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THE FOURIER STIELTJES ALGEBRA OF A TOPOLOGICAL SEMIGROUP WITH INVOLUTION

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Let S be a topological semigroup with a continuous involution. We study a subalgebra $F(S)$ of the algebra of continuous weakly almost periodic functions on S. $F(S)$ is translation invariant, closed under conjugation and contains constants. When S has an identity, then $F(S)$ is the linear span of the cone of continuous positive definite functions on S. We show that there exists a norm $\|\cdot\|_{\Omega}$ on $F(S)$ such that $(F(S), \|\cdot\|_{\Omega})$ is a commutative Banach algebra which can be identified with the predual of a W^*-algebra $W^*(S)$. When S is a locally compact group, then $F(S)$ is precisely the Fourier Stieltjes algebra of S. We also show that $\sigma(F(S))$, the spectrum of $F(S)$, is a *-semigroup in $W^*(S)$, and study the relation of $\sigma(F(S_1))$ and $\sigma(F(S_2))$ when $F(S_1)$ and $F(S_2)$ are isometric isomorphic Banach algebras.

1. Introduction. Recently, Dunkl and Ramirez [5] defined a subalgebra $R(S)$ of the algebra $WAP(S)$ of complex-valued continuous weakly almost periodic functions on S. The algebra $R(S)$, called the *representation algebra* of S, is constructed by considering continuous representations of S into the unit ball of $L_\infty(X, \mu)$ with the weak*-topology, where (X, μ) is some probability measure space. They showed that $R(S)$ is translation invariant, closed under conjugation and contains all bounded continuous semi-characters on S. Furthermore $R(S)$, with an appropriate norm, becomes a commutative Banach algebra and the dual of $R(S)$ can be identified with a weak*-closed subalgebra of a commutative W^*-algebra. If G is a commutative locally compact group, then $R(G) = M(\hat{G})\hat{}$, the Fourier Stieltjes transform of the measure algebra on the dual group $\hat{G}$ (see [6, p. 80]).

Our present work deals with the study of the subalgebra $F(S)$ of $WAP(S)$ of a topological *-semigroup S (i.e., a topological semigroup with a continuous involution). If S has an identity, then $F(S)$ is the linear span of continuous positive definite function on S. Also if S is a commutative, then $F(S)$ is contained in the representation algebra $R(S)$. We show that $F(S)$ can be identified with the predual of a W^*-algebra, $W^*(S)$. Furthermore $F(S)$ with the predual norm is a commutative Banach algebra, called the Fourier Stieltjes algebra of S. The algebra $F(S)$ is also translation invariant, closed under conjugation and contains all continuous *-semi-characters of

S. Also there exists an ultra-weakly continuous *-representation of S into the unit ball of $W^*(S)$ "containing" all other ultra-weakly continuous *-representations of S into the unit ball of a W^*-algebra. In particular, if G is a locally compact group (with involution $g \to g^{-1}$), then $W^*(G)$ is the big group algebra defined by John Ernest [9] (see also [8]). Furthermore, if S is commutative and has an identity, then $F(S)$ is isometric and algebra isomorphic to a weak*-dense subalgebra of the measure algebra of a compact topological commutative semigroup.

This paper is organized in the following way: In §2 we list some notations and preliminary properties of topological *-semigroups S; definitions and properties of $F(S)$ and $W^*(S)$ as stated in the previous paragraph will be made precise in §3 and 4. Analysis of the spectrum $\sigma(F(S))$ of $F(S)$ is taken up in §5. We show that $\sigma(F(S))$ is a *-semigroup in $W^*(S)$ and study the relation of $\sigma(F(S_1))$ and $\sigma(F(S_2))$ when S_1, S_2 are topological *-semigroups, and $F(S_1)$ and $F(S_2)$ are isometric isomorphic Banach algebras.

Continuous positive definite functions on topological *-semigroups S have been studied by R. J. Lindahl and P. H. Maserick [15], and more recently by C. Berg and J. Christensen [3] for commutative S with involution on S given by the identity map. Our analysis of the spectrum of $F(S)$ is inspired and motivated by the work of Martin E. Walter in [18] and [19].

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2. Preliminaries and some notations. Let A be a subset of a linear space E, then $\langle A \rangle$ will denote the linear span of A. If E is also a normed linear space, then the closure of A and the closed linear span of A will be denoted by $\bar{A}$ and $\langle A \rangle^-$ respectively if the closure is taken with respect to the norm topology, or by $\bar{A}^\tau$ and $\langle A \rangle^{-\tau}$ respectively if the closure is taken with respect to a topology τ on E different from the norm topology.

The continuous dual of a normed linear space E will be denoted by E^*. If $x \in E$ and $\phi \in E^*$, then the value of ϕ at x will be denoted by $\phi(x)$ or $\langle \phi, x \rangle$. Also if $F \subseteq E^*$, then $\sigma(E, F)$ will denote the locally convex topology on E determined by the semi-norms $\{p_\phi; \phi \in F\}$, where $p_\phi(x) = |\phi(x)|$ for all $x \in E$.

If M is a W^*-algebra, then M_* will denote its unique predual. For each $x \in M$, and $\phi \in M_*$, write $L_x\phi$, $R_x\phi$ and ϕ^* as the functionals in M_* defined by $L_x\phi(y) = \phi(xy)$, $R_x\phi(y) = \phi(yx)$ and $\phi^*(y) = \overline{\phi(y^*)}$ for each $y \in M$. Also the ultraweak topology on M (i.e., the $\sigma(M, M_*)$-topology) will often be written as the σ-topology.

By a *topological semigroup* S, we shall mean a semigroup S with a Hausdorff topology such that for each $a \in S$, the mappings $s \to as$ and $s \to sa$ from S into S are continuous.

Let S be a topological semigroup and let $C(S)$ be the space of bounded continuous complex-valued functions on S. For each $a \in S$, define the left and right translation operators ℓ_a, r_a on $C(S)$ by:

$$(\ell_a f)(s) = f(as)$$

$$(r_a f)(s) = f(sa)$$

for each $s \in S$. A function $f \in C(S)$ is *weakly almost periodic* if $\{\ell_a f; a \in S\}$ is relatively compact in the weak topology of $C(S)$. Then, as known, the space $WAP(S)$ of continuous weakly almost periodic functions on S is a translation invariant closed subalgebra of $C(S)$ containing constants.

