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ON COUNTABLE PRODUCTS AND ALGEBRAIC CONVEXIFICATIONS OF PROBABILISTIC METRIC SPACES

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Two different ways of defining a probabilistic metric on the countable product of a family of probabilistic metric spaces are studied and compared. The algebraic convexification of probabilistic metric spaces is also investigated.

0. Introduction. Finite products of probabilistic metric (PM) spaces have been studied previously by R. Egbert [1], R. Tardiff [10], A. Xavier [13] and V. Istratescu and I. Vaduva [2]. In this paper we turn to the study of countable products.

If $\{(S_i, \mathscr{F}^i, \tau_i) \mid i \in N\}$ is a family of PM spaces and if we form the generalized metric space $(\prod_{i=1}^{\infty} S_i, \prod_{i=1}^{\infty} \Delta^+, \prod_{i=1}^{\infty} \tau_i)$ in the sense of E. Trillas [11, 12], then the problem is to choose the most satisfactory assignment of a probability distribution function in Δ^+ to each member of the family $(\mathscr{F}^i)$, i.e., to each sequence $(F_i) \in \prod_{i=1}^{\infty} \Delta^+$. Two natural assignments are considered:

(a) The series $\sum_{i=1}^{\infty} (1/2^i) F_i$ as the weak limit of the pointwise nondecreasing sequence $\{\sum_{i=1}^{n} (1/2^i) F_i \mid n \in N\}$ in Δ^+.

(b) The product $\tau_{i=1}^{\infty} F_i$ as the weak limit of the pointwise nonincreasing sequence $\{\tau(F_1, \cdots, F_n) \mid n \in N\}$ in Δ^+, where τ is an arbitrary triangle function.

In case (a) we speak of *Σ-products* and in case (b) of *τ-products*.

In addition we also consider the question of the algebraic convexification of a PM space, which involves the embedding of the given space a in convex subspace of a suitably defined countable product.

Throughout the paper we assume that the reader is familiar with the basic definitions and concepts of the theory of PM spaces as given, e.g., in [8] or [10].

1. On Σ-products. We begin with the following:

DEFINITION 1.1. Let $\{(S_i, \mathscr{F}^i, \tau_i) \mid i \in N\}$ be a countable family of PM spaces. The Σ-product of this family is the space $(\prod_{i=1}^{\infty} S_i, \mathscr{F}^{\Sigma})$, where $\mathscr{F}^{\Sigma}: \prod_{i=1}^{\infty} S_i \times \prod_{i=1}^{\infty} S_i \to \Delta^+$, is the mapping given by $\mathscr{F}^{\Sigma}((p_i), (q_i)) = \sum_{i=1}^{\infty} (1/2^i) \mathscr{F}^i(p_i, q_i)$, for any sequences (p_i) and (q_i) in $\prod_{i=1}^{\infty} S_i$.

In this section we will use the abbreviations: $S = \prod_{i=1}^{\infty} S_i$, $F = \mathscr{F}^{\Sigma}$, $F_{\overline{pq}} = \mathscr{F}^{\Sigma}((p_i), (q_i))$ and $F_{p_i q_i} = \mathscr{F}^i(p_i, q_i)$.

THEOREM 1.1. *The Σ-product (S, F) is a PM space, more precisely, a Menger space under the t-norm T_w.*

Proof. We have to show: (1) $F_{\bar{p}\bar{q}} = \varepsilon_0$ if and only if $\bar{p} = \bar{q}$, where $\varepsilon_0 \in \Delta^+$ is given by

$$\varepsilon_0(x) = \begin{cases} 0, & \text{if } x \leqq 0\,, \\ 1, & \text{if } x > 0\,; \end{cases} \tag{1.1}$$

(2) $F_{\bar{p}\bar{q}} = F_{\bar{q}\bar{p}}$, and (3) if $F_{\bar{p}\bar{q}}(x) = 1$ and $F_{\bar{q}\bar{r}}(y) = 1$ then $F_{\bar{p}\bar{r}}(x + y) = 1$. Since $\sum_{i=1}^{\infty} (a_i/2^i) = 1$ if and only if each $a_i = 1$ (when $a_i \in [0, 1]$, for each $i \in N$), the verification of (1), (2), and (3) is immediate.

