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KERNEL DILATION IN REPRODUCING KERNEL HILBERT SPACE AND ITS APPLICATION TO MOMENT PROBLEMS

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We give a reformulation of Nagy's Principal Theorem in terms of a dilation of a family of operators in reproducing kernel Hilbert space. In this setting we are able to generalize Nagy's result to obtain dilations $\tilde{K}$ of reproducing kernels K derived from certain families of operators. We define the concept of positive type for kernels K whose values are unbounded operators on a Hilbert space. The construction of $\tilde{K}$ is such that it possesses a property, which we call splitting, not enjoyed by K. We show that the splitting property constitutes the utility of dilation theory and use it to solve moment problems.

Our main result in this work generalizes Nagy's Principal Theorem (cf. Nagy [14]) the central conclusion of which we give here for reference.

THEOREM. *Let Γ be a *-semi-group with identity ε and suppose $\{T_\gamma\}_{\gamma\in\Gamma}$ is a family of bounded liner transformations in the complex Hilbert space H satisfying:* (a) $T_\varepsilon = I$, (b) *T_γ considered as a function of γ is of positive type, and* (c) *T_γ is completely admissible, i.e., the following inequality obtains for all finite sums*

$$(0.1) \qquad \sum_{i=1}^{n}\sum_{j=1}^{n}\langle T_{\gamma_j^*\alpha^*\alpha\gamma_i}x_i, x_j\rangle \leqq M_\alpha^2\sum_{i=1}^{n}\sum_{j=1}^{n}\langle T_{\gamma_j^*\gamma_i}x_i, x_j\rangle$$

with constant $M_\alpha > 0$. Then there exists a representation $\{D_\gamma\}_{\gamma\in\Gamma}$ of Γ in an extension space H such that

$$T_\gamma = \tilde{P}D_{\gamma|H}\,.$$

Furthermore $\tilde{H}$ may be chosen minimal in the sense that it is spanned by elements of the form $D_\gamma x$ where $x\in H$ and $\gamma\in\Gamma$.

Recently P. Masani [15] and independently F. H. Szafraniec [16] have been able to substantially weaken Nagy's condition (0.1). The weakened versions of (0.1) are:

$$(0.2)\ (\mathrm{S}) \qquad |T(\alpha)| \leqq C\cdot p(\alpha)\,, \qquad \text{where} \quad 0 \leqq p(\alpha\beta) \leqq p(\alpha)p(\beta)\,,$$

$$(0.2)\ (\mathrm{M}) \qquad 0 \leqq T_{\gamma^*\alpha^*\alpha\gamma} < M_\alpha^2\cdot T_{\gamma^*\gamma}\,.$$

These can be shown to be equivalent.

As we will show, Nagy's theorem properly belongs to the theory of reproducing Kernel Hilbert spaces and the dilation of such spaces. Let H be a complex Hilbert space and H_0 a dense linear manifold of H. Let $L(H_0)$ denote the collection of linear operators T (possibly unbounded) in H whose domains $\operatorname{dom}(T) \supset H_0$ contain H_0. By a *reproducing Kernel Hilbert space* we mean a Hilbert space $\mathscr{H}$ of functions $\phi: \Gamma \to H$ defined on a set Γ to H along with a function $K: \Gamma^2 \to L(H_0)$ such that for $\gamma \in \Gamma$ and $x \in H_0$, $K(\cdot, \gamma)x \in \mathscr{H}$ and for $\phi \in \mathscr{H}$ the reproducing property holds,

$$(\phi(\cdot), K(\cdot, \gamma)x)_{\mathscr{H}} = \langle \phi(\gamma), x \rangle_H .$$

The function K is referred to as a *reproducing kernel* (cf. Aronszajn [1], MacNearney [9, 10]). Throughout this paper, in addition to other possible structure, Γ will possess an idempotent unary operation $*$, i.e., $\gamma^{**} = \gamma$ for all $\gamma \in \Gamma$, and a distinguished element ε satisfying $\varepsilon^* = \varepsilon$, Γ need not have a neutral element.

A reproducing kernel is always of *positive type*, i.e., for all $n = 1, 2, \cdots$ and all $\alpha_1, \cdots, \alpha_n \in \Gamma$ and $x_1, \cdots, x_n \in H_0$ the inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \langle K(\alpha_j, \alpha_i)x_i, x_j \rangle \geqq 0$$

holds. In fact, since the norm of the vector $\sum_{i=1}^{n} K(\cdot, \alpha_i)x_i \in \mathscr{H}$ is nonnegative, we can write

$$\begin{aligned} 0 \leqq (\sum_i K(\cdot, \alpha_i)x_i, \sum_j K(\cdot, \alpha_j)x_j) &= \sum_{i,j} (K(\cdot, \alpha_i)x_i, K(\cdot, \alpha_j)x_j) \\ &= \sum_{i,j} \langle K(\alpha_j, \alpha_i)x_i, x_j \rangle . \end{aligned}$$

Furthermore, as is well-known, a densely defined kernel of positive type acting in a complex Hilbert space necessarily *conjugates* in the following sense: the adjoint K^* exists and

$$K^*(\alpha, \beta) \supset K(\beta, \alpha) ,$$

with equality when K is a family of bounded operators.