By an *involution* on a topological semigroup S we shall mean a map from S into S, denoted by $s \to s^*$, such that

(1) $$(ab)^* = b^*a^*$$

(2) $$a^{**} = a$$

for all $a \in S$. A topological $*$-semigroup is a topological semigroup with a fixd continuous involution.

REMARK 2.1. (a) Not all topological semigroups admit an involution (see [15, p. 771]).

(b) If S is commutative, then the identity map on S defines an involution on S.

(c) If S has an identity u, then $u^* = u$.

(d) If M is a W^*-algebra, then the unit ball of M with the σ-topology is a compact topological $*$-semigroup with the multiplication and involution of M.

If S is a topological $*$-semigroup, $f \in C(S)$, define $f^* \in C(S)$ by $f^*(s) = \overline{f(s^*)}$ for all $s \in S$. Then the map $f \to f^*$ defines an involution on the Banach algebra $C(S)$.

A complex-valued function f on a topological $*$-semigroup S is called *positive definite* if for any complex numbers $\lambda_1, \cdots, \lambda_n$ and any $s_1, \cdots, s_n$ in S, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \bar{\lambda}_i \lambda_j f(s_i^* s_j) \geqq 0 .$$

The collection of continuous positive definite functions on S will be denoted by $P(S)$. The next proposition can be proved by an argument similar to that in [12, 32.9] (see also [15, Theorem 3.4]).

PROPOSITION 2.2. *Let S be a topological *-semigroup. Then $P(S)$ is a cone in $C(S)$ closed under conjugation, involution and pointwise product.*

When G is a group, then the involution on G, unless otherwise specified, will be the one defined by the inversion map $g \to g^{-1}$, $g \in G$.

3. **The Fourier Stieltjes algebra $F(S)$.** Throughout this section, S will denote a topological *-semigroup.

By a representation of S we shall mean a pair (ω, M), where M is a W^*-algebra and ω is a homomorphism of S into $M_1 = \{x \in M; ||x|| \leqq 1\}$ regarded as a semigroup with multiplication from M i.e., $\omega(ab) = \omega(a)\omega(b)$ for all $a, b \in S$. The representation (ω, M) is a *-representation if $\omega(a^*) = \omega(a)^*$ for all $a \in S$; it is σ-continuous if ω is continuous when M_1 has the σ-topology.

REMARK 3.1. If S has an identity u, and (ω, M) is a *-representation of S, then $\omega(u) = p$ is a projection in M, and $\omega(S)$ is contained in the W^*-algebra pMp for which p is the identity. Also if $\langle\omega(S)\rangle^{-\sigma} = M$, then $\omega(u)$ is the identity of M.

If (ω, M) is a σ-continuous *-representation of S such that $\langle\omega(S)\rangle$ is σ-dense in M, then $\text{card}(M_*) \leqq c^{\text{card}(S)}$, where c is the cardinality of the real numbers. Hence we may form the collection $\Omega(S)$ of all σ-continuous *-representations $\alpha = (\omega, M)$ of S such that $\langle\omega(S)\rangle^{\sigma} = M$. Let $F(S)$ denote all complex-valued functions f such that $f = \hat{\psi}$ for some $\omega \in M_*$ and some $\alpha = (\omega, M)$ in $\Omega(S)$. For each $f \in F(S)$, let

$$||f||_\infty = \sup\{|f(s)|; s \in S\}$$
$$||f||_\Omega = \inf\{||\psi||; \psi \in M_*, \hat{\psi} = f \text{ and } (\omega, M) \in \Omega(S)\}\ .$$

THEOREM 3.2. (a) *$F(S)$ is a subalgebra of $WAP(S)$ containing the constant functions. Furthermore, $||\cdot||_\Omega$ is a norm on $F(S)$ and $(F(S), ||\cdot||_\Omega)$ is a commutative normed algebra with unit.*

(b) *If $f \in F(S)$ and $a \in S$, then the functions $r_a f$, $\ell_a f$, f^*, $\bar{f}$ are all in $F(S)$ and $||r_a f||_\Omega \leqq ||f||_\Omega$, $||\ell_a f||_\Omega \leqq ||f||_\Omega$, $||f^*||_\Omega = ||f||_\Omega$, $||\bar{f}||_\Omega = ||f||_\Omega$ and $||f||_\infty \leqq ||f||_\Omega$.*

Proof. That $F(S) \subseteq WAP(S)$ follows from [14, Lemma 6.3]. The remainder of the theorem can be proved quite similarly to [6, Theorem 2.1.6], we omit the details.

We shall call $(F(S), ||\cdot||_\Omega)$ the *Fourier Stieltjes algebra* of S.

REMARK 3.3. (a) The algebra $F(S)$ cannot be enlarged and the norm on $F(S)$ cannot be decreased by considering a collection $\mathscr{C}$ of σ-continuous *-representations of S containing $\Omega(S)$. Indeed, if $f \in F(S)$ and $\alpha = (\omega, M) \in \mathscr{C}$ such that $f = \hat{\psi}$ for some $\psi \in M_*$, let $N = \langle \omega(S) \rangle^{-\sigma}$ and ψ_0 be the restriction of ψ to N. Then $(\omega, N) \in \Omega(S)$, $\hat{\psi}_0 = f$ and $||\psi_0|| \leqq ||\psi||$.

(b) If S is commutative, then $F(S) \subseteqq R(S)$, where $R(S)$ is the representation algebra of S defined by Dunkl and Ramirez [5]. To see this, let $f \in F(S)$. Choose $(\omega, M) \in \Omega(S)$ such that $\hat{\phi} = f$ for some $\phi \in M_*$. Let X be the spectrum of M. Then $\operatorname{card}(X) \leqq c^{\operatorname{card}(S)}$. By the Riesz representation theorem, there exists a probability measure μ_ϕ on X such that $\phi(a) = \int_X a(t)d\mu_\phi(t)$ for each $a \in M$. Consider the mapping Φ_ϕ from M into $L_\infty(X, \mu_\phi)$ defined by $\Phi_\phi(a) = \hat{a}$, where $\hat{a}$ is the Gelfand transform of a. Then Φ_ϕ is a W^*-homomorphism of M into $L_\infty(X, \mu_\phi)$ (see [16, p. 46]). Define a presentation $(\omega_\phi, L_\infty(X, \mu_\phi))$ of S by $\omega_\phi(S) = \Phi_\phi(\omega(s))$. Then $\hat{\phi}(s) = \langle 1, \omega_\phi(s) \rangle$ for all $s \in S$. Hence $f = \hat{\phi} \in R(S)$.