THEOREM 1.2. *The Σ-product (S, F) is a PM space under the triangle function τ_{T_m} whenever each $(S_i, \mathscr{F}^i, \tau_i)$ is such that $\tau_i \geqq \tau_{T_m}$.*

Proof. In view of (1) and (2) of Theorem 1.1, we need only prove the triangle inequality. Let $x, y \geqq 0$ and $\bar{p}, \bar{q}, \bar{r}$ in S be given. Then

$$\begin{aligned} T_m(F_{\bar{p}\bar{q}}(x), F_{\bar{q}\bar{r}}(y)) &= \operatorname{Max}\,(F_{\bar{p}\bar{q}}(x) + F_{\bar{q}\bar{r}}(y) - 1, 0) \\ &= \operatorname{Max}\left(\sum_{i=1}^{\infty} 2^{-i}(F_{p_iq_i}(x) + F_{q_ir_i}(y) - 1), 0\right) \\ &\leqq \sum_{i=1}^{\infty} 2^{-i} \operatorname{Max}\,(F_{p_iq_i}(x) + F_{q_ir_i}(y) - 1, 0) \\ &= \sum_{i=1}^{\infty} 2^{-i} T_m(F_{p_iq_i}(x), F_{q_ir_i}(y)) \leqq \sum_{i=1}^{\infty} 2^{-i} F_{p_ir_i}(x + y) = F_{\bar{p}\bar{r}}(x + y)\,, \end{aligned}$$

where in the last inequality we have used the fact that for every $i \in N$,

$$\begin{aligned} T_m(F_{p_iq_i}(x), F_{q_ir_i}(y)) &\leqq \tau_{T_m}(F_{p_iq_i}, F_{q_ir_i})(x + y) \\ &\leqq \tau_i(F_{p_iq_i}, F_{q_ir_i})(x + y) \leqq F_{p_ir_i}(x + y)\,. \end{aligned}$$

Thus for any $t \geqq 0$, $\tau_{T_m}(F_{\bar{p}\bar{q}}, F_{\bar{q}\bar{r}})(t) = \sup_{x+y=t} T_m(F_{\bar{p}\bar{q}}(x), F_{\bar{q}\bar{r}}(y)) \leqq F_{\bar{p}\bar{r}}(t)$.

Following the lines of the above proof it is easy to see that the Σ-product (S, F) is a PM space under the triangle function Π_{T_m} whenever each $(S_i, \mathscr{F}^i, \tau_i)$ is such that $\tau_i \geqq \Pi_{T_m}$.

Since the most common t-norms are stronger than T_m (e.g., $T_m \leqq \text{Prod} \leqq \text{Min}$) it follows that Theorem 1.2 applies to a large class of PM spaces. However T_m cannot be replaced by a stronger t-norm, whence, for triangle functions of the form τ_T, the result of Theorem 1.2 is best-possible. This is a consequence of:

THEOREM 1.3. *Let T be a t-norm and suppose that*

$$T\left(\sum_{i=1}^{\infty}\frac{a_i}{2^i}, \sum_{i=1}^{\infty}\frac{b_i}{2^i}\right) \leqq \sum_{i=1}^{\infty}\frac{1}{2^i}T(a_i, b_i)\ , \tag{1.2}$$

for any sequences (a_i), (b_i) *in* $[0, 1]$. *Then* $T_w \leqq T \leqq T_m$.

Proof. Note first that T_w satisfies (1.2). Similarly, the fact that T_m satisfies (1.2) is the crucial point in the proof of Theorem 1.2. Now suppose T satisfies (1.2). Since T is always stronger than T_w and since $T = T_m$ on the boundary of the unit square, we must show $T \leqq T_m$ on $(0, 1) \times (0, 1)$. To this end, let $B_0 = 0$ and, for any $n \geqq 1$, let $B_n = 1/2 + \cdots + 1/2^n$ and consider the partition $(0, 1) \times (0, 1) = R_1 \cup \bigcup_{n=2}^{\infty} R_n$, where

$$R_1 = \{(x, y) \mid 0 < x, y < 1, x + y \leqq 1\}$$

and

$$R_n = \{(x, y) \mid 0 < x, y < 1, 1 + B_{n-2} < x + y \leqq 1 + B_{n-1}\}\ .$$

Let $(x, y) \in R_1$ be such that $x + y = 1$, and let $\sum_{i=1}^{\infty}(a_i/2^i)$ be any binary expansion of x, i.e., $x = \sum_{i=1}^{\infty}(a_i/2^i)$, where $a_i \in \{0, 1\}$, for each i. Then noting that $T(1, 0) = T(0, 1) = 0$ and using (1.2) we have

$$T(x, y) = T(x, 1 - x) = T\left(\sum_{i=1}^{\infty}\frac{a_i}{2^i}, \sum_{i=1}^{\infty}\frac{1 - a_i}{2^i}\right) \leqq \sum_{i=1}^{\infty}\frac{1}{2^i}T(a_i, 1 - a_i) = 0\ .$$

