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NATURALLY TOTALLY ORDERED SEMIGROUPS

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Naturally totally ordered semigroups which are o-Archimedean are completely characterized by Hölder and Clifford. The purpose of this note is to provide a complete characterization and thus a construction of a class of one-sided naturally totally ordered semigroups, which are not o-Archimedean.

A totally ordered (t.o.) semigroup S is positively ordered if $ab \geqq a$ and $ab \geqq b$ for all a, b in S. A positively totally ordered semigroup S is said to be right (left) naturally totally ordered (n.t.o.) if $a < b$ implies that $b = ax$ $(b = ya)$ for some $x, y \in S$. A right and left n.t.o. semigroup is a n.t.o. semigroup, as defined in [1; 154]. Clifford and Hölder completely characterized order Archimedean (o-Archimedean) n.t.o. semigroups. In [4] the study of one-sided n.t.o. semigroups has been initiated. It is observed that right n.t.o. semigroups which are o-Archimedean are n.t.o. Hence their structure is known. In this paper we provide a construction of a class of right n.t.o. semigroups which are not o-Archimedean. This is obtained by using the ideas in the proof of Tamura's theorem [3; 55]. We refer the reader to [1] for all the undefined terms mentioned in this paper. An element x in S is called o-Archimedean if, for every $y \in S$, there exists an integer n such that $x^n \geqq y$. An ideal P in S is called completely prime if, for all a, b in S, $ab \in P$, implies that either $a \in P$ or $b \in P$. An ideal Q in S is called prime if, for all right (two-sided) ideals A and B in S, $AB \subseteqq P$ implies that either $A \subseteqq P$ or $B \subseteqq P$. Clearly every completely prime ideal is prime. We denote the intersection of all prime ideals by P^*. An element in a semigroup S is said to be central if it commutes with every element in S.

LEMMA 1. *Let every right ideal in a semigroup S contain an ideal. If $a \in P^*$, then for every $x \in S$, there exists a positive integer m such that $a^m \in xS$.*

Proof. Suppose no power of a is in xS. Since xS contains an ideal, say T, no power of a is in T. Then the collection P of all ideals containing T and not containing any power of a is nonempty. So by Zorn's lemma, there exists an ideal P, which is a maximal element in P. Now we claim that P is a prime ideal, which establishes the conclusion. Suppose there exist ideals B and C such that $BC \subseteq P$, without B and C being contained in P. By maximality of P, there exist natural numbers m and n such that $a^m \in P \cup B$ and $a^n \in P \cup C$, so that $a^{m+n} \in P$, which is not true.

LEMMA 2. *Let S be a semigroup in which every right ideal contains an ideal. Suppose P^* contains a central cancellable element a such that $\bigcap_{n=1}^{\infty} a^n S = \Box$ and $a \notin aS$. If N is the additive semigroup of nonnegative integers; G is a group and $I: G \times G \rightarrow N$ be a function satisfying:*

(i) $I(\alpha, \beta) + I(\alpha\beta, \gamma) = I(\alpha, \beta\gamma) + I(\beta, \gamma)$, *where α, β, γ are the elements in G.*

(ii) $I(\varepsilon, \varepsilon) = 1$, *where ε is the identity of G (these two conditions also imply $I(\alpha, \varepsilon) = I(\varepsilon, \alpha) = 1$ for every $\alpha \in G$) then S isomorphic with the semigroup $N \times G$, where the multiplication in $N \times G$ being defined by:*

