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LOCALLY UNIVALENT FUNCTIONS AND COEFFICIENT DISTORTIONS

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We look at functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfying $\sum_{n=2}^{\infty} n\,|a_n| > 1$ and determine conditions for which the arguments of the coefficients may vary without affecting the univalence of the function. A bound on the radius of starlikeness for the convolution of functions taken from the closed convex hull of convex functions and a special subclass of starlike functions is also obtained.

1. **Introduction.** Let LS denote the class of functions of the form $z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and locally univalent in the unit disk U, and let S denote the subclass of univalent functions. It is well known that a sufficient condition for $z + \sum_{n=2}^{\infty} a_n z^n$ to be in S is that

$$\sum_{n=2}^{\infty} n\,|a_n| \leqq 1 \,. \tag{1}$$

For functions of the form

$$z - \sum_{n=2}^{\infty} |a_n|\, z^n \tag{2}$$

the condition (1) is also necessary. This follows because functions that fail to satisfy (1) are not even in LS. The necessary and sufficient condition (1) for functions of the form (2) to be in S makes extremal problems much more manageable. Very little is known for functions in S of the form

$$f_\lambda(z) = z - e^{i\lambda} \sum_{n=2}^{\infty} |a_n|\, z^n \,, \tag{3}$$

where the coefficients are not necessarily real but have constant argument. In [6] it is asked if a function $g(\lambda, n)$ can be found for which the inequality $|a_n| \leqq g(\lambda, n)$ is sharp. Note that $g(0, n) = 1/n$ and $g(\pi, n) = n$, with extremal functions $z - z^n/n$ and $z/(1-z)^2$ respectively.

In this paper we show that a function in LS must satisfy (1) when its coefficients are "close to" negative. Since the degree of closeness depends on both λ and the coefficients in (3), rather than on λ alone, we cannot conclude that $g(\lambda, n) \leqq 1/n$ for any positive λ. We also examine the extent to which a violation of condition (1) enables us to distort the arguments of some of the coefficients to construct functions that are not in LS.

The Hadamard product or convolution of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. For

$$h_\lambda(z) = z + e^{i\lambda} \sum_{n=2}^{\infty} z^n , \tag{4}$$

we may express (3) as $f_0 * h_\lambda$. When λ is sufficiently small $h_\lambda(z)$ is starlike and we find a bound on the radius of starlikeness for $h_\lambda * f$, where $\operatorname{Re} f(z)/z > 1/2$ $(z \in U)$. This generalizes a result of MacGregor [4].

2. Coefficient distortions. Given a function $f(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the function $g(z) = z + \sum_{n=2}^{\infty} e^{i\lambda_n} b_n z^n$ is said to be in $F_\varepsilon(f)$ for some $\varepsilon > 0$ if $-\varepsilon \leqq \lambda_n \leqq \varepsilon$ for all n.

LEMMA 1. *If $f(z) = z + \sum_{n=2}^{\infty} b_n z^n \notin LS$, then there exists an $\varepsilon > 0$ such that $g \in F_\varepsilon(f)$ implies that $g \notin LS$.*

Proof. Suppose, on the contrary, that there is a sequence $\varepsilon(n)$ tending to 0 for which we can find a corresponding sequence of functions $g_n \in F_{\varepsilon(n)}$ such that $g_n \in LS$ for all n. Since $f \notin LS$, there exists a point $z_0 \in U$ such that $f'(z_0) = 0$. Note that $\{g'_n\}$ converges uniformly to f' in some neighborhood D of z_0. Since $g'_n \neq 0$ in D for any n, it follows by Hurwitz's theorem that $f' \neq 0$ in D. This contradicts our assumption that $f'(z_0) = 0$.

THEOREM 1. *If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ with $\sum_{n=2}^{\infty} n|a_n| > 1$, then there exists an $\varepsilon > 0$ such that $g \in F_\varepsilon(f)$ implies that $g \notin LS$.*

Proof. Since $f'(0) = 1$ and $f'(r) < 0$ for $r(<1)$ sufficiently close to 1, there must exist a point r_0, $0 < r_0 < 1$, such that $f'(r_0) = 0$. The result now follows from the lemma, with $b_n = -|a_n|$.

COROLLARY. *If $\sum_{n=2}^{\infty} n|a_n| > 1$, then $f_\lambda(z)$, defined by (3), is not in LS for λ sufficiently small.*

Our next theorem shows that if (1) is violated, then we can always construct a nonunivalent function by distorting the arguments of finitely many coefficients.

THEOREM 2. *If $\sum_{n=2}^{N} n|a_n| > 1$, then there exist real numbers $\alpha_2, \cdots, \alpha_N$ $(-\pi < \alpha_j \leqq \pi)$ such that*

$$f(z) = z + \sum_{n=2}^{N} a_n e^{i\alpha_n} z^n + \sum_{n=N+1}^{\infty} a_n z^n \notin LS .$$

Proof. Let $C = z(t)$ be an arc of increasing modulus from the origin to the boundary of U such that

$$g(z) = \sum_{n=N+1}^{\infty} n a_n z^{n-1} \leqq 0 \qquad \text{for} \quad z \in C .$$

For each point $z = z(t) \in C$, $|z| = r$, choose $\alpha_n(t)$ so that

$$n a_n e^{i\alpha_n(t)} z^{n-1} = -n |a_n| r^{n-1} \quad (n = 2, \cdots, N) .$$

Next define $f_t(z)$ by

$$f_t(z) = z + \sum_{n=2}^{N} a_n e^{i\alpha_n(t)} z^n + \sum_{n=N+1}^{\infty} a_n z^n ,$$

where $\alpha_n(t)$ varies continuously with t. Since $g(z) \leqq 0$ for $z \in C$, $|z| = r$, we have for all t that

$$f_t'(z) \leqq 1 - \sum_{n=2}^{N} n |a_n| r^{n-1} \qquad (z \in C) .$$

Since $f_t'(0) = 1$ and $f_t'(z) < 0$ for $z \in C$ sufficiently close to the boundary of U, it follows for some $\xi = z(t_0)$ on C that $f_{t_0}'(\xi) = 0$. Setting $f(z) = f_{t_0}(z)$ and $\alpha_n = \alpha_n(t_0)$, the result follows.

COROLLARY 1. *If $\sum_{n=2}^{N} n |a_n| > 1$, then there exist real numbers $\alpha_2, \cdots, \alpha_N$ $(-\pi < \alpha_j \leqq \pi