Thus since T is nondecreasing, $T(x, y) = T_m(x, y) = 0$ for all (x, y) in R_1.

Now fix $n \geqq 2$ and consider any point $(x, y) \in R_n$. Then $x + y = 1 + B_{n-2} + a$, where $0 < a \leqq 1/2^{n-1}$, so that at least one of x, y must be greater than B_{n-1}.

Suppose $x = B_{n-1} + \sum_{i=n}^{\infty}(x_i/2^i)$, where $x_i \in \{0, 1\}$ for each i. Then since $1 - B_{n-1} = 1/2^{n-1}$, we have

$$\begin{aligned} y &= 1 + B_{n-2} + a - x = B_{n-2} + a + 1/2^{n-1} - \sum_{i=n}^{\infty} x_i/2^i \\ &= B_{n-2} + 2^{n-1}a/2^{n-1} + \sum_{i=n}^{\infty}(1 - x_i)/2^i\ . \end{aligned}$$

Consequently, writing

$$x = B_{n-2} + 1/2^{n-1} + \sum_{i=n}^{\infty} x_i/2^i\ ,$$

and then using (1.2) and the fact that $T(1, 1) = 1$, yields

$$\begin{aligned} T(x, y) &\leqq B_{n-2} + \frac{1}{2^{n-1}}T(1, 2^{n-1}a) + \sum_{i=n}^{\infty}\frac{1}{2^i}T(x_i, 1 - x_i) \\ &= B_{n-2} + a = x + y - 1 = T_m(x, y)\ . \end{aligned}$$

If $x < B_{n-1}$, then reversing the roles of x and y yields the same conclusion, and this completes the proof.

It should be noted that neither the commutativity nor associativity of T was used in the above proof. Thus we have in fact established:

COROLLARY 1.1. *Let* $T\colon [0, 1] \times [0, 1] \to [0, 1]$ *be nondecreasing in each place and such that* $T(0, x) = T(x, 0) = 0$ *and* $T(x, 1) = T(1, x) = x$, *for any* x *in* $[0, 1]$. *Suppose* T *satisfies* (1.2). *Then* $T_w \leqq T \leqq T_m$.

The converse of Corollary 1.1 is false as the following example shows.

EXAMPLE 1.1. For any $\lambda \in [0, 1]$ consider the function $T_\lambda\colon [0, 1] \times [0, 1] \to [0, 1]$ defined by

$$T_\lambda(x, y) = \begin{cases} T_m(x, y), & \text{if} \quad x + y \leqq 1 + \lambda \quad \text{or} \quad x = 1 \quad \text{or} \quad y = 1\,, \\ \lambda, & \text{otherwise.} \end{cases}$$

If $0 \leqq \mu < \lambda \leqq 1$, we have $T_w = T_0 \leqq T_\mu < T_\lambda \leqq T_1 = T_m$. Let $0 < \lambda < 1$. Then there is an $n \in N$ such that $\lambda < 1 - 2^{-(n-1)} < 1$ and consequently an $a \in (0, 1]$ such that $(1 + \lambda)(2 - 2^{-(n-1)})^{-1} < a < 1$. Hence

$$T_\lambda\Big(\sum_{i=1}^n \frac{a}{2^i}, \sum_{i=1}^n \frac{a}{2^i}\Big) = T_\lambda(a(1 - 2^{-n}), a(1 - 2^{-n}))$$
$$= \lambda = T_\lambda(a, a) > \sum_{i=1}^n \frac{1}{2^i} T_\lambda(a, a)\,.$$

Thus whenever $0 < \lambda < 1$, (1.2) fails for T_λ.