$$(m, \alpha)(n, \beta) = (m + n + I(\alpha, \beta), \alpha\beta) .$$

Proof. Essentially the proof is similar to Tamura's proof [3; 56] except for some minor changes. We sketch now the important facts. Set $T_n = a^n S \backslash a^{n+1} S$, $n = 1, 2, \cdots$ and $T_0 = S \backslash aS$. Since a is a central element, every element in S can be written uniquely as $a^n z$, $z \notin aS$. Define a relation σ on S by: $x \sigma y$ iff $a^m x = a^n y$ for some $m, n \geqq 0$. Since a is central, σ is a congruence. Denote the classes of σ by S_α, where $\alpha \in G$. G is a semigroup by defining that the product $\alpha\beta$ in G is the index of the class containing xy, where $x \in S_\alpha$ and $y \in S_\beta$. All positive powers of a constitute a single σ-class. For, if $x \sigma a$ and if x is not a power of a, then $x = a^m z$, $z \notin aS$. Now $a^p(a^m z) = a^q(a)$ for some $p, q \geqq 0$. If $p + m \leqq q$, then $z = a^{q-p-m+1}$ since a is cancellable and so x is a power of a, which is not true. If $p + m > q$, then $a \in aS$, which is again not true. Denote the class containing of all positive powers of a by S_ε. Then ε is the identity of G. We observe that G is a group. For, let $\alpha \in G$. If $x \in S_\alpha$, they by Lemma 1, $a^m = xy$ for some natural number m and for some y in S. So $a \sigma xy$ and thus if S_β is the σ-class containing y, then $S_\varepsilon = S_\alpha S_\beta$ and $\varepsilon = \alpha\beta$. By the same reason $\varepsilon = \beta\gamma$ for some $\gamma \in S$. Thus $\varepsilon = \alpha\beta = \beta\alpha$ and hence G is a group. The rest of the proof is similar to that given in the above reference.

The following lemma is of independent interest. This proves the converse of Lemma 2 under some additional hypotheses. The two lemmas do provide structure theorem when the semigroup contains proper prime ideals, which is not the case with Tamura's theorem and cancellative condition is obtained as a consequence. This supplements Nordhal's result [2].

LEMMA 3. *Let $T = N \times G$ be the semigroup as described in Lemma 2, and let I satisfy the additional property that there exists an $\alpha \in G$ such that $I(\alpha^n, \alpha) = 0$ for all n. Then every right ideal in T contains an ideal and P^*, which is the intersection of all prime ideals in T, contains a central cancellable element a such that $\bigcap_{n=1}^{\infty} a^n T = \square$ and $a \notin aT$.*

Proof. As noted in [3; 55] the multiplication defined in Lemma 2 is associative. By direct verification one can show that $(0, \varepsilon)$ is a central cancellable element; $\bigcap_{n=1}^{\infty} (0, \varepsilon)^n T = \square$ and $(0, \varepsilon) \in (0, \varepsilon)T$. For the rest of the proof we need the following result:

$(*)$ for any $(n, \beta) \in N \times G$, there exists an m such that $(0, \varepsilon)^m \in (n, \beta)T$.

This is true because we can choose positive integers m and k such that $m > n + I(\beta, \beta^{-1})$ and $k = m - 1 - n - I(\beta, \beta^{-1})$ and hence $(0, \varepsilon)^m = (n, \beta)(k, \beta^{-1})$.

Now let A be a right ideal. By $(*)$, $(0, \varepsilon)^m = (m - 1, \varepsilon) \in A$ for some positive integer m. Clearly $(0, \varepsilon)^m = (m - 1, \varepsilon)$ is central. Thus A contains a two-sided ideal $T(m - 1, \varepsilon)T$.

We claim now that T has proper prime ideals. By hypothesis there exists an α such that $I(\alpha^n, \alpha) = 0$ for all n. It can be verified easily $(0, \alpha)^n \notin (1, \alpha)T$ for every n. But $(1, \alpha)T$ contains an ideal Q from above. So $(0, \alpha)^n \notin Q$ for every n. Then by Zorn's lemma the nonempty collection of ideals containing Q and not containing any power of $(0, \alpha)$ has a maximal element P. We prove now that P is a prime ideal. For, let A and B be ideals such that $AB \subseteq P$, $A \not\subseteq P$, and $B \not\subseteq P$. Then by maximality of P there exist integers m and n such that $(0, \alpha)^m \in P \cup A$ and $(0, \alpha)^n \in P \cup B$, so that $(0, \alpha)^{m+n} \in P$, which is not true.