However our principal objective is to obtain kernels which *split*, meaning for all $\alpha, \beta \in \Gamma$ and $x \in H_0$, the equation

$$K(\alpha, \beta) = K(\alpha, \varepsilon)K(\varepsilon, \beta)$$

holds for a fixed distinguished element ε as described above. Generally a given kernel K does not possess this property and we are led to replace it by a kernel $\tilde{K}$ called a dilation of K, which does split and has the same "weak values" as K. Specifically let $K(\alpha, \beta) \in L(H_0)$ for $\alpha, \beta \in \Gamma$. Then the kernel $\tilde{K}(\alpha, \beta) \in L(\tilde{H}_0)$ for $\alpha, \beta \in \Gamma$ is a *dilation* of K, denoted $K = prK$, if H_0 is a dense linear manifold of a Hilbert superspace $\tilde{H} \supset H$, dom $\tilde{K}(\varepsilon, \beta) \supset$ dom $K(\varepsilon, \beta)$ and for $x \in$ dom $K(\varepsilon, \beta)$,

$$K(\varepsilon, \beta)x = \tilde{P}\tilde{K}(\varepsilon, \beta)x$$

where $\tilde{P}$ is the orthogonal projection of $\tilde{H}$ onto H. Thus for $x \in$ dom $K(\varepsilon, \beta)$ and $y \in H$

$$\langle K(\varepsilon, \beta)x, y\rangle = \langle \tilde{K}(\varepsilon, \beta)x, y\rangle .$$

The kernel $\tilde{K}$ need not be a reproducing kernel. Here $\tilde{H}$ is referred to as a *dilation space* of H and for emphasis the projection equation above is referred to as *projectivity*.

As the following proposition shows, a kernel which conjugates and splits is necessarily of positive type.

PROPOSITION. *Let $H_0 =$ dom $K(\alpha, \beta)$, $a, \beta \in \Gamma$ be dense in H. If K conjugates and splits, then K is of positive type on H_0.*

Proof. For all $n = 1, 2, \cdots, \alpha_1, \cdots, \alpha_n \in \Gamma$, and $\phi_1, \cdots, \phi_n$ H_0 we have

$$\sum_{i,j} (K(\alpha_j, \alpha_i)\phi_i, \phi_j) = \sum_{i,j} (K(\alpha_j, \varepsilon)K(\varepsilon, \alpha_i)\phi_i, \phi_j)$$
$$= \sum_{i,j} (K(\varepsilon, \alpha_i)\phi_i, K(\varepsilon, \alpha_j)\phi_j) = ||\sum_i K(\varepsilon, \alpha_i)\phi_i||^2 \geqq 0 .$$

The following simple lemma will be useful in the sequel.

LEMMA. *If the kernel K splits and conjugates when one of its arguments is ε, then it conjugates.*

1. Kernel dilation theorem. The setting for our main result consists of a complex Hilbert space H and an operator family $T: \Gamma \to \mathscr{B}(H)$. In addition the various cases for which our result is valid can collectively be described by a real-valued measure space $(W, \mathscr{W}, \lambda)$ and functions c and g where

$$c: \Gamma^2 \times W \longrightarrow C \quad \text{and} \quad g: \Gamma^2 \times W \longrightarrow \Gamma .$$

Finally define $K: \Gamma^2 \to L(H_0)$ in terms of c, g, T and λ by

$$K(\alpha, \beta) = \int_W c(\alpha^*, \beta, w)T(g(\alpha^*, \beta, w))d\lambda(w) \tag{1.1}$$

where H_0 is a dense linear manifold of H defined below. It is implicit in this definition that T be such that the integral converges absolutely.

There are four cases corresponding to specific choices of the parameter functions and other special restrictions.

Case I. (Semi-group kernel) Let Γ be a *-semi-group with associative binary operation $\cdot$, $(\alpha\cdot\beta)^* = \beta^*\alpha^*$. Let $W = \{1\}$, $\lambda(w) = 1$, $c = 1$ and $g(\alpha, \beta, 1) = \alpha\cdot\beta$. Then K becomes

$$K(\alpha, \beta) = T(\alpha^*\cdot\beta) .$$

Here $K(\alpha, \beta)$ is bounded and linear so $K(\alpha, \beta) \in B(H) \subset L(H_0)$ where $H_0 = H$.