Note that the inclusion $F(S) \subseteqq R(S)$ may be proper (see Example 4.2).

(c) If S is an idempotent commutative topological semigroup with involution $s^* = s$ for all $s \in S$, then any representation (ω, M) of S, where M is a commutative W^*-algebra, is a *-representation. In particular $F(S) = R(S)$. Indeed, we may assume that $M = L_\infty(X, \mu)$ for some measure space (X, μ). Since $\omega(s)^2 = \omega(s^2) = \omega(s)$ for all $s \in S$, it follows that $\omega(s)$ is a characteristic function on some subset of Ω. Hence $\omega(s)^* = \omega(s^*)$.

(d) Let G be an abelian group. Then for any representation (μ, M) where M is a commutative W^*-algebra, is a *-representation of G. Consequently $F(G) = R(G)$. Indeed, write $M = L_\infty(X, \mu)$ for some measure space (X, μ). We may assume that $\omega(u) = 1$, where u is the identity of G. Then for each $g \in G$, $\omega(g)\omega(g^{-1}) = \omega(u) = 1$. Hence $|\omega(g)| = 1$ and $\omega(g^{-1}) = \overline{\omega(g)} = \omega(g)^*$.

(e) If S is the unit ball of a W^*-algebra M, then the restriction map is a linear isometry from M_* into $F(S)$.

(f) A function $\chi: S \to C$ is called a *semi-character* if $|\chi(s)| \leqq 1$ and $\chi(s \cdot t) = \chi(s)\chi(t)$ for all $s, t \in S$. A continuous semi-character χ is in $F(S)$ if and only if $\chi(s^*) = \overline{\chi(s)}$ for all $s \in S$. In this case $\chi \in P(S)$ and $||\chi||_\Omega = 1$ whenever χ is nonzero, (see [6, Remark 2.1.8]).

The next proposition follows easily from [15, Theorem 3.2] and Remark 3.3(a):

PROPOSITION 3.4. *If S has an identity, then* $F(S) = \langle P(S) \rangle$.

REMARK 3.5. (a) Let S_u denote the semigroup formed by adjoining to S an identity u. Equip S_u with the topology η that a subset $0 \subseteqq S_u$ is in η if and only if $0 \cap S$ is open in S. Then (S_u, η) is a topological semigroup. Also the involution on S can be extended to an involution on S_u by defining $u^* = u$. Let r denote the restriction map from $F(S_u)$ into $F(S)$. Then r is norm decreasing, onto and $r(P(S_u)) = F(S) \cap P(S)$.

(b) The assumption that S has an identity cannot be removed from Proposition 3.4. Indeed, let S be a set with at least two elements. Let $z \in S$ be fixd. Define on S the multiplication $ab = z$ for all $a, b \in S$. Equip S with the discrete topology and involution $a = a^*$ for all $a \in S$. Pick $w \in S$, $w \neq z$. Let f be the characteristic function on the set $\{w\}$. Then $f \in P(S)$, but $f \notin F(S)$. Indeed, there exists no k such that

$$\left|\sum_{i=1}^{n} c_i f(s_i)\right|^2 \leqq k \sum_{i,j=1}^{n} c_i \bar{c}_j f(s_i s_j^*)$$

for any $s_1, \cdots, s_n$ in S and complex numbers $c_1, \cdots, c_n$. Hence by Corollary 1.2 in [15], f is not extendable to a function in $P(S_u)$. By (a), $f \in F(S)$.

4. **The operator algebra $W^*(S)$.** Let S be *a* topological *-semigroup and write $M_\Omega = \sum \oplus M_\alpha$, the direct summand of the W^*-algebras M_α, $\alpha \in \Omega(S)$. (See [16, p. 2].) Define a *-homomorphism of S into M_Ω by: $\omega_\Omega(s)(\alpha) = \omega_\alpha(s)$ for each $\alpha = (\omega_\alpha, M_\alpha)$ in $\Omega(S)$. Then

$$||\omega_\Omega(s)|| = \sup\{||\omega_\alpha(s)||; \alpha \in \Omega(S)\} \leqq 1$$

for each $s \in S$. Also if s_n is a net in S converging to some $s \in S$, then the net $\langle \omega_\Omega(s_n)(\alpha), \psi\rangle = \langle \omega_\alpha(s_n), \psi\rangle$ converges to $\langle \omega_\Omega(s)(\alpha), \psi\rangle$ for each $\alpha \in \Omega(S)$ and $\psi \in (M_\alpha)_*$. Since the σ-topology on M_Ω agrees with the topology determined by the semi-norms $\{P_{\alpha,\psi}; \alpha \in \Omega(S), \psi \in (M_\alpha)_*\}$ on the unit ball, where

$$|P_{\alpha,\psi}(x)| = |\langle x(a), \psi\rangle|$$

for each $x \in M_\Omega$, it follows that $(\omega_\Omega, M_\Omega)$ is a σ-continuous *-representation of S. Write

$$W^*(S) = \langle \omega_\Omega(S)\rangle^{-\sigma} .$$

THEOREM 4.1. *Let S be a topological *-semigroup. Then:*

(a) *The mapping π: $W^*(S)_* \to F(S)$ defined by $\pi(\psi) = \hat{\psi}$, $\psi \in W^*(S)_*$, is a linear isometry from $W^*(S)_*$ onto $F(S)$. Consequently, the normed algebra $F(S)$ is complete. Furthermore, $\pi(\psi)$ is positive definite if and only if ψ is positive.*

(b) *If* (ω, M) *is any* σ*-continuous* $*$*-representation of* S*, then there exists a* W^**-homomorphism* h_ω *from* $W^*(S)$ *into* M *such that the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\omega_\Omega} & W^*(S) \\ & \searrow_{\omega} & \downarrow h_\omega \\ & & M \end{array}$$

is commutative. Also if $\psi \in M_*$*, then* $\langle x, \hat{\psi}\rangle = \langle h_\omega(x), \psi\rangle$ *for all* $x \in W^*(S)$.

Proof of Theorem 4.1 is rather routine. We omit the details.

EXAMPLE 4.2. Let $\boldsymbol{Z}$ be the group of integers with addition and involution $n \to -n$. Then $F(\boldsymbol{Z}) = R(\boldsymbol{Z}) = \langle P(\boldsymbol{Z})\rangle$ (Remark 3.3(d) and Proposition 3.4), and $W^*(\boldsymbol{Z})$ is the commutative W^*-algebra $C(T)^{**}$, where T is circle group (see Remark 4.3(b)).