Finally we shall show that $(0, \varepsilon)$ belongs to every prime ideal. Let P be a prime ideal and $x \in P$. By $(*)$ there exists an integer m such that $(0, \varepsilon)^m \in xS \subseteq P$. Since $(0, \varepsilon)$ is central, $(0, \varepsilon)^{m-1}S^1(0, \varepsilon)S^1 \subseteq (0, \varepsilon)^m S^1 \subseteq P$. So $(0, \varepsilon)^{m-1} \in P$. Continuing in this manner we have $(0, \varepsilon) \in P$.

LEMMA 4. *Let S be a positively* t.o. *semigroup. Then the set A of all o-Archimedean elements is a completely prime ideal. If S is right* n.t.o., *then A is included in every completely prime ideal if $A \neq S$.*

Proof. It is easy to check that A is an ideal. To prove that A is a completely prime ideal, assume $ab \in A$ with a and $b \notin A$. We may set $a \leqq b$. Then $ab \leqq b^2$ and $b^m < z$ for some z and for every

m. This implies $(ab)^n \leqq b^{2n} < z$ for every n, which is not true. Suppose now that P is a completely prime ideal in a right n.t.o. semigroup S. Let $x \in A$ and $y \in P$. Then there exists an n such that $x^n \geqq y$. If $y = x^n$, $x \in P$ since P is completely prime. If $y < x^n$, then $x^n \in yS \subseteqq P$, so that $x \in P$, as before.

LEMMA 5. *Let S be a right* n.t.o. *semigroup. Then every right ideal is an ideal and every prime ideal is completely prime.*

Proof. Let A be a right ideal. If $a \in A$ and $s \in S$, then $a \leqq sa$ by positive order. If $a = sa$, then $sa \in A$. If $a < sa$, by right n.t.o. condition, for some t we have $sa = at$ and so $sa \in A$. Thus A is an ideal. Suppose now that P is a prime ideal and $ab \in P$. Then from the above,

$$(aS^1)(bS^1) \subseteqq abS^1 \cup abS \cup aSbS \subseteqq abS^1 \cup abS \cup abS^2 \subseteqq P ,$$

which implies a or b belongs to P.

THEOREM. *Let N be the set of all nonnegative integers and G be a group. Suppose I is a function from $G \times G \rightarrow \{0, 1\}$ satisfying the following properties*:

(i) $I(\alpha, \beta) + I(\alpha\beta, \gamma) = I(\alpha, \beta\gamma) + I(\beta, \gamma)$ *for every* α, β, γ *in* G.

(ii) $I(\varepsilon, \varepsilon) = 1$, ε *being the identity of* G.

(iii) *There exists a nonperiodic element* $\alpha \in G$ *such that for all but a finite number of* n, $I(\alpha^n, \alpha) = 0$.

(iv) $I(\alpha, \alpha^{-1}) = 0$ *for every* $\alpha \neq \varepsilon$ *in* G.

(v) *For* $\alpha \neq \beta$, $\alpha \neq \varepsilon$, $\beta \neq \varepsilon$ *one of* $I(\beta, \beta^{-1}\alpha)$ *and* $I(\alpha, \alpha^{-1}\beta)$ *is zero and the other is* 1.

(vi) *If* $(\beta, \beta^{-1}\alpha) = 0$, *then for every* γ *we have only the following possibilities*:

(a) $I(\beta, \gamma) = 0$ *and* $I(\alpha, \gamma) = 1$

(b) $I(\beta, \gamma) = I(\alpha, \gamma) = I(\beta\gamma, (\beta\gamma)^{-1}\alpha\gamma) = 0$

(c) $I(\beta, \gamma) = 1 = I(\alpha, \gamma)$ *and* $I(\beta\gamma, (\beta\gamma)^{-1}\alpha\gamma) = 0$.