Case II. (Cosine kernel) Let $\Gamma = \boldsymbol{R}$ the real numbers with $\varepsilon = 0$ and $\gamma^* = -\gamma$. Take $W = \{-1, 1\}$, $\lambda(\{-1\}) = \lambda(\{1\}) = 1/2$, and $c = 1$. Finally put

$$g(\alpha, \beta, -1) = \beta + \alpha \quad \text{and} \quad g(\alpha, \beta, 1) = \beta - \alpha .$$

Then

$$K(\alpha, \beta) = \frac{1}{2}T(\beta + \alpha) + \frac{1}{2}T(\beta - \alpha) .$$

In this case we also require T to satisfy $T(-\gamma) = T(\gamma)$. Again $K(\alpha, \beta) \in B(H)$ here.

Case III. (Resolvent kernel) Let $\Gamma = (\boldsymbol{C} - \boldsymbol{R}) \cup \{0\}$ the complex numbers with $\varepsilon = 0$ and $\gamma^* = \bar{\gamma}$. Take $W = \{-1, 1\}$, $\lambda(\{-1\}) = 1/2$, $\lambda(\{1\}) = 1/2$ and g to be the projections,

$$g(\alpha, \beta, -1) = \alpha \quad \text{and} \quad g(\alpha, \beta, 1) = \beta .$$

Finally, in this case, c is only defined for $\operatorname{dom} c = \{(\alpha, \beta, w): \alpha \neq \beta\} \subset \Gamma^2 \times W$ by

$$c(\alpha, \beta, -1) = \frac{2\alpha}{\alpha - \beta} \quad \text{and} \quad c(\alpha, \beta, 1) = \frac{2\beta}{\beta - \alpha} .$$

Then K becomes

$$K(\alpha, \beta) = \frac{\bar{\alpha}}{\bar{\alpha} - \beta}T(\bar{\alpha}) + \frac{\beta}{\beta - \bar{\alpha}}T(\beta) \tag{1.2}$$

for those α and β for which $\bar{\alpha} - \beta \neq 0$. In this case we also require that $T(\gamma)$ be strongly differentiable on a dense linear manifold H_0[1] (of course $T'(\gamma)$ may be unbounded). Therefore $Q(\gamma) = \gamma T(\gamma)$ is also differentiable there and we may define K on H_0 for the exceptional values by

[1] This means for $x \in H_0$ there is a vector $T'(\gamma)x \in H$ for which the limit $\lim \alpha \to \gamma \|((T(\alpha)x - T(\gamma)x)/(\alpha - \gamma)) - T'(\gamma)x\| = 0$ is valid.

$$K(\bar{\beta}, \beta) = Q'(\beta) = \lim_{\bar{\alpha}\to\beta} \frac{\bar{\alpha}T(\bar{\alpha}) - \beta T(\beta)}{\bar{\alpha} - \beta} . \tag{1.3}$$

Hence for all $\alpha, \beta \in \boldsymbol{\Gamma}$, $K(\alpha, \beta) \in L(H_0)$.

Case IV. (Hankel kernel) Let $\boldsymbol{\Gamma} = \boldsymbol{R}^+ \cup \{0\}$ with $\varepsilon = 0$ and $\gamma^* = \gamma$. Here the Hilbert space H is the unitary space $\boldsymbol{C}$ of complex numbers. For $\nu > 0$, let $W = \boldsymbol{\Gamma}$, ω be the Borel subsets of $\boldsymbol{\Gamma}$ and

$$d\lambda(w) = (2^{\nu-1/2}\Gamma(\nu + 1/2))^{-1}w^{2\nu}dw \tag{1.4}$$

where Γ is the familiar gamma function. Letting $\Delta(\alpha, \beta, \gamma)$ denote the area of the triangle with sides of length α, β, and γ when such a triangle exists, put

$$c(\alpha, \beta, w) = \frac{2^{2\nu-5/2}[\Gamma(\nu + 1/2)]^2}{\pi^{1/2}\Gamma(\nu)}(\alpha\beta w)^{1-2\nu}[\Delta(\alpha, \beta, w)]^{2\nu-2} , \tag{1.5}$$

for $\alpha, \beta, w \in \boldsymbol{R}^+$ when $\Delta(\alpha, \beta, w)$ is defined, and set $c(\alpha, \beta, w)$ to zero otherwise. In general $c(\alpha, \beta, \cdot)$ is unbounded. Finally set $g(\alpha, \beta, w) = w$. In this case K becomes

$$K(\alpha, \beta) = \int_0^\infty T(w)c(\alpha, \beta, w)d\lambda(w) .$$

We remark that the integral converges for $T \in L(0, \infty)$, cf. Haimo [7]. Here $K(\alpha, \beta) \in B(H)$ since H is finite dimensional.

KERNEL DILATION THEOREM. *In the four cases above, if K satisfies:*

(a) $K(\varepsilon, \varepsilon) = I$

(b) *K is of positive type,*

then there exists a dilation space $\tilde{H} \supset H$ and a dilated kernel $\tilde{K}$ in $\tilde{H}$ which conjugates and splits.