Since the functions T_λ defined above are not associative, Example 1.1 is not a complete counterexample of Theorem 1.3. A t-norm weaker than T_m violating (1.2) remains to be found. Indeed, there is good reason to conjecture that any continuous t-norm weaker than T_m, satisfies (1.2).

As a consequence of Theorem 1.3 it is to be expected that, even in the case of a family of PM spaces under the same t-norm T, the Σ-product need not be a PM space under T. The next two examples show that this is indeed the case.

EXAMPLE 1.2. The Σ-product of Wald spaces is not necessarily a Wald space.

For any $a \geqq 0$, let $\varepsilon_a(x) = \varepsilon_0(x - a)$, where ε_0 is given by (1.1). Consider the metric space $(\boldsymbol{R}^+, |\ \ |)$ as a Wald space $(\boldsymbol{R}^+, G, *)$, where $G_{pq} = \varepsilon_{|p-q|}$ for all $p, q \in \boldsymbol{R}^+$. Let $(S_i, \mathscr{F}^i, \tau_i) = (\boldsymbol{R}^+, G, *)$ for each i and form the Σ-product $(\prod_{i=1}^\infty \boldsymbol{R}^+, F, \tau_{T_m})$. Choose $\bar{p} = (0)$, $\bar{q} = (1, 0, 0, \cdots)$, $\bar{r} = (1)$. Then $F_{\bar{p}\bar{q}} = F_{\bar{q}\bar{r}} = 1/2(\varepsilon_1 + \varepsilon_0)$, $F_{\bar{p}\bar{r}} = \varepsilon_1$, and $F_{\bar{p}\bar{q}} * F_{\bar{q}\bar{r}} = 1/4\varepsilon_2 + 1/2\varepsilon_1 + 1/4\varepsilon_0$, whence for $0 < x < 1$, $F_{\bar{p}\bar{q}} * F_{\bar{q}\bar{r}} > F_{\bar{p}\bar{r}}$.

EXAMPLE 1.3. The Σ-product of simple spaces is not necessarily a simple space.

Let each component space be the simple space $(\boldsymbol{R}^+, d, G)$ generated by the metric $d(x, y) = |x - y|/1 + |x - y|$ and a distribution function $G \in \mathscr{D}^+$ such that $G(1) < 1/2$.

In the Σ-product $(\prod_{i=1}^\infty \boldsymbol{R}^+, F, \tau_{T_m})$ we have $F_{\bar{p}\bar{q}}(x) = \sum_{i=1}^\infty 1/2^i G_{p_i q_i}(x)$, where for every $i \geqq 1$, $G_{p_i q_i}(x) = G(x/d(p_i, q_i))$ if $p_i \neq q_i$ and $G_{p_i q_i}(x) = \varepsilon_0(x)$ if $p_i = q_i$. Choose $\bar{p} = (0)$, $\bar{q} = (0, 2, 3, 4, \cdots, n, \cdots)$ and $\bar{r} = (1, 2, 3, \cdots, n, \cdots)$. Then $F_{\bar{p}\bar{q}}(1/4) \geqq 1/2$, $F_{\bar{q}\bar{r}}(1/4) \geqq 1/2$ but $F_{\bar{p}\bar{r}}(1/2) < 1/2$. Thus the Σ-product is not a Menger space under Min. Consequently it cannot be a simple space.

One of the most interesting facts about Σ-products is given in the following:

THEOREM 1.4. *Let $\{(S_i, \mathscr{F}^i, \tau_i) | i \in \boldsymbol{N}\}$ and (S, F) be as in Theorem 1.2. Let each S_i be endowed with the ε, λ-topology induced by $\mathscr{F}^i$. Then the ε, λ-topology on S induced by F is the product topology.*

Proof. Since T_m is continuous the system of neighborhoods $B = \{N_{\bar{p}}(\varepsilon, \lambda) | \bar{p} \in S, \varepsilon, \lambda > 0\}$, where $N_{\bar{p}}(\varepsilon, \lambda) = \{\bar{q} | \bar{q} \in S, F_{\bar{p}\bar{q}}(\varepsilon) > 1 - \lambda\}$, is a basis for the ε, λ-topology in (S, F). Similarly, for every $i \in \boldsymbol{N}$, the system $B_i = \{N_p(\varepsilon, \lambda) | p \in S_i, \varepsilon, \lambda > 0\}$ where $N_p(\varepsilon, \lambda) = \{q | q \in S_i, F^i_{pq}(\varepsilon) > 1 - \lambda\}$, is a basis for the ε, λ-topology in (S_i, F^i). Thus we have to show that B and the system of neighborhoods

$$C = \left\{\prod_{i=1}^n N_{p_i}(\varepsilon_i, \lambda_i) \times \prod_{i=1}^\infty S_{i+n} | n \in \boldsymbol{N}, (p_1, \cdots, p_n) \in \prod_{i=1}^n S_i\right\}$$

which is a basis for the product topology in S, are equivalent.