On the other hand, if $\boldsymbol{Z}$ has involution $n \to n$, then $F(\boldsymbol{Z}) = C^2$, $W^*(\boldsymbol{Z}) = C^2$ and hence $F(\boldsymbol{Z})$ is a proper subset of $R(\boldsymbol{Z})$ (see Remark 3.3(a)). To see this, consider any $(\omega, M) \in \Omega(\boldsymbol{Z})$. Then M is a commutative W^*-algebra. Hence $\omega(n) = \overline{\omega(n)} = \omega(-n)$ by Remark 3.3 (d). Consequently $\omega(n)^2 = \omega(0) = 1$ for all $n \in \boldsymbol{Z}$ and $\omega(\boldsymbol{Z})$ has at most two elements. However, if M is the subalgebra of $L_\infty[0, 1]$ generated by the functions 1, h, where

$$h(t) = \begin{cases} 1 & \text{if } 0 \leqq t < \dfrac{1}{2} \\ -1 & \text{if } \dfrac{1}{2} \leqq t \leqq 1 \end{cases}$$

then $\omega(0) = 1$ and $\omega(1) = \omega(-1) = h$ defines a representation of $\boldsymbol{Z}$ in $\Omega(\boldsymbol{Z})$, and $\langle \omega(\boldsymbol{Z})\rangle = M$, which is two-dimensional. Hence $W^*(\boldsymbol{Z}) = C^2$ and $F(\boldsymbol{Z}) = C^2$.

REMARK 4.3. (a) Let S be a locally compact topological $*$-semigroup. Let $M(S)$ be the Banach algebra of complex, finite, regular Borel measures on S with multiplication of two elements μ, ν in $M(S)$ defined by

$$\int f d\mu * \nu = \iint f(st)d\mu(s)d\nu(t)$$

for all $f \in C_0(S)$, and total variation norm, where $C_0(S)$ is the space of all functions $f \in C(S)$ vanishing at infinity (see [11]). For each $\mu \in M(S)$, define $\mu^* \in M(S)$ to be the measure representing the functional $f \to \overline{\int f^*(t)d\mu(t)}$ on $C_0(S)$. Then as observed in [14, Lemma 6.8], the map $\mu \to \mu^*$ is an involution on $M(S)$ with $||\mu|| = ||\mu^*||$ and $(\varepsilon_a)^* = \varepsilon_{a^*}$ for each $a \in S$, where ε_a is the point measure at a.

Define

$$\langle \tilde{\omega}_\Omega(\mu), \phi \rangle = \int \hat{\phi}(t) d\mu(t)$$

for each $\mu \in M(S)$, and $\phi \in W^*(S)_*$. Clearly $\tilde{\omega}_\Omega(\varepsilon_a) = \omega(a)$ for each $a \in S$. Consequently $\tilde{\omega}_\Omega$ defines a norm-decreasing *-homomorphism of $M(S)$ into $W^*(S)$ which is continuous when $M(S)$ has the $\sigma(M(S), F(S))$ topology and $W^*(S)$ has the σ-topology. Furthermore, if β is any norm-decreasing *-homomorphism from $M(S)$ into a W^*-algebra M which is continuous when $M(S)$ has the $\sigma(M(S), F(S))$ topology and M has the σ-topology, then there exists a W^*-homomorphism h from $W^*(S)$ into M such that the diagram

$$\begin{array}{ccc} M(S) & \xrightarrow{\tilde{\omega}_\Omega} & W^*(S) \\ & \searrow^{\beta} & \downarrow h \\ & & M \end{array}$$

is commutative.

(b) Let G be a locally compact group. Let $C^*(G)$ denote the completion of $L_1(G)$ with the norm

$$||h||_c = \sup \{||T_h||\} , \qquad h \in L_1(G) ,$$

where the suprenum is taken over all no-where trivial *-representation T of $L_1(G)$ as an algebra of bounded linear operators on a Hilbert space. Then as well known (see [10, Chapter 2]), $\langle P(G) \rangle$ can be identified with the dual of $C^*(G)$, and $P(G)$ is precisely the positive linear functionals on the C^*-algebra $C^*(G)$. In this case $F(G) = \langle P(G) \rangle$ (see Proposition 3.4) and $||f||_\Omega$ is precisely the norm of f regarded as a linear functional on $C^*(G)$ for each $f \in F(G)$. Furthermore, $W^*(G)$ is isomorphic to the second conjugate algebra of $C^*(G)$ with the Arens product. (See [8, Remark 2.6 and Proposition 2.8].)

(c) Let S be a topological *-semigroup and let $C_\Omega^*(S) = \langle \omega_\Omega(S) \rangle^-$. Then $C_\Omega^*(S)$ is a C^*-subalgebra of $W^*(S)$. For each $f \in F(S)$, $x \in C_\Omega^*(S)$, define $f \cdot x \in W^*(S)$ by

$$\langle f \cdot x, g \rangle = \langle x, f \cdot g \rangle$$

for all $g \in F(S)$. Then $f \cdot x \in C_\Omega^*(S)$. Also if $m \in C_\Omega^*(S)^*$ and $x \in C_\Omega^*(S)$, then the element $m_L(x)$ in $W^*(S)$ defined by

$$\langle m_L(x), f\rangle = \langle m, f\cdot x\rangle$$

for all $f \in F(S)$ is also in $C^*_\Omega(S)$. Hence we may define on $C^*_\Omega(S)^*$ the Arens product [1] by

$$\langle n\cdot m, x\rangle = \langle n, m_L(x)\rangle$$

for any $n, m \in C^*_\Omega(S)$, $x \in C^*_\Omega(S)$. Then $C^*_\Omega(S)^*$ with product and the dual norm is a commutative Banach algebra containing an isometric copy of $F(S)$. Furthermore, multiplication in the unit ball of $C^*_\Omega(S)^*$ is jointly continuous with respect to the weak*-topology. (This follows from our [14, Theorem 6.11] when S has an identity; otherwise just use an argument similar to the proof given there.)