Proof. Let $\tilde{H}_0$ be the linear space of all H-valued functions $\phi(\tau)$, $\tau \in \boldsymbol{\Gamma}$, of the form

$$\phi(\cdot) = \sum_{i=1}^{n} K(\cdot, \gamma_i)x_i , \qquad \gamma_i \in \boldsymbol{\Gamma},\ x_i \in H_0 ,$$

for some $n = 1, 2, \cdots$. If also $\psi(\cdot) = \sum_{j=1}^{m} K(\cdot, \delta_j)y_j \in \tilde{H}_0$, then define the form $(\cdot, \cdot)_\sim$ by

$$\begin{aligned}(\phi(\cdot), \psi(\cdot))_\sim &= \sum_{i,j} \langle K(\delta_j, \gamma_i)x_i, y_j\rangle \\ &= \sum_{j=1}^{m} \langle \phi(\delta_j), y_j\rangle = \sum_{i=1}^{n} \langle x_i, \psi(\gamma_i)\rangle .\end{aligned}$$

The last member of the string above follows from the fact that K conjugates. We immediately see that this form is independent of the particular representations of ϕ and ψ and instead depends only on their values. Also it is easy to see that the form is positive, sesqui-linear, and symmetric. To see that the form is definite, suppose $(\phi(\cdot), \phi(\cdot))_{\sim} = 0$. By the Cauchy-Schwarz inequality

$$|\langle\phi(\alpha), y\rangle|^2 = |(\phi(\cdot), K(\cdot, \alpha)y)_{\sim}|^2 \leqq (\phi(\cdot), \phi(\cdot))_{\sim}\cdot(K(\cdot, \alpha)y, K(\cdot, \alpha)y)_{\sim} = 0 .$$

Since this holds for all $y \in H_0$ and $\alpha \in \Gamma$, $\phi(\cdot)$ must be the zero function. Henceforth we omit the subscript $\sim$ in this inner-product.

Next let $\tilde{H}$ be the completion of the inner-product space $\tilde{H}_0$. It is possible to realize $\tilde{H}$ as a space of H-valued functions on Γ. In fact let $\{\psi_n\}_1^\infty$ be a fundamental sequence in $\tilde{H}_0$ converging to the ideal element $[\{\psi_n\}] \in \tilde{H}$. Fix $\alpha \in \Gamma$ and for $y \in H_0$ put

$$G_0(y) = \lim_n \langle y, \psi_n(\alpha)\rangle = \lim_n (K(\cdot, \alpha)y, \psi_n(\cdot)) = (K(\cdot, \alpha)y, [\{\psi_n\}]) .$$

It is easy to see that G_0 is a bounded linear functional on H_0 and therefore has a unique continuous extension to a bounded linear functional, G, on H. By the Riesz theorem there exists a vector in H, call it $\psi(\alpha)$, associated with G in the sense that

$$\langle y, \psi(\alpha)\rangle = G(y) = \lim_n \langle y, \psi_n(\alpha)\rangle$$

holds for all $y \in H$. If also $\{\phi_n\}_1^\infty$ is a fundumental sequence in $\tilde{H}_0$ converging to $[\{\psi_n\}]$, then

$$\lim_n (K(\cdot, \alpha)y, \phi_n(\cdot)) = (K(\cdot, \alpha)y, [\{\psi_n\}]) = G(y)$$

so that $\psi(\alpha)$ is independent of the particular sequence conve rging to $[\{\psi_n\}]$. If $[\{\psi_n\}] \in \tilde{H}_0$ so that $[\{\psi_n\}](\alpha) \in H$ exists *a priori*, t hen for all $y \in H_0$,

$$\langle y, [\{\psi_n\}](\alpha)\rangle = G(y) = \langle y, \psi(\alpha)\rangle .$$

Hence the value derived above agrees with the *a priori* value of this element. Thus we may identify the ideal element $[\{\psi_n\}]$ in an unambiguous and consistent way with function $\psi: \Gamma \to H$. Furthermore, the values of this function are reproduced by K since

$$([\{\psi_n\}], K(\cdot, \alpha)y) = (\psi(\cdot), K(\cdot, \alpha)y) = \langle\psi(\alpha), y\rangle .$$

Therefore $\tilde{H}$ is a reproducing kernel Hilbert space with kernel K.

The space H may be embedded in $\tilde{H}$ by identifying

$$x \longleftrightarrow K(\cdot, \varepsilon)x .$$

This is possible since

$$(K(\tau, \varepsilon)x, K(\tau, \varepsilon)y) = \langle K(\varepsilon, \varepsilon)x, y\rangle = \langle x, y\rangle .$$

The orthogonal projection $\tilde{P}$ of $\tilde{H}$ onto H is given on $\tilde{H}_0$ by

$$\tilde{P}K(\tau, \gamma)x = K(\tau, \varepsilon)K(\varepsilon, \gamma)x .$$

In fact for all $K(\tau, \varepsilon)y \in H$,

$$\begin{aligned}(K(\tau, \gamma)x &- K(\tau, \varepsilon)K(\varepsilon, \gamma)x, K(\tau, \varepsilon)y) \\ &= \langle K(\varepsilon, \gamma)x, y\rangle - \langle K(\varepsilon, \varepsilon)K(\varepsilon, \gamma)x, y\rangle = 0 .\end{aligned}$$