Given $N_{\bar{p}}(\varepsilon, \lambda)$ in B, choose $k \in \boldsymbol{N}$ such that $\lambda' = 1 - (1 - \lambda)/\sum_{i=1}^k 2^{-i} > 0$ and note that $\lambda' < 1$ if $\lambda < 1$. Then if $\bar{q} \in U = \prod_{i=1}^k N_{p_i}(\varepsilon, \lambda') \times \prod_{i=1}^\infty S_{i+k}$ we have $F_{p_i q_i}(\varepsilon) > 1 - \lambda'$, for $i = 1, 2, \cdots, k$. Thus

$$F_{\bar{p}\bar{q}}(\varepsilon) \geqq \sum_{i=1}^k 2^{-i} F_{p_i q_i}(\varepsilon) > \sum_{i=1}^k 2^{-i}(1 - \lambda') = 1 - \lambda\ ,$$

and $U \subset N_{\bar{p}}(\varepsilon, \lambda)$. In the other direction, let $V = \prod_{i=1}^n N_{p_i}(\varepsilon_i, \lambda_i) \times$

$\prod_{i=1}^{\infty} S_{i+n}$, where $0 < \lambda_i < 1$ for $i = 1, 2, \cdots, n$, be a given neighborhood in C. Choose $\varepsilon = \operatorname{Min}\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$ and

$$\lambda = 1 - \operatorname{Max}\left\{2^{-i}(1-\lambda_i) + \sum_{\substack{k=1\\k\neq i}}^{\infty} 2^{-k} \,\middle|\, i = 1, \cdots, n\right\} .$$

If $\bar{q} \in N_{\bar{p}}(\varepsilon, \lambda)$, we have for each $i = 1, 2, \cdots, n$,

$$\begin{aligned} F_{\bar{p}\bar{q}}(\varepsilon) > 1 - \lambda &= \operatorname{Max}\left\{2^{-i}(1-\lambda_i) + \sum_{\substack{k=1\\k\neq i}}^{\infty} 2^{-k} \,\middle|\, i = 1, \cdots, n\right\} \\ &\geqq 2^{-i}(1-\lambda_i) + \sum_{\substack{k=1\\k\neq i}}^{\infty} 2^{-k} F_{p_k q_k}(\varepsilon) , \end{aligned}$$

whence

$$F_{p_i q_i}(\varepsilon_i) \geqq F_{p_i q_i}(\varepsilon) > 1 - \lambda_i$$

thus $N_{\bar{p}}(\varepsilon, \lambda) \subset V$, and the proof is complete.

Recalling some elementary theorems of general topology, it is immediate that the ε, λ-topology induced by F on S is the least topology making the projections $\pi_i: S \to S_i$ continuous for all $i \in N$. We also have:

COROLLARY 1.2. *If (S', F', τ') is a PM space with $\tau' \geqq \tau_{T_m}$ then the mapping f from S' into $\prod_{i=1}^{\infty} S'$ given by $f(p) = (p)$ is an isometry and is continuous with respect to the ε, λ-topology.*

From [5] we know that the ε, λ-topology of a PM space with an Archimedean t-norm T is metrizable by the metrics $d_z(p, q) = -\log C_T F_{pq}(z)$, for any $z > 0$, where C_T is the T-conjugate transform for the semigroup (Δ^+, τ_T), i.e., C_T is defined for any $F \in \Delta^+$ via:

$$C_T F(z) = \sup_{x \geqq 0} e^{-xz} hF(x) , \quad \text{for all} \quad z \geqq 0 ,$$

where h is a fixed multiplicative generator of T and $hF \in \Delta^+$ is given by

$$hF(x) = \begin{cases} 0, & x \leqq 0 , \\ h(F(x)), & 0 < x . \end{cases}$$

Combining this with Theorem 1.2 and using the T_m-conjugate transform $(h(x) = e^{x-1})$, we obtain:

COROLLARY 1.3. *The product topology in S is metrizable by the metric $d_z(\bar{p}, \bar{q}) = -\log \sup_{x \geqq 0} \exp\left(\sum_{i=1}^{\infty} 2^{-i}(F_{p_i q_i}(x) - zx - 1)\right)$, for* *any* $z > 0$. *This metric is equivalent to the metric* $d'(\bar{p}, \bar{q}) = \sum_{i=1}^{\infty} 2^{-i} \operatorname{Min}[-\log \sup_{x \geqq 0} \exp(F_{p_i q_i}(x) - z_i x - 1), 1]$, *where* $z_i > 0$ *for all* $i \in N$.

2. On τ-products.

DEFINITION 2.1. Let $\{(S_i, \mathscr{F}^i, \tau_i) \mid i \in N\}$ be a countable family of PM spaces. The τ-product is the space $(\prod_{i=1}^{\infty} S_i, G)$, where $G: \prod_{i=1}^{\infty} S_i \times \prod_{i=1}^{\infty} S_i \to \Delta^+$, is the mapping given by $G((p_i), (q_i)) = \tau_{i=1}^{\infty} \mathscr{F}^i(p_i, q_i) = w - \lim_{n\to\infty} \tau(\mathscr{F}^1(p_1, q_1), \cdots, \mathscr{F}^n(p_n, q_n))$, for any sequences (p_i), (q_i) in $\prod_{i=1}^{\infty} S_i$.

As in the preceding section, we adopt the conventions, $S = \prod_{i=1}^{\infty} S_i$, $G_{\bar{p}\bar{q}} = G((p_i), (q_i))$, $F_{p_i q_i} = \mathscr{F}^i(p_i, q_i)$.

THEOREM 2.1. *If each of the PM spaces* $(S_i, \mathscr{F}^i, \tau_i)$ *is such that* $\tau_i \geqq \tau$, *where* τ *is a continuous triangle function, then the* τ-*product* (S, G) *is a PM space under* τ.

Proof. If $G_{\bar{p}\bar{q}} = \varepsilon_0$ then $F_{p_i q_i} = \varepsilon_0$, for any i, so $(p_i) = (q_i)$. Conversely $G_{\bar{p}\bar{q}} = \tau_{i=1}^{\infty} \varepsilon_0 = \varepsilon_0$. The symmetry of G is obvious and the triangle inequality follows from

$$\begin{aligned}\tau(G_{\bar{p}\bar{q}}, G_{\bar{q}\bar{r}}) &= \tau(w - \lim_{n\to\infty} \tau(F_{p_1 q_1}, \cdots, F_{p_n q_n}), w - \lim_{n\to\infty} \tau(F_{q_1 r_1}, \cdots, F_{q_n r_n})) \\ &= w - \lim_{n\to\infty} \mathop{\tau}_{i=1}^{n} \tau(F_{p_i q_i}, F_{q_i r_i}) \leqq w - \lim_{n\to\infty} \mathop{\tau}_{i=1}^{n} \tau_i(F_{p_i q_i}, F_{q_i r_i}) \\ &\leqq w - \lim_{n\to\infty} \mathop{\tau}_{i=1}^{n} F_{p_i r_i} = G_{\bar{p}\bar{r}} .\end{aligned}$$

At first this result, which is a straightforward generalization from finite products to countably infinite ones, seems to be satisfactory. However, two difficulties arise immediately. The first is the fact that since the sequence $\{\tau_{i=1}^{n} F_{p_i q_i} \mid n \in N\}$ is nonincreasing its weak limit may be zero everywhere, i.e., the infinite product may diverge. This question has recently been studied by R. Moynihan [4]. The second difficulty is of a topological nature.

THEOREM 2.2. *Let each of the PM spaces* $(S_i, \mathscr{F}^i, \tau_i)$ *be endowed with the* ε, λ-*topology. Then the product topology is weaker than the* ε, λ-*topology in* (S, G).