(vii) *If* $I(\beta, \beta^{-1}\alpha) = 0$, *then for every* γ *we have only the following possibilities*:

(a) $I(\gamma, \beta) = 0$ *and* $I(\gamma, \alpha) = 1$

(b) $I(\gamma, \beta) = I(\gamma, \alpha) = I(\gamma\beta, (\gamma\beta)^{-1}\gamma\alpha) = 0$

(c) $I(\gamma, \beta) = 1 = I(\gamma, \alpha)$ *and* $I(\gamma\beta, (\gamma\beta)^{-1}\gamma\alpha) = 0$.

(viii) *If* $I(\beta, \gamma) = 1$ *and* $I(\alpha, \gamma) = 0$, *then* $I(\beta\gamma, (\beta\gamma)^{-1}\alpha\gamma) = 0$.

(ix) *If* $I(\gamma, \beta) = 1$ *and* $I(\gamma, \alpha) = 0$, *then* $I(\gamma\beta, (\gamma\beta)^{-1}\gamma\alpha) = 0$.

Define multiplication in $N \times G$ *by*:

$$(m, \alpha)(n, \beta) = (m + n + I(\alpha, \beta), \alpha\beta) .$$

Define order in $N \times G$ *by: if* $m \geqq 1$, $(m, \alpha) > (0, \beta)$; $(0, \varepsilon) > (0, \beta)$ *if* $\beta \neq \varepsilon$; *for* $\alpha \neq \beta$, $\alpha \neq \varepsilon$, $\beta \neq \varepsilon$, $(0, \alpha) > (0, \beta)$ *if* $I(\beta, \beta^{-1}\alpha) = 0$; *if* $m > n$, $(m, \alpha) > (n, \beta)$; $(m, \alpha) > (m, \beta)$ *if* $I(\beta, \beta^{-1}\alpha) = 0$. *Then* $N \times G$ *is a right* n.t.o. *semigroup not containing* 1 *and is non-o-Archimedean but contains a central cancellable o-Archimedean element. Conversely every such semigroup can be constructed in this way.*

Proof. By direct verification one can establish that $N \times G$ is a right n.t.o. semigroup not containing 1. By (iii), there exists an α such that all but a finite number, say m of $I(\alpha^i, \alpha)$ are zero. Then for any n $(0, \alpha)^n = (t, \alpha^n)$ and $t < m + 1$ since α is not periodic and hence $(0, \alpha)^n < (m + 1, \varepsilon)$. Therefore $N \times G$ is not o-Archimedean. It is easy to verify that $(0, \varepsilon)$ is a central o-Archimedean element.

Conversely let S be a right n.t.o. semigroup containing a central cancellable o-Archimedean element a. Then by Lemmas 4 and 5, P^* contains a. $a \notin aS$, since otherwise we have for some s, $a = as = as^2$, so that $s = s^2$ by cancellative condition of a. Now if $y \in S$, $ya = yas = ysa$, which implies $y = ys$. But $(as)y = ay$. So $(sy)a = ya$ and $sy = y$. Thus s is an identity, which is a contradiction. Hence $a \notin aS$. Also $\bigcap_{n=1}^{\infty} a^n S = \square$. For if $y \in \bigcap_{n=1}^{\infty} a^n S$, $y = a^n s_n$ and so $y \geqq a^n$. Since a is o-Archimedean, we have $a^n = y$. Now $a^n \in \bigcap_{n=1}^{\infty} a^n S$ implies $a^n = a^{n+1}$ by positive order and hence $a = a^2$ by cancellation, which is again untrue as before. Combining Lemmas 2, 4, and 5, we have that S is isomorphic with $N \times G$, with the I-function satifying the first two properties stated in the theorem. If $\alpha \neq \varepsilon$ and if $(0, \alpha)$ is in $(0, \varepsilon)S$, then we must have $(0, \alpha) = (0, \varepsilon)(0, \alpha)$, which implies $I(\varepsilon, \alpha) = 0$, which contradicts the consequence of the properties (i) and (ii), as noted in Lemma 2. Since S is right n.t.o. we must have then $(0, \varepsilon)$ is in $(0, \alpha)S$, which implies $(0, \varepsilon) = (0, \alpha)$ $(0, \alpha^{-1})$ and so $I(\alpha, \alpha^{-1}) = 0$. Now set $\gamma = \beta^{-1}$ in (i). Then if $\beta \neq \varepsilon$,

$$I(\alpha, \beta) + I(\alpha\beta, \beta^{-1}) = I(\alpha, \varepsilon) + I(\beta, \beta^{-1}) = 1 + 0 .$$

Thus $I(\alpha, \beta) = 0$ or 1 and hence $I(\alpha, \beta) = 0$ or 1 for every $\beta \in G$. Therefore I maps $G \times G$ into $\{0, 1\}$.

To prove (iii), observe that S contains non-o-Archimedian elements. Therefore there exist (m, α) and (n, β) in $N \times G$ such that $(m, \alpha)^r < (n, \beta)$ for every natural number r. If $m \neq 0$ and $n \neq 0$, then there exists an k such that $km > n$. Then

$$(m, \alpha)^{k+1} = ((k + 1)m + I(\alpha, \alpha) + \cdots + I(\alpha^{k+1}, \alpha), \alpha^{k+1})$$

and hence $(m, \alpha)^{k+1} > (n, \beta)$, which is not true. If $m = 0$ and $n \neq 0$, then $(0, \alpha)^r < (n, \beta)$ for all r implies $I(\alpha, \alpha) + \cdots + I(\alpha^r, \alpha) \leqq n$ for every natural number r. Therefore all but a finite number of

$I(\alpha^i, \alpha)$ are zero. If $m = n = 0$, then it is clear that $I(\alpha^i, \alpha) = 0$ for every i. The case $m \neq 0$ and $n = 0$ is inadmissible. α is not periodic since otherwise if $\alpha^s = \varepsilon$, then $(0, \alpha)^s = (N, \varepsilon)$ where $N = I(\alpha, \alpha) + \cdots + I(\alpha^s, \alpha)$ and $(0, \alpha)^{ts} = (t(N+1) - 1, \varepsilon)$. Choose k such that $k(N+1) - 1 > n$. Therefore

$$(0, \alpha)^{ks} = (k(N+1) - 1, \varepsilon) > (n, \varepsilon) > (n, \beta)\ ,$$

which is not true. It is routine to check that the remaining conditions are necessary for the admissibility of right n.t.o. structure on $N \times G$.

In commutative case, the I-function satisfies the property $I(\alpha, \beta) = I(\beta, \alpha)$, in addition to the above. The commutative example supporting the hypothesis of the theorem is $N \times G$, where G is an infinite cyclic group generated by x and the I-function is defined by: $I(x^n, x^m) = 0$ if $n, m > 0$; $I(x^n, x^{-m}) = 0$ if $0 < n \leqq m$ and in all other cases the value of I-function is 1.

Since $N \times G$ is cancellative, the right n.t.o. semigroup containing central cancellable o-Archimedean elements, which is not o-Archimedean, is in fact cancellative. By condition (iii) of the theorem, the group G is necessarily nonperiodic and $N \times G$ is not finitely generated since by corollary of [4; 11] cancellative finitely generated right n.t.o. semigroups are infinite cyclic semigroups (possibly adjoined with 1) and hence o-Archimedean. More than this we do not know the structure of G. So the problem "what groups admit the I-function with the prescribed properties in the theorem" remains open. It may be remarked that a particular case of this theorem has been solved by Etterbeck in his thesis, written at the University of California, Davis.

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