Next define the dilated kernel $\tilde{K}$ on $\tilde{H}_0$ as follows:

$$\tilde{K}(\varepsilon, \beta)K(\tau, \gamma)x = \int_W c(\beta, \gamma, w)K(\tau, g(\beta, \gamma, w))xd\lambda(w) \tag{1.6}$$

and

$$\tilde{K}(\alpha, \beta)K(\tau, \gamma)x = \int_W c(\alpha^*, \beta, w)\tilde{K}(\varepsilon, g(\alpha^*, \beta, w))K(\tau, \gamma)xd\lambda(w) , \tag{1.7}$$

or altogether

$$\begin{aligned}\tilde{K}(\alpha, \beta)K(\tau, \gamma)x = \int_W \int_W & c(\alpha^*, \beta, w)c(g(\alpha^*, \beta, w), \gamma, v) \\ & \times K(\tau, g(g(\alpha^*, \beta, w), \gamma, v))xd\lambda(v)d\lambda(w) .\end{aligned}$$

We show in each of the four separate cases that $\tilde{K}$ is consistently defined ($\tilde{K}(\varepsilon, \beta)$ from (1.6) agrees with that from (1.7)), conjugates, splits, and projects. However we show for all the cases at once that $\tilde{K}$ is well-defined assuming the conjugation property. We avoid circular reasoning by regarding $\tilde{K}$ as only formally defined by (1.6) and (1.7) until conjugacy has been proved. Thus suppose $\phi(\tau) = \sum_{i=1}^{n} K(\tau, \gamma_i)x_i$ is a representation of zero. Then for $\psi(\tau) = K(\tau, \delta)y$ we have

$$\begin{aligned}(\tilde{K}(\alpha, \beta)\phi(\cdot), \psi(\cdot)) &= \sum_{i=1}^{n} (\tilde{K}(\alpha, \beta)K(\tau, \gamma_i)x_i, K(\tau, \delta)y) \\ &= \sum_{i=1}^{n} (K(\tau, \gamma_i)x_i, \tilde{K}^*(\alpha, \beta)K(\tau, \delta)y) \\ &= (\phi(\tau), \tilde{K}(\beta, \alpha)K(\tau, \delta)y) = 0 .\end{aligned}$$

Since this equation holds for arbitrary $\psi \in \tilde{H}_0$, it follows that $\tilde{K}(\alpha, \beta)\phi(\cdot) = 0$ and therefore $\tilde{K}(\alpha, \beta)$ is well-defined.

2. Proof for the semi-group kernel.

Here

$$\tilde{K}(\varepsilon, \beta)K(\tau, \gamma)x = K(\tau, \beta\cdot\gamma)x , \tag{2.1}$$

and

$$\tilde{K}(\alpha, \beta)K(\tau, \gamma)x = \tilde{K}(\varepsilon, \alpha^* \cdot \beta)K(\tau, \gamma)x \ . \tag{2.2}$$

Note that

$$\tilde{K}(\alpha, \varepsilon)K(\tau, \gamma)x = K(\tau, \alpha^* \cdot \gamma)x \ . \tag{2.3}$$

The definition of $\tilde{K}(\varepsilon, \beta)$ is obviously consistent between the first two equations.

Splitting is established by direct computation. By the Lemma, in the presence of splitting, conjugacy follows from the equations

$$\tilde{K}^*(\varepsilon, \beta) \supset \tilde{K}(\beta, \varepsilon) \quad \text{and} \quad \tilde{K}^*(\alpha, \varepsilon) \supset \tilde{K}(\varepsilon, \alpha) \ .$$

Both of these follow from the direct calculation that

$$(\tilde{K}(\varepsilon, \beta)K(\tau, \gamma)x, K(\tau, \delta)y) = (K(\tau, \gamma)x, \tilde{K}(\beta, \varepsilon)K(\tau, \delta)y)$$

in which the fact that $K(\varepsilon, \beta) = T(\beta)$ is used. Finally projectivity is also established by the following direct calculation.

$$\begin{aligned}\tilde{P}\tilde{K}(\alpha, \beta)K(\tau, \varepsilon)x &= \tilde{P}[\tilde{K}(\varepsilon, \alpha^* \cdot \beta)K(\tau, \varepsilon)x] = \tilde{P}K(\tau, \alpha^* \cdot \beta \cdot \varepsilon)x \\ &= K(\tau, \varepsilon)K(\varepsilon, \alpha^* \cdot \beta)x = K(\tau, \varepsilon)T(\alpha^* \cdot \beta)x = K(\tau, \varepsilon)K(\alpha, \beta)x \ .\end{aligned}$$