Proof. Let $U = \prod_{i=1}^{n} N_{p_i}(\varepsilon_i, \lambda_i) \times \prod_{i=n+1}^{\infty} S_i$ be a standard neighborhood in the product topology. Choose $\varepsilon = \operatorname{Min}\{\varepsilon_1, \cdots, \varepsilon_n\}$, $\lambda = \operatorname{Min}\{\lambda_1, \cdots, \lambda_n\}$ and let $\bar{q} \in N_{\bar{p}}(\varepsilon, \lambda)$. Then, since $G_{\bar{p}\bar{q}} \leqq F_{p_i q_i}$ for all

i, we have $1 - \lambda_i \leqq 1 - \lambda < G_{\bar{p}\bar{q}}(\varepsilon) \leqq F_{p_iq_i}(\varepsilon) \leqq F_{p_iq_i}(\varepsilon_i)$. Whence $N_{\bar{p}}(\varepsilon, \lambda) \subset \prod_{i=1}^{\infty} N_{p_i}(\varepsilon, \lambda) \subset U$.

From the above proof it is clear that, in general, the two topologies are not equal. For if this were the case, given $N_{\bar{p}}(\varepsilon, \lambda)$ there would exist a product neighborhood $U = \prod_{i=1}^{m} N_{p_i}(\varepsilon_i, \lambda_i) \times \prod_{i=1}^{\infty} S_{i+m}$ such that $U \subset N_{\bar{p}}(\varepsilon, \lambda) \subset \prod_{i=1}^{\infty} N_{p_i}(\varepsilon, \lambda)$, which implies that $S_i = N_{p_i}(\varepsilon, \lambda)$ for all $i \geqq m$, a very strong condition. It follows that statements such as Corollary 1.2 also fail in general.

The reason for the difference between Theorems 1.4 and 2.2 is easily understood if one pays attention to the probabilistic interpretation of the ε, λ-neighborhoods in the respective products spaces: If $N_{\bar{p}}(\varepsilon, \lambda)$ is a neighborhood in the Σ-product then $(q_i) \in N_{\bar{p}}(\varepsilon, \lambda)$ implies that, with probability greater than $1 - \lambda$, at least one of the p_i is at a distance less than ε from the corresponding q_i. On the other hand, if $N_{\bar{p}}(\varepsilon, \lambda)$ is a neighborhood in the τ-product, and $(q_i) \in N_{\bar{p}}(\varepsilon, \lambda)$ then, with probability greater than $1 - \lambda$, all the p_i are at a distance less than ε from the corresponding q_i.

3. Algebraic convexifications. For a PM space $(S, \mathscr{F}, \tau)$ the Wald-betweenness relation which is defined by $W(p, q, r)$ if and only if $\tau(F_{pq}, F_{qr}) = F_{pr}$ has recently been studied in [5]. In accordance with the concepts developed there, we make the following:

DEFINITION 3.1. A probabilistic semi-metric space is τ-convex if, for every pair of distinct points p, r in S, there exists a point q S, $p \neq q \neq r$, such that $\tau(F_{pq}, F_{qr}) = F_{pr}$.

DEFINITION 3.2. An algebraic convexification [12] of a PM space $(S, \mathscr{F}, \tau)$ is any extension of this space which is τ-convex.

THEOREM 3.1. *If $(S, \mathscr{F})$ is a probabilistic semi-metric space, then there exists an extension $(S^*, \mathscr{F}^*)$ which is Π_{T_m}-convex.*

Proof. For any $\bar{p}(n) = (p_1, p_2, \cdots, p_n)$ in S^n, let $(\bar{p}(n), *)$ denote the element of $\prod_{i=1}^{\infty} S$ obtained by repeating the finite string $\bar{p}(n)$ infinitely often: thus $(\bar{p}(n), *) = (p_1, p_2, \cdots, p_n, p_1, p_2, \cdots, p_n, \cdots)$. Let

$$S^* = \{(\bar{p}(2^k), *) \mid k \in N \quad \text{and} \quad \bar{p}(2^k) \in S^{2^k}\} .$$

In the Σ-product $(\prod_{i=1}^{\infty} S, \mathscr{F})$ let F^* be the restriction of F to $S^* \times S^*$ and $f: S \rightarrow S^*$ the injection given by $f(p) = (p, p, p, \cdots)$. Note that f is distance preserving in the sense that $F_{pq} = F^*_{f(p)f(q)}$ for any $p, q \in S$. Thus $(S^*, \mathscr{F}^*)$ is an extension of $(S, \mathscr{F})$. To establish the Π_{T_m}-convexity of S^* let $\bar{p}(2^i) = (p_1, p_2, \cdots, p_{2^i})$ and

$\bar{r}(2^j) = (r_1, r_2, \cdots, r_{2^j})$ be any two fixed elements of S^{2^i} and S^{2^j}, respectively, and assume, without loss of generality, that $i \leqq j$. Let $\alpha = (\bar{p}(2^i), *)$ and $\gamma = (\bar{r}(2^j), *)$; and note that $\alpha = \gamma$ if and only if $\bar{r}(2^j)$ is the string obtained by repeating the string $\bar{p}(2^i)$ exactly 2^{j-i} times. Now suppose $\alpha \neq \gamma$. Let

$$\bar{q}(2^{j+1}) = (\overbrace{\bar{p}(2^i), \cdots, \bar{p}(2^i)}^{2^{i-j}\text{ times}}, \bar{r}(2^j))$$

and let $\beta = (\bar{q}(2^{j+1}), *)$. If $\beta = \alpha$ then, as one readily sees, $\alpha = \gamma$, which cannot be. Thus $\beta \neq \alpha$ and, similarly $\beta \neq \gamma$. Since $\bar{q}(2^{j+1})$ breaks up into two strings, each of length 2^j, it follows that for any $k \in N$, either $\beta_k = \alpha_k$ or $\beta_k = \gamma_k$, whence we have that for any $x > 0$, $F_{\alpha_k\beta_k}(x) + F_{\beta_k\gamma_k}(x) - 1$ is equal to either $F_{\beta_k\gamma_k}(x)$ or $F_{\alpha_k\beta_k}(x)$. An appeal to Definition 1.1 then yields that $T_m(F^*_{\alpha\beta}(x), F^*_{\beta\gamma}(x)) = F^*_{\alpha\gamma}(x)$, i.e., $\Pi_{T_m}(F^*_{\alpha\beta}, F^*_{\beta\gamma}) = F^*_{\alpha\gamma}$.

COROLLARY 3.1. *For each PM space, $(S, \mathscr{F}, \Pi_{T_m})$, there exists a convex extension $(S^*, \mathscr{F}^*, \Pi_{T_m})$.*

An analogous result also holds for τ-products (that is again subject to the defect that the infinite τ-products involved may diverge).

THEOREM 3.2. *Let $(S, \mathscr{F}, \tau)$ be a PM space with τ continuous. Then there exists a pair of mappings (f, g) from $(S, \mathscr{F}, \tau)$ into a τ-convex PM space $(S^*, \mathscr{F}^*, \tau)$ such that $f: S \to S^*$ is an injection and $g: \Delta^+ \to \Delta^+$ is a τ-morphism that satisfies $\mathscr{F}^* \circ f \times f = g \circ \mathscr{F}$.*

Proof. Consider the space S^* constructed in Theorem 3.1 endowed with the relative structure of the τ-product $(\prod_{i=1}^\infty S, G, \tau)$. Let $f: S \to S^*$ be the injection defined in the preceding proof; and let $g: \Delta^+ \to \Delta^+$ be given by $g(F) = \tau_{i=1}^\infty F$, for every $F \in \Delta^+$. Clearly the pair (f, g) satisfies the required properties. Let α, β, γ be as in the preceding proof. As above, for any $k \in N$, either $\beta_k = \alpha_k$ or $\beta_k = \gamma_k$, so that, $\tau(F_{\alpha_k\beta_k}, F_{\beta_k\gamma_k}) = F_{\alpha_k\gamma_k}$. Since τ is continuous, we have

$$\begin{aligned}\tau(F^*_{\alpha\beta}, F^*_{\beta\gamma}) &= \tau\Big(\mathop{\tau}_{k=1}^{\infty} F_{\alpha_k\beta_k}, \mathop{\tau}_{k=1}^{\infty} F_{\beta_k\gamma_k}\Big) = \mathop{\tau}_{i=1}^{\infty} \tau(F_{\alpha_k\beta_k}, F_{\beta_k\gamma_k}) \\ &= \mathop{\tau}_{i=1}^{\infty} F_{\alpha_k\gamma_k} = F^*_{\alpha\gamma},\end{aligned}$$

whence $(S^*, \mathscr{F}^*, \tau)$ is τ-convex.

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