(d) Let S be a topological *-semigroup and let S_d denote S with the discrete topology. For each $m \in C^*_\Omega(S)^*$, define $\hat{m}(s) = m(\omega_\Omega(s))$ for all $s \in S$. Then clearly $\hat{m} \in \langle P(S_d)\rangle$. Since $(\omega_\Omega, M) \in \Omega(S)$, where M is the enveloping W^*-algebra of $C^*_\Omega(S)$, it follows that $\hat{m} \in F(S_d)$ and $||\hat{m}||_\Omega \leqq ||m||$. A simple computation shows that the map $\tilde{\pi}: C^*_\Omega(S)^* \to F(S_d)$ defined by $\tilde{\pi}(m) = \hat{m}$, $m \in C^*_\Omega(S)^*$, is norm-decreasing algebra isomorphism from $C^*_\Omega(S)^*$ into $F(S_d)$. In particular if S is discrete, then $F(S)$ is isometric and algebra isomorphic to $C^*_\Omega(S)^*$.

(e) Let S be a commutative topological *-semigroup with an identity u, and let $\hat{S}_d$ (resp. $\hat{S}$) denote the set of all (resp. continuous) nonzero *-semi-characters on S i.e., semi-characters χ on S such that $\chi(s^*) = \overline{\chi(s)}$ for all $s \in S$. Note that $\chi(u) = 1$ for each $\chi \in \hat{S}_d$. Equip $\hat{S}_d$ with the topology of pointwise convergence. Then $\hat{S}_d$ with pointwise multiplication is a compact togological semigroup. Let $\Delta(S)$ denote the spectrum of the commutative C^*-algebra $C^*_\Omega(S)$ and write $\Delta(S)^\wedge = \tilde{\pi}(\Delta(S))$ where $\tilde{\pi}$ is as defined in (d) above. Then $\Delta(S)^\wedge$ is a compact subsemigroup of $\hat{S}_d$ containing $\hat{S}$. Furthermore, it follows from [14, Theorem 6.12] that there exists a linear isometry and algebra homomorphism from $F(S)$ into a weak*-dense subalgebra of the measure algebra $M(\Delta(S)^\wedge)$. Also if S is a discrete commutative *-semigroup with an identity, then $F(S)$ is isometric and algebra isomorphic to $M(\hat{S})$.

5. The spectrum of $F(S)$. Throughout this section S will denote a topological *-semigroup and $\sigma(F(S))$ will denote the spectrum of $F(S)$ i.e., the collection of all nonzero multiplicative linear functionals on $F(S)$.

Recently Martin E. Walter [18] [19] has given detailed analysis on the spectrum of the Fourier Stieltjes algebra of a locally compact group. In this section, we shall generalize some of Walter's results to the spectrum of $F(S)$. We begin with the following simple observations.

Let $x \in W^*(S)$ and $f \in F(S)$. Define a bounded complex-valued function $x_l(f)$ on S by

$$x_l(f)(s) = \langle x, l_s f \rangle$$

for each $s \in S$. Let $\phi \in W^*(S)_*$ such that $\hat{\phi} = f$. Then $(L_{\omega_\Omega(s)}\phi)\hat{} = l_s f$. Hence

$$\begin{aligned} x_l(f)(s) &= \langle x, L_{\omega_\Omega(s)}\phi \rangle \\ &= \langle R_x\phi, \omega_\Omega(s) \rangle \\ &= (R_x\phi)\hat{}(s) \end{aligned}$$

for all $s \in S$. Consequently $x_l f \in F(S)$ and $||x_l f||_\Omega \leqq ||x|| \, ||f||_\Omega$. Hence if $y \in W^*(S)$, we may define an element $y \circ x$ in $W^*(S)$ by

$$\langle y \circ x, f \rangle = \langle y, x_l(f) \rangle$$

for all $f \in F(S)$.

LEMMA 5.1. *If $x, y \in W^*(S)$, then $y \circ x = y \cdot x$, where $y \cdot x$ denotes the product of y, x in $W^*(S)$. Consequently $(x \cdot y)_l(f) = y_l(x_l(f))$ for each $f \in F(S)$.*

Proof. Let $x \in W^*(S)$ be fixed. The equation $y \circ x = y \cdot x$ clearly holds for all $y = \omega_\Omega(s)$, $s \in S$. Hence it holds for all $y \in \langle \omega_\Omega(S) \rangle$. Now if $y \in W^*(S)$ and y_α is a net in $\langle \omega_\Omega(S) \rangle$ converging to y in the σ-topology, then for each $f \in F(S)$,

$$\begin{aligned} \langle y \circ x, f \rangle &= \langle y, x_l(f) \rangle \\ &= \lim_\alpha \langle y_\alpha, x_l(f) \rangle \\ &= \lim_\alpha \langle y_\alpha \circ x, f \rangle \\ &= \lim_\alpha \langle y_\alpha \cdot x, f \rangle \\ &= \langle y \cdot x, f \rangle \end{aligned}$$

by [16, p. 18]. The final assertions from direct computation.

LEMMA 5.2. (a) *If $x \in \sigma(F(S))$, then x_l is an algebra homomorphism from $F(S)$ into $F(S)$.*

(b) *If S has an identity, and x is a nonzero element in $W^*(S)$ such that x_l is an algebra homomorphism, then $x \in \sigma(F(S))$.*

Proof. (a) If $f, g \in F(S)$, then

$$x_l(f \cdot g)(s) = \langle x, l_s(f \cdot g) \rangle = \langle x, (l_s f)(l_s g) \rangle = \langle x, l_s f \rangle \langle x, l_s g \rangle$$

for all $s \in S$. Hence x_l is an algebra homomorphism.

(b) follows simply by evaluation at the identity.

REMARK 5.3. Note that Lemma 5.2 (b) is false when S does not have an identity as the following example shows. Let $S = \{s_1, s_2, s_3\}$ with multiplication defined by

$$x_i \cdot x_j = \begin{cases} x_i & \text{if} \quad i = j \\ x_1 & \text{if} \quad i \neq j \,. \end{cases}$$

Then S is commutative and has no identity. Let S have the discrete topology and involution defined by the identity map. Then $F(S)$ separate points. In fact, let $M = L_\infty[0, 1]$ and define a *-representation ω of S into M by $\omega(s_1) = 0$, $\omega(s_2) = 1_{[0,1/2)}$, and $\omega(s_3) = 1_{[1/2,1]}$. Then $F_\omega(S)$ clearly separate points and contained in $F(S)$. Hence $\sigma(F(S)) = \omega_\Omega(S)$ consists of three distinct points, and the identity e of $W^*(S)$ is not in $\sigma(F(S))$. However e_l, being the identity operator on $F(S)$, is an algebra homomorphism.