3. Proof for the cosine kernel. In this case

$$\tilde{K}(0, \beta)K(\tau, \gamma)x = \frac{1}{2}K(\tau, -\beta + \gamma)x + \frac{1}{2}K(\tau, \beta + \gamma)x \tag{3.1}$$

and

$$\begin{aligned}\tilde{K}(\alpha, \beta)K(\tau, \gamma)x &= \frac{1}{2}\tilde{K}(0, \alpha + \beta)K(\tau, \gamma)x \\ &\quad + \frac{1}{2}\tilde{K}(0, -\alpha + \beta)K(\tau, \gamma)x \ .\end{aligned} \tag{3.2}$$

Note that

$$\tilde{K}(\alpha, 0)K(\tau, \gamma)x = \frac{1}{2}K(\tau, \alpha + \gamma)x + \frac{1}{2}K(\tau, \alpha + \gamma)x \ . \tag{3.3}$$

By simple direct calculation and using the fact that $K(0, \beta) = T(\beta)$, conjugacy and the splitting property are seen to hold. Recalling that $T(-\beta) = T(\beta)$ here, projectivity follows from the chain of equations

$$\begin{aligned}\tilde{P}\tilde{K}(0, \beta)K(\tau, 0)x &= \tilde{P}\left[\frac{1}{2}K(\tau, -\beta)x + \frac{1}{2}K(\tau, \beta)x\right] \\ &= \frac{1}{2}K(\tau, 0)K(0, -\beta)x + \frac{1}{2}K(\tau, 0)K(0, \beta)x = K(\tau, 0)K(0, \beta)x \ .\end{aligned}$$

4. Proof for the resolvent kernel. Here the dilated kernel is given as

$$\tilde{K}(0, \beta)K(\tau, \gamma)x = \frac{\beta}{\beta - \gamma}K(\tau, \beta)x + \frac{\gamma}{\gamma - \beta}K(\tau, \gamma)x \tag{4.1}$$

and

$$\begin{aligned}\tilde{K}(\alpha, \beta)K(\tau, \gamma)x &= \frac{\bar{\alpha}}{\bar{\alpha} - \beta}\tilde{K}(0, \bar{\alpha})K(\tau, \gamma) \\ &+ \frac{\beta}{\beta - \bar{\alpha}}\tilde{K}(0, \beta)K(\tau, \gamma)x ,\end{aligned} \tag{4.2}$$

when denominators do not vanish.

We note that in particular

$$\tilde{K}(\alpha, 0)K(\tau, \gamma)x = \frac{\bar{\alpha}}{\bar{\alpha} - \gamma}K(\tau, \bar{\alpha}) + \frac{\gamma}{\gamma - \bar{\alpha}}K(\tau, \gamma)x \tag{4.3}$$

if $\bar{\alpha} \neq \gamma$. By the assumed differentiability of T, $\tilde{K}$ may be defined for the exceptional cases by operator limits of the above equations. Thus, for example, when $\tau \neq \gamma$,

$$\begin{aligned}\tilde{K}(0, \gamma)K(\tau, \gamma)x &= \lim_{\beta \to \gamma} \tilde{K}(0, \beta)K(\tau, \gamma)x \\ &= \lim_{\beta \to \gamma}\left[\frac{\bar{\tau}^2}{(\bar{\tau} - \beta)(\bar{\tau} - \gamma)}T(\bar{\tau}) + \frac{\beta^2(\beta - \bar{\tau})^{-1}T(\beta) - \gamma^2(\gamma - \bar{\tau})^{-1}T(\gamma)}{\beta - \gamma}\right]x\end{aligned}$$

which exists.

That $\tilde{K}(0, \beta)$ is consistently defined by (4.2) is immediate. Using the fact that $K(0, \beta) = T(\beta)$, lengthy but straightforward calculations based upon (4.1)-(4.3) confirm conjugacy, projectivity and the splitting property for the nonexceptional values of the argument. From their validity we obtain these properties for the exceptional values as limiting cases.

5. Proof for the Hankel kernel. Here

$$\tilde{K}(0, \beta)K(\tau, \gamma)x = \int_0^\infty c(\beta, \gamma, \varepsilon, w)K(\tau, w)x d\lambda(w) \tag{5.1}$$

and

$$\tilde{K}(\alpha, \beta)K(\tau, \gamma)x = \int_0^\infty c(\alpha, \beta, w)\tilde{K}(0, w)K(\tau, \gamma)x d\lambda(w) , \tag{5.2}$$

which exist, cf. Haimo [7].

