of numbers $\{c_r\}$ defined in (8), let $x_\sigma(t) = c(\sigma) = \Sigma c_\alpha z_\alpha$ be the spline of degree 2 (broken line segment) such that $x_\sigma(a_k) = c_k$. The vector $x_\sigma(t)$ is the Euler-Lagrange solution of $D(\sigma)$ in the sense that the number of times it crosses the axis is the number of negative eigenvalues (see Theorem 2). Furthermore we will prove that $x_\sigma(t) \to x_0(t)$ in the strong derivative sense described above.

The next theorem is obtained by noting that the product of $D(\sigma)$ with vectors of the form $x_1 = \sum_{k=1}^{n-1} c_k z_k$ and $x_2 = \sum_{k=n_1+1}^{n_2} c_k z_k$ is almost the zero vector because of (8). In fact $x_1^T D(\sigma) x_1 = -c_{n+1}c_n e_{n,n+1}$ and $x_2^T D(\sigma) x_2 = -c_{n_1}c_{n_1+1}e_{n_1+1,n_1} - c_{n_2}c_{n_2+1}e_{n_2,n_2+1}$. These results are easily obtained by "visualizing" the effect of $D(\sigma)$ on the given vectors. Hence our remark about the Euler-Lagrange equation of tridiagonal matrices.

THEOREM 4. *If $c_l c_{l+1} \leqq 0$ for exactly the values $l = n_1, n_2, \cdots$ then the vectors $(c_1, c_2, \cdots, c_{n_1}, 0, 0, 0, \cdots)^T$, $(0, 0, 0, \cdots, c_{n_1+1}, c_{n_1+2}, \cdots, c_{n_2}, 0, 0, 0, \cdots)^T$, etc. are "negative vectors" for $D(\sigma)$ in the sense that $x^T D(\sigma) x \leqq 0$.*

Since $c_l c_{l+1} \leqq 0$ and $-e_{n,n+1} > 0$ we are done.

Note that we have constructive negative vectors for $D(\sigma)$. We wish to show that we have the correct number. This will be done by showing that the sequences $\{c_k\}$ and $\{p_k\}$ are very closely related. Thus choose M, a positive integer, so large that $b - a < M\sigma$. Define $g_1 = \prod_{i=1}^{M-1} e_{i,i+1}$ and $g_{k+1} = g_k/e_{k,k+1}$ $(k = 1, 2, \cdots, M - 1)$. Note that $g_n g_{n+1} < 0$. Define the vector $y = (y_1, y_2, y_3, \cdots)^T$ by $y_k = (-1)^{k+1} p_{k-1} g_k$. The kth component of the vector $D(\sigma)y$ is the quantity

$$[e_{k,k-1}p_{k-2}g_{k-1} - e_{kk}p_{k-1}g_k + e_{k,k+1}p_k g_{k+1}](-1)^k$$
$$= [e_{k,k-1}^2 p_{k-2} - e_{kk}p_{k-1} + p_k](-1)^k g_k = 0$$

by the definition of p_k in (7).

THEOREM 5. *There exists a constant $\gamma \neq 0$ so that $c_k = \gamma(-1)^{k-1}p_{k-1}g_k$. Thus the results in Theorem 2 (part d) hold.*

Let $E^k(\sigma)$ denote the matrix formed from the first k rows and $k + 1$ columns of $D(\sigma)$. By removing the first column of $E^k(\sigma)$ we note that the rank of $E^k(\sigma)$ is k. Let z be in E^{k+1}. Then the solution space of $E^k(\sigma)z = 0$ is one dimensional. By construction $z_1 = (c_1, c_2, \cdots, c_k, c_{k+1})^T$ and $z_2 = (y_1, y_2, \cdots, y_k, y_{k+1})^T$ are in the null

space of $E^{(k)}(\sigma)$ when $y_k = (-1)^{k+1}p_{k-1}g_k$. This completes the proof.

We note that by considering the matrix $D(\sigma) - \lambda I$ (or more generally the matrix $D(\sigma) - \lambda F(\sigma)$ in our setting) we are able to obtain results which count the number of eigenvalues of $D(\sigma)$ which are greater than or equal to λ (or solve a "generalized" eigenvalue problem).

The final result in this section is to show that $\int_a^b [c'(\sigma) - x_0'(t)]^2 dt \to$ 0 as $\sigma \to 0$. The proof is rather technical. It relies on the hypotheses (1) and (2) in [3] and Theorems 4 and 5 in [4] which show that these hypotheses hold in (more general circumstances than) our problem setting with $\sigma_0 = 0$. For convenience and without loss of generality we assume ξ_0 is an oscillation point of (2), i.e., $x_0(t)$ is a nonzero solution of (2) such that $x_0(a) = 0 = x_0(\xi_0)$. Choose $b = \xi + \varepsilon$ (for ε sufficiently small), $\sigma_r = 1/r$ for $1/r < \varepsilon$ and $r > M$.

Let (3) denote the inner product (with $x(a) = 0$), let $J(x; \sigma) = J(x, x; \sigma)$ be given in (5) and let $\mathscr{A}(\sigma)$ be as defined above except that $a_k = a + \sigma k$ and $x(t)$ in $\mathscr{A}(\sigma)$ vanishes if $t > \xi + \sigma$. For continuity of exposition we state conditions (1) and (2) of [3]: Strong convergence is denoted by $x_q \Rightarrow x_0$ and weak convergence by $x_q \to x_0$ in the sense of [3].

Let Σ be a metric space with metric ρ. A sequence $\{\sigma_r\}$ in Σ converges to σ_0 in Σ, written $\sigma_r \to \sigma_0$, if $\lim_{r=\infty} \rho(\sigma_r, \sigma_0) = 0$. For each σ in Σ, let $\mathscr{A}(\sigma)$ be a closed subspace of $\mathscr{A}(0)$ such that:

(9a)   if $\sigma_r \to \sigma_0$, x_r in $\mathscr{A}(\sigma_r)$, and $x_r \to y_0$ then y_0 is in $\mathscr{A}(0)$;

(9b)   if $\bar{x}_0$ is in $\mathscr{A}(0)$ and $\varepsilon > 0$, there exists $\delta > 0$

such that, whenever $\rho(\sigma, \sigma_0) < \delta$, there exists x_σ in $\mathscr{A}(\sigma)$ satisfying $\| \bar{x}_0 - x_\sigma \| < \varepsilon$.

For each σ in Σ let $J(x; \sigma)$ be a quadratic form defined on $\mathscr{A}(\sigma)$ with $J(x, y; \sigma)$ the associated bilinear form. For $r = 0, 1, 2, \cdots$, let x_r be in $\mathscr{A}(\sigma_r)$, y_r in $\mathscr{A}(\sigma_r)$ such that: If $x_r \to \bar{x}_0$, $y_r \Rightarrow y_0$, and $\sigma_r \to \sigma_0$, then

(10a) $\lim\limits_{r=\infty} J(x_r, y_r; \sigma_r) = J(\bar{x}_0, y_0; \sigma_0)$,

(10b) $\liminf\limits_{r=\infty} J(x_r; \sigma_r) \geqq J(\bar{x}_0; \sigma_0)$,

and

(10c) $\lim\limits_{r=\infty} J(x_r; \sigma_r) = J(\bar{x}_0; \sigma_0)$ implies $x_r \Longrightarrow \bar{x}_0$.

We begin by proving an interesting theorem concerning $J(x, y; \sigma)$.

THEOREM 6. *There exists a $\delta > 0$ and $M > 0$ such that if $\sigma < \delta$ then $J(x, y; \sigma) \leqq M \| x \| \| y \|$ for all x and y in $\mathscr{A}(\sigma)$.*

Suppose not. Then for $r = N, N + 1, \cdots$ we may choose x_r, y_r in $\mathscr{A}(\sigma_r)$ such that $\| x_r \| = \| y_r \| = 1$, $\sigma_r < 1/r$, and $a_r^2 = | J(x_r, y_r; \sigma_r) | > r$. Now $\bar{x}_r = x_r/a_r \Rightarrow 0$, $\bar{y}_r = y_r/a_r \Rightarrow 0$ so by (10a), $1 = J(\bar{x}_r, \bar{y}_r; \sigma_r) \to J(0, 0; 0) = 0$. This contradiction establishes the result.

For the next result let k_σ satisfy $a_{k_\sigma} < \xi \leqq a_{k_\sigma + 1} = b_0$. Let $x_\sigma(t) = (c_1, c_2, \cdots, c_{k_\sigma}, 0)_\sigma = \sum_{i=1}^{k_\sigma} c_i z_i(t)$ where $z_i(t)$ is the ith spline basis element and assume $J(x_\sigma(t); \sigma) < 0$. By Theorems 2 and 3, $c_{k_\sigma} c_{k_\sigma + 1} \leqq 0$ so we may choose c_1 such that $J(x_\sigma(t); \sigma) < 0$.

LEMMA 7. *For each $\sigma = 1/m$ we have $J(c_\sigma^*; \sigma) < 0$ where $c_\sigma^* = (c_1, c_2, \cdots, c_{n_\sigma}, 0)_\sigma$; c_i is chosen by our algorithm. Furthermore if $\| c_\sigma^* \| = 1$ then there exists $y_0(t)$ in $\mathscr{A}(b)$ such that $c_{\sigma_n} \Rightarrow y_0(t)$.*

The first part was proven above. The second part follows since Hilbert spaces are weak sequentially compact; that is, if $\{z_k\} \subset S$ (bounded) then there exists z_0 in S such that $z_{k_n} \to z_0$ for some subsequence $\{z_{k_n}\}$ of $\{z_k\}$.

THEOREM 8. *The vector $c_\sigma^* = (c_1, c_2, \cdots, c_\sigma, 0)$ given by the algorithm converges strongly to $x_0(t)$ (as $\sigma \to 0$) in the derivative norm sense of (3).*

By lemma let $\{c_{\sigma_q}^*\}$ be any weakly convergent subsequence of $\{c_{\sigma_n}^*\}$ so that $\lim\sup_{\sigma_q \to 0} J(c_{\sigma_q}^*; \sigma_q) \leqq 0$. By (10b) $\lim\inf_{\sigma_q \to 0} J(c_{\sigma_q}^*; \sigma_q) \geqq J(y_0(t))$. Thus

$$0 \geqq \lim_{\sigma_q \to 0} \sup J(c_{\sigma_q}^*; \sigma_q) \geqq \lim_{\sigma_q \to 0} \inf J(c_{\sigma_q}^*; \sigma_q) \geqq J(y_0(t)) \geqq 0$$

and

$$\lim_{\sigma_q \to 0} J(c_{\sigma_q}^*; \sigma_q) = J(y_0(t)) = 0 \text{ implies by (10c) that } c_{\sigma_q}^* \Longrightarrow y_0(t) .$$

Furthermore $y_0(t) = x_0(t)$ on $[a, b_0]$ or $y_0(t) \equiv 0$. The latter is impossible since $1 = \lim_{q \to \infty} \| c_{\sigma_q} \| = \| y_0 \| = 0$ is impossible. This completes the proof.

IV. More general problems. In future papers we will construct the Euler-Lagrange equations for "higher order" banded symmetric matrices, such as for $2k$th order symmetric differential equations and partial differential equations, by modifying the above arguments. In addition we have been able to adapt our existing

ideas to eigenvalue problems (and double eigenvalue problems), integral differential equations, and to some singular differential equations.

The last result is somewhat unexpected! We have been able to numerically construct the Bessel functions $J_0(x)$ and $J_1(x)$ although our quadratic form is singular as $r(0) = 0$. In fact, at this time the author must confess that we have not developed the abstract results to justify the results we have obtained. It is evident that this will come when the work of Stein [7] has been properly interpreted. It thus appears that our algorithm may be applicable to many of the singular differential equations and quadratic forms which occur in mathematical physics. It also appears that by a change of the independent variable such as $s = 1/t$ we may be able to handle problems where $b = \infty$ such as limit point limit circle problems. Needless to say we will not only obtain qualitative results for second order problems but also quantitative and numerical results for more general problems.

Finally a remark about symmetric matrices: While it appears that tridiagonal results are rather limited we note the well known "Jacobi-Given's Method" for reduction of symmetric matrices to tridiagonal form [6, pp. 352-354] by finite product of orthogonal matrices of the form $0_{lm} = (a_{ij})$ where $a_{ll} = a_{mm} = \cos \phi$, $a_{lm} = -a_{ml} = \sin \phi (m < l)$, $a_{kk} = 1$ if $k \neq l$ or m, and $a_{ij} = 0$ otherwise. Far more efficient for banded matrices are methods (for example) due to Schwartz [8, pp. 273-283]. In fact, it appears from references such as [8] that the "state of the art" for numerical solutions of symmetric matrices has sufficiently progressed to make the numerical solutions of symmetric differential systems and quadratic functionals a very feasible concept.

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SOUTHERN ILLINOIS UNIVERSITY
CARBONDALE, IL 62901