PROPOSITION 5.4. *If* $x, y \in \sigma(F(S))$, *then* x^* *and* $x \cdot y$ *are in* $\sigma(F(S))$.

Proof. Let $f, g \in P(S) \cap F(S)$. Then $f \cdot g \in P(S) \cap F(S)$ by Proposition 2.2. Hence if $x \in \sigma(F(S))$, then $x^* \neq 0$ and

$$\langle x^*, f \cdot g \rangle = \overline{\langle x, f \cdot g \rangle} = \overline{\langle x, f \rangle \langle x, g \rangle} = \langle x^*, f \rangle \langle x^*, g \rangle$$

by Theorem 4.1 (b). Since $\langle P(S) \cap F(S) \rangle = F(S)$, it follows that $x^* \in \sigma(F(S))$.

If $x, y \in \sigma(F(S))$ and $f, g \in F(S)$, then

$$\begin{aligned} \langle x \cdot y, f \cdot g \rangle &= \langle x, y_l(f \cdot g) \rangle = \langle x, y_l(f) y_l(g) \rangle = \langle x, y_l(f) \rangle \langle x, y_l(g) \rangle \\ &= \langle x \cdot y, f \rangle \langle x \cdot y, g \rangle \end{aligned}$$

using Lemmas 5.1 and 5.2. To see that $x \cdot y \neq 0$, we observe that if 1 is the constant one function on S, then $\langle x, 1 \rangle = 1$. Hence

$$\langle x \cdot y, 1 \rangle = \langle x, y_l(1) \rangle = \langle x, 1 \rangle = 1$$

using Lemma 5.1 again. Hence $x \cdot y \in \sigma(F(S))$.

If (ω_i, M), $i = 1, 2$, are σ-continuous *-representations of S, let $(\omega_1 \otimes \omega_2, M_1 \overline{\otimes} M_2)$ denote the σ-continuous representation of S into the W^*-tensor product $M_1 \otimes M_2$ by

$$(\omega_1 \otimes \omega_2)(s) = \omega_1(s) \otimes \omega_2(s)$$

for each $s \in S$.

PROPOSITION 5.5. *Let* x *be a nonzero element in* $W^*(S)$. *Then the followings are equivalent:*

(a) $x \in \sigma(F(S))$.

(b) $h_{\omega_1 \otimes \omega_2}(x) = h_{\omega_1}(x) \otimes h_{\omega_2}(x)$ *for any* σ*-continuous* **-representations* (ω_i, M_i), $i = 1, 2$, *of* S.

(c) $h_{\omega_\Omega \otimes \omega_\Omega}(x) = x \otimes x$.

Proof. (a) $\Rightarrow$ (b). Let $\phi_i \in (M_i)_*$. Then $(\phi_1 \otimes \phi_2)\hat{} = \hat{\phi}_1 \cdot \hat{\phi}_2$. Hence

$$\begin{aligned}\langle h_{\omega_1 \otimes \omega_2}(x), \phi_1 \otimes \phi_2 \rangle &= \langle x, (\phi_1 \otimes \phi_2)\hat{}\,\rangle \\ &= \langle x, \hat{\phi}_1 \cdot \hat{\phi}_2 \rangle \\ &= \langle x, \hat{\phi}_1 \rangle \langle x, \hat{\phi}_2 \rangle \\ &= \langle h_{\omega_1}(x), \phi_1 \rangle \langle h_{\omega_2}(x), \phi_2 \rangle \\ &= \langle h_{\omega_1}(x) \otimes h_{\omega_2}(x), \phi_1 \otimes \phi_2 \rangle\end{aligned}$$

using Theorem 4.1 (b). Since $\{\phi_1 \otimes \phi_2, \phi_i \in (M_i)_*\}$ is total in $(M_1 \otimes M_2)_*$, (b) follows.

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a). Let $f_1, f_2 \in F(S)$ and ψ_1, ψ_2 be the unique elements in $W^*(S)_*$ such that $\hat{\psi}_i = f_i$. Then

$$\begin{aligned}\langle f_1, x \rangle \langle f_2, x \rangle &= \langle \psi_1, x \rangle \langle \psi_2, x \rangle \\ &= \langle \psi_1 \otimes \psi_2, x \otimes x \rangle \\ &= \langle \psi_1 \otimes \psi_2, h_{\omega_\Omega \otimes \omega_\Omega}(x) \rangle \\ &= \langle (\psi_1 \otimes \psi_2)\hat{}, x \rangle \qquad \text{(by Theorem 4.1(b))} \\ &= \langle \hat{\psi_1} \cdot \hat{\psi_2}, x \rangle \\ &= \langle f_1 \cdot f_2, x \rangle \,.\end{aligned}$$

Since $x \neq 0$, $x \in \sigma(F(S))$.

REMARK 5.6. (a) Both Propositions 5.4 and 5.5 are due to Martin E. Walter ([18, Theorem 1(ii) and (iii)] and [19, Corollary to Theorem 2]) when S is a locally compact group. Our proof of Proposition 5.4 is completely different from that of Walter. However, using Proposition 5.5 and an argument similar to that in [18, Theorem 1(iii)] we can also obtain a part of Proposition 5.4, i.e., if $x, y \in \sigma(F(S))$, then $x^* \in \sigma(F(S))$ and $x \cdot y \in \sigma(F(S)) \cup \{0\}$.

(b) It follows from Proposition 5.5 that $\sigma(F(S))$ with the σ-topology is a compact topological *-semigroup. Also, $\overline{\omega_\Omega(S)}^\sigma$ is a *-subsemigroup of $\sigma(F(S))$; $\overline{\omega_\Omega(S)}^\sigma$ is precisely the largest *-compactification of S as defined in [15, Theorem 5.1].

PROPOSITION 5.7. *Let* $T = \overline{\omega_\Omega(S)}^\sigma$. *Then there exists a linear isometry and algebra isomorphism* U *from* $F(T)$ *onto* $F(S)$ *such that* U^* *is a* W^**-isomorphism from* $W^*(S)$ *onto* $W^*(T)$.

Proof. Let ω'_Ω denote the *-representation of T into $W^*(T)$ and let $\omega'(s) = \omega'_\Omega(\omega_\Omega(s))$ for all $s \in S$. Then $(\omega', W^*(T))$ is a σ-continuous *-representation of S, and $\langle \omega'(S) \rangle^{-\sigma} = W^*(T)$. Let $h = h_{\omega'}$ (Theorem 4.1). Then h is onto and

$$h(\omega_\Omega(s)) = \omega'_\Omega(\omega_\Omega(s))$$

for all $s \in S$. Consequently $h(t) = \omega'_\Omega(t)$ for all $t \in T$. On the other hand, if $k = h_{\omega_0}$ where ω_0 is the injection map of T into $W^*(S)$, then

$$k(\omega'_\Omega(t)) = t$$

for all $t \in T$. Hence $k(h(t)) = t$ for all $t \in T$. Since $\langle T \rangle^{-\sigma} = W^*(S)$, it follows that $k(h(x)) = x$ for all $x \in W^*(S)$. Consequently h is a W^*-isomorphism. Define $U(\psi\hat{}) = (h^*\psi)\hat{}$ for all $\psi \in W^*(T)_*$. Then $U(f)(s) = f(\omega'(s))$ for all $s \in S$, $f \in F(T)$. Hence U is a linear isometry and algebra homomorphism from $F(T)$ onto $F(S)$, and $U^* = h$ is a W^*-isomorphism from $W^*(S)$ onto $W^*(T)$.

Martin Walter proved in [18] the following beautiful duality theorem: If the Fourier algebras of two locally compact groups G_1 and G_2 are isometric isomorphic, then G_1 and G_2 are topologically isomorphic. This result, as pointed out in [18, p. 18] is equivalent to B. E. Johnson's isomorphism theorem for the measure algebras of the locally compact groups when G_1 and G_2 are abelian. It is easy to see from Proposition 5.7 that Walter's result is no longer valid when G_1, G_2 are topological *-semigroups. However we shall show in the next theorem that if S_1 and S_2 are topological *-semigroups with identity and $F(S_1)$ and $F(S_2)$ are isometric isomorphic, then the compact topological *-semigroups $\sigma(F(S_1))$ and $\sigma(F(S_2))$ are strongly related.

Let $\sigma_u(F(S))$ denote all unitary elements in $\sigma(F(S))$ and let $\sigma_c(F(S))$ denote the centre of the semigroup $\sigma(F(S))$, i.e., all $x \in \sigma(F(S))$ such that $x \cdot y = y \cdot x$ for all $y \in \sigma(F(S))$. Then $\sigma_u(F(S))$ is a group and $\sigma_c(F(S))$ is a closed *-subsemigroup of $\sigma(F(S))$.

THEOREM 5.8. *Let S_1, S_2 be topological *-semigroups with identity. If the Banach algebras $F(S_1)$ and $F(S_2)$ are isometric isomorphic, then there exists a homeomorphism ϕ from $\sigma(F(S_1))$ onto $\sigma(F(S_2))$ such that*

(a) *$\phi(x^*) = \phi(x)^*$ for all $x \in \sigma(F(S_1))$.*

(b) *For each $x, y \in \sigma(F(S_1))$, either $\phi(x \cdot y) = \phi(x)\phi(y)$ or $\phi(x \cdot y) = \phi(y)\phi(x)$.*

(c) *ϕ is a *-isomorphism from $\sigma_c(F(S_1))$ onto $\sigma_c(F(S_2))$.*

(d) *ϕ is either a *-isomorphism or a *-anti-isomorphism from $\sigma_u(F(S_1))$ onto $\sigma_u(F(S_2))$.*

Furthermore, if for each $x \in \sigma(F(S_1))$,

$$H_x = \{y \in \sigma(F(S_1)); \phi(x \cdot y) = \phi(y)\phi(x)\} ;$$
$$K_x = \{y \in \sigma(F(S_1)); \phi(x \cdot y) = \phi(x)\phi(y)\}$$

and if

$$H = \cap \{H_x; x \in \sigma_u(F(S_1))\} ; \qquad K = \cap \{K_x; x \in \sigma_u(F(S_2))\}$$

then

(e) H_x, K_x *are* σ*-closed subsemigroups of* $\sigma(F_1(S))$ *such that* $y \in H_x$ (resp. $y \in K_x$) *if and only if* $y^* \in K_{x^*}$(resp. $y^* \in K_{x^*}$).

(f) H *and* K *are* σ*-closed* **-subsemigroups of* $\sigma(F(S_1))$ *such that* $H \cup K = \sigma(F_1(S))$.

Proof. We follow an idea Martin Walter in the proof of Theorem 2 in [18]. Let U be the isomorphism from $F(S_2)$ onto $F(S_1)$. Since S_1 has an identity, it follows that e_1, the identity of $W^*(S_1)$, is in $\sigma(F(S_1))$. Hence $u = U^*(e_1)$ and $v = u^*$ are in $\sigma(F(S_2))$ (by Proposition 5.4) and v_l is an algebra homomorphism from $F(S_2)$ into $F(S_2)$ (Lemma 5.1) such that $||v_l(f)||_\Omega \leqq ||v|| \, ||f||_\Omega = ||f||_\Omega$ for all $f \in F(S_2)$. On the other hand, since u is unitary [13, Lemma 12], it follows that

$$||v_l(f)|| \leqq ||u_l(v_l(f))|| = ||(v \cdot u)_l(f)|| = ||f||$$

for each $f \in F(S_2)$ by Lemma 5.1, i.e., v is an isometry. Also if $f \in F(S_2)$, then $v_l(u_l(f)) = f$. Hence v_l is onto. Consequently $U \circ v_l$ is also an isometric isomorphism from $F(S_2)$ onto $F(S_1)$. Let $\Phi = (U \circ v_l)^*$. Then Φ is an isometry from $W^*(S_1)$ onto $W^*(S_2)$. Also

$$\langle \Phi(e_1), f \rangle = \langle U^*(e_1), v_l(f) \rangle = \langle u, v_l(f) \rangle = \langle e_2, f \rangle$$

for all $f \in F(S_1)$, where e_2 is the identity of $W^*(S_2)$, by Lemma 5.1. Hence $\Phi(e_1) = e_2$. By Theorem 7 in [13], Φ is a Jordan *-isomorphism from $W^*(S_1)$ onto $W^*(S_2)$. Let ϕ be the restriction of Φ to $\sigma(F(S_1))$. Then clearly ϕ is a homomorphism from $\sigma(F(S_1))$ onto $\sigma(F(S_2))$. We shall show that ϕ has all desired properties.

That (a) and (c) hold follow from Theorem 5 and Lemma 8 in [13].

To prove (b), we first note that if $xy = yx$, then (b) holds by [13, Theorem 5]. Otherwise, using [13, Lemma 6], we have

$$\phi(x)\phi(y) + \phi(y)\phi(x) = \phi(xy) + \phi(yx) .$$

If $\phi(xy) \neq \phi(y)\phi(x)$ and $\phi(xy) \neq \phi(x)\phi(y)$, then $\phi(xy)$, $\phi(yx)$, $\phi(x)\phi(y)$ and $\phi(y)\phi(x)$ are pairwise distinct elements in $\sigma(F(S_2))$. However, elements in $\sigma(F(S_2))$ are linearly independent [4, p. 93], which is impossible. Hence (b) holds.

If (e) holds, then clearly H and K are *-subsemigroups of $\sigma(F(S_1))$. Also, if $x \in \sigma(F(S_1))$, then $H_x \cap \sigma_u(F(S_1))$ and $K_x \cap \sigma_u(F(S_1))$ are subgroups of $\sigma_u(F(S_1))$ with union equal to $\sigma_u(F(S_1))$ by (b). Hence either $\sigma_u(F(S_1)) \subseteqq H_x$ or $\sigma_u(F(S_1)) \subseteqq K_x$. Hence $H \cup K = \sigma(F(S_1))$ and (f) holds. Also a similar argument shows that either $\sigma_u(F(S_1)) \subseteqq H$ or $\sigma_u(F(S_1)) \subseteqq K$. Hence (d) follows readily from [13, Lemma 12].

It remains to prove (e). By Theorem 10 in [13], there exists a central projections $z_i \in W^*(S_i)_*$ such that Φ is a *-isomorphism from $W^*(S_1)z_1$ onto $W^*(S_2)z_2$ and a *-anti-isomorphism from $W^*(S_1)(e_1 - z_1)$ onto $W^*(S_2)(e_2-z_2)$. Then $\Phi(xz_1) = \Phi(x)z_1$ and $\Phi(x(e_1-z_1)) = \Phi(x)(e_2-z_2)$ for all $x \in W^*(S_1)$. Also observe that

$$(1) \qquad y \in H_x \quad \text{if and only if} \quad (xy - yx)z_1 = 0$$

and

$$(2) \qquad y \in K_x \quad \text{if and only if} \quad (xy - yx)z_2 = 0\,.$$

To prove (1), let $y \in H_x$. Then

$$\begin{aligned}\Phi((xy - yx)z_1) &= \Phi(xy - yx)z_2 \\ &= \Phi(y)\Phi(x)z_2 - \Phi(yz_1)\Phi(xz_1) \\ &= 0\,.\end{aligned}$$

Hence $(xy - yx)z_1 = 0$. Conversely, if $(xy - yx)z_1 = 0$ and $y \notin K_x$, then $y \in H_x$ by (b). If $y \in K_x$, then $(xy - yx)(e_1 - z_1) = 0$. Hence $xy = yx$. So $y \in H_x$ by [13, Theorem 5]. (2) can be proved similarly.

Now if $y_1, y_2 \in H_x$, then

$$x(y_1y_2)z_1 = (xy_1)z_1y_2 = (y_1x)z_1y_2 = y_1(xy_2)z_1 = (y_1y_2)xz_1\,.$$

Hence $y_1y_2 \in H_x$ by (1). Similarly we show that $y \in H_x$ if and only if $y^* \in H_{x^*}$ and that H_x is σ-closed. The assertions on K_x can be proved by using (2).

REMARK 5.9. (a) Martin Walter [18, Theorem 1(i)] proved that if G is a locally compact group, then $\sigma_u(F(G))$ is topologically isomorphic to G.

(b) Let S be a topological *-semigroup and $T = \overline{\omega_\Omega(S)}$. It follows from Proposition 5.7 and its proof that there exists a homeomorphism and *-isomorphism ϕ from $\sigma(F(S))$ onto $\sigma(F(T))$ such that $\phi(\omega_\Omega(s)) = \omega'_\Omega(\omega_\Omega(s))$ for all $s \in S$, where ω_Ω and ω'_Ω denote the *-representations of S, T into $W^*(S)$ and $W^*(T)$ respectively.

PROPOSITION 5.10. *Let S_1, S_2 be topological *-semigroups with identity. If there exists a Banach algebra isomorphism U from $F(S_2)$ onto $F(S_1)$ such that U maps $P(S_2)$ onto $P(S_1)$ and $\|Uf\|_\infty = \|f\|_\infty$*

for all $f \in F(S_2)$, *then there exists a homeomorphism* ϕ *from* $\overline{\omega_\Omega(S_1)}^\sigma$ *onto* $\overline{\omega_\Omega(S_2)}^\sigma$ *such that* (i) $\phi(x^*) = \phi(x)^*$ *for all* $x \in \overline{\omega_\Omega(S_1)}^\sigma$ *and* (ii) *for any* $x, y \in \overline{\omega_\Omega(S_1)}$, *either* $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$ *or* $\phi(x \cdot y) = \phi(y) \cdot \phi(x)$.

Proof. The assumption implies that U^* takes the identity of $W^*(S_1)$ to the identity of $W^*(S_2)$. Hence if ϕ is the restriction of U^* to $\overline{\omega_\Omega(S_1)}^\sigma$, then it follows from the proof of Theorem 5.8 that ϕ has properties (i) and (ii). Also an argument similar to that in [7, p. 99] shows that the $\overline{\omega_\Omega(S_i)}^\sigma = \{x \in \sigma(F(S_i));\ |\langle f, x\rangle| \leqq ||f||_\infty$ for all $f \in F(S_i)\}$. Hence ϕ is a homeomorphism mapping $\overline{\omega_\Omega(S_1)}^\sigma$ onto $\overline{\omega_\Omega(S_2)}^\sigma$.

REMARK 5.11. (a) Theorem 5.8 remains valid when either S_1 or S_2 has identity. Do the conclusions of Theorem 5.8 still hold when both S_1 and S_2 are assumed not to have an identity?

(b) The following questions are posted to us by the referee: Do the hypotheses of Theorem 5.8 imply anything about a sup-norm isometry between $F(S_1)$ and $F(S_2)$? (It is true for groups by Walter's result.) Also can one deduce any relationship between $\overline{\omega_\Omega(S_1)}^\sigma$ and $\overline{\omega_\Omega(S_2)}^\sigma$? (See Proposition 5.10.)

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