Put

$$I(t) = 2^{\nu - 1/2}\Gamma(\nu + 1/2)t^{(1/2) - \nu}J_{\nu - 1/2}(t)$$

where $J_\rho(t)$ is Bessel's function of order ρ. Then (cf. Cholewinski and Haimo [3] and Cholewinski, Haimo, and Nussbaum [2])

$$\int_0^\infty f(w)c(0, \beta, w)d\lambda(w) = f(w) , \tag{5.3}$$

$$\int_0^\infty I(tw)c(\alpha, \beta, w)d\lambda(w) = I(\alpha t)I(\beta t) \tag{5.4}$$

and

$$\int_0^\infty I(\alpha t)I(\beta t)I(\alpha t)d\lambda(t) = c(\alpha, \beta, \gamma) . \tag{5.5}$$

From (5.3) we obtain

$$K(0, \beta) = T(\beta) .$$

By use of the following lemma and Fubini's theorem, consistency, conjugacy, and the splitting property are seen to be valid through direct calculation. Also we obtain the equation

$$\tilde{K}(\alpha, 0)K(\tau, \gamma)x = \int_0^\infty c(\alpha, \gamma, w)K(\tau, w)xd\lambda(w)$$

from the lemma and Fubini's theorem.

LEMMA. *With notation as above*

$$\int_0^\infty c(\alpha_1, \alpha_2, t)c(\alpha_3, \alpha_4, t)d\lambda(t) = \int_0^\infty \prod_{i=1}^4 I(\alpha_i w)d\lambda(w) ,$$

and

$$\int_0^\infty \int_0^\infty c(\alpha_1, \alpha_2, s)c(\alpha_3, \alpha_4, t)c(\alpha_5, s, t)d\lambda(t)d\lambda(s) = \int_0^\infty \prod_{i=1}^5 I(\alpha_i w)d\lambda(w) .$$

Proof. By (5.5), Fubini's theorem, and (5.4),

$$\begin{aligned}\int_0^\infty & c(\alpha_1, \alpha_2, t)c(\alpha_3, \alpha_4, t)d\lambda(t) \\ &= \int_0^\infty \int_0^\infty c(\alpha_1, \alpha_2, t)I(\alpha_3 w)I(\alpha_4 w)I(tw)d\lambda(w)d\lambda(t) \\ &= \int_0^\infty I(\alpha_3 w)I(\alpha_4 w)\int_0^\infty c(\alpha_1, \alpha_2, t)I(tw)d\lambda(t)d\lambda(w) \\ &= \int_0^\infty \prod_{i=1}^4 I(\alpha_i w)d\lambda(w) .\end{aligned}$$

The second equation can be proved in a similar fashion.

6. Nagy's principal theorem. The setting of Nagy's theorem without the complete admissibility assumption is included in Case I. Let $\tilde{K}$ be a dilation kernel acting in dilation space $\tilde{H}$ guaranteed by the Kernel Dilation Theorem. Put $\tilde{T}(\alpha^*\beta) = \tilde{K}(\alpha, \beta)$. That this assignment is well-defined follows immediately from the definition of $\tilde{K}$ in §2. By conjugacy $\tilde{T}$ preserves *, and the splitting property implies $\tilde{T}$ preserves the binary operation of $\boldsymbol{\Gamma}$. In fact

$$\tilde{T}(\alpha\cdot\beta) = \tilde{K}(\alpha^*, \beta) = \tilde{K}(\alpha^*, \varepsilon)\tilde{K}(\varepsilon, \beta) = \tilde{T}(\alpha)\tilde{T}(\beta) .$$

We remark that the kernel Dilation Theorem continues in force for Case I under the replacement of the binary operation $\cdot$ by a function $b\colon \boldsymbol{\Gamma}^2 \to \boldsymbol{\Gamma}$ and satisfying:

$$b^*(\alpha, \beta) = b(\beta^*, \alpha^*)$$

and

$$b(b(\alpha, \varepsilon), b(\beta, \gamma)) = b(b(\alpha, \beta), b(\varepsilon, \gamma)) .$$

But these conditions trivially imply that b is an associative *-semi-group binary operation. Therefore we see that Nagy's theorem is the most general one possible with (W, ω, λ), c and g as in Case I.

7. Application to moment problems. In this section we give only a brief description of the technique. Further details may be found in Nagy [14], Devinatz [4], Shonkwiler [11-13], Faulkner and Shonkwiler [5, 6], and Korányi [8].

By a moment problem we mean, given a function $h\colon S \times E \subset \boldsymbol{C} \times \boldsymbol{R} \to \boldsymbol{C}$, find necessary and sufficient conditions on a function $f\colon S \to \boldsymbol{C}$ in order that there exists a nondecreasing bounded function $\mu\colon E \to \boldsymbol{R}$ so that

$$f(s) = \int_E h(s, t)d\mu(t) .$$

For our first example let $h(s, t) = t^s$, $-\infty < t < \infty$, $s \in \boldsymbol{\Gamma}$ where $\boldsymbol{\Gamma}$ is the *-semi-group of nonnegative integers under addition with $\varepsilon = 0$ and $\gamma^* = \gamma$. Let T be a function $T\colon \boldsymbol{\Gamma} \to \boldsymbol{R}$ and interpret $T(\gamma)$ as a bounded operator on the unitary space $\boldsymbol{C}$ under the ordinary product of a complex number by a real number. Setting $K(\alpha, \beta) = T(\alpha^* + \beta)$, conditions (a)-(c) of Nagy's theorem are then necessary and sufficient in order that

$$T(s) = \int_{\boldsymbol{R}} t^s d\mu(t)$$

for some nondecreasing bounded function μ.

For under these conditions there exists a dilated *-representation $\tilde{T}$ of $\boldsymbol{\Gamma}$ consisting of bounded operators acting on a Hilbert space $\tilde{H} \supset \boldsymbol{C}$. Thus, for all s,

$$T^*(s) = T(s) \quad \text{and} \quad T(s) = T^s(1) .$$

Therefore by the spectral theorem there exists a resolution of the identity $\tilde{E}_t$ such that

$$\tilde{T}(s) = \int_{\boldsymbol{R}} t^s d\tilde{E}_t .$$

Finally by projectivity,

$$T(s) = \langle T(s)1, 1\rangle_C = (\tilde{T}(s)1, 1)_{\tilde{H}} = \int_{\boldsymbol{R}} t^s d(\tilde{E}_t 1, 1)_{\tilde{H}} ,$$

and the required representation holds with $\mu(t) = (\tilde{E}_t 1, 1)_{\tilde{H}}$.

For our last example let $h(s, t) = s/(1 - st)$, $t > 0$, $s \in \boldsymbol{\Gamma}$. Here $\boldsymbol{\Gamma} = (\boldsymbol{C} - \boldsymbol{R}) \cup \{0\}$ with $\varepsilon = 0$ and $\gamma^* = \bar{\gamma}$. Again let T be a function $T: \boldsymbol{\Gamma} \to \boldsymbol{C}$ and interpret $T(\gamma)$ as an operator on the unitary space $\boldsymbol{C}$. Setting

$$K(\alpha, \beta) = \frac{\bar{\alpha}}{\bar{\alpha} - \beta} T(\alpha) + \frac{\beta}{\beta - \bar{\alpha}} T(\beta) ,$$

then conditions (a) and (b) of the Kernel Dilation Theorem are necessary and sufficient in order that

$$T(s) = \int_{\boldsymbol{R}} \frac{s}{1 - st} d\mu(t) , \qquad s \in (\boldsymbol{C} - \boldsymbol{R}) \cup \{0\} ,$$

for some nondecreasing bounded function μ. In fact under these condition s by Case III there exists a dilation kernel $\tilde{K}$ acting in some dilation space $\tilde{H}$. It may be seen directly from the definition of $\tilde{K}$ in §4 that

$$\tilde{K}(\alpha, 0) = \tilde{K}(0, \bar{\alpha}) .$$

Hence if $\tilde{Q}(\beta) = \beta\tilde{K}(0, \beta)$ then

$$\tilde{Q}^*(\beta) = (\beta\tilde{K}(0, \beta))^* \supset \bar{\beta}\tilde{K}(\beta, 0) = \bar{\beta}\tilde{K}(0, \bar{\beta}) = \tilde{Q}(\bar{\beta}) .$$

Also by the differentiability of T it follows that $\tilde{K}(0, \beta)$ is continuous in β, and so $\tilde{Q}$ satisfies the limiting condition

$$\lim_{\beta \to 0} \frac{1}{\beta} \tilde{Q}(\beta) = \lim_{\beta \to 0} \tilde{K}(0, \beta) = \tilde{K}(0, 0) = \tilde{I}|_{\tilde{H}_0} ,$$

where the last member follows directly from (4.1).

Furthermore by the definition of $\tilde{K}$, (4.2), and splitting property

$$\frac{\tilde{Q}(\bar{\alpha})}{\bar{\alpha}-\beta}+\frac{\tilde{Q}(\beta)}{\beta-\bar{\alpha}}=\tilde{K}(\alpha,\beta)=\tilde{K}(\alpha,0)\tilde{K}(0,\beta)=\frac{\tilde{Q}(\bar{\alpha})}{\bar{\alpha}}\frac{\tilde{Q}(\beta)}{\beta}.$$

Or rewritten

$$\bar{\alpha}\beta(\tilde{Q}(\bar{\alpha})-\tilde{Q}(\beta))=(\bar{\alpha}-\beta)\tilde{Q}(\bar{\alpha})\tilde{Q}(\beta),$$

which is the resolvent equation.

Hence $\tilde{Q}(\beta)$ is the resolvent of a self-adjoint operator and by the spectral theorem

$$\tilde{Q}(\beta)=\beta(\tilde{I}-\beta\tilde{A})^{-1}=\int_R\frac{\beta}{1-\beta t}d\tilde{E}_t.$$

Finally, as above, projectivity gives the calculation

$$T(s)=\langle T(s)1,1\rangle=(\tilde{T}(s)1,1)_{\tilde{H}}=\int_R\frac{s}{1-st}d(\tilde{E}_t1,1)_{\tilde{H}}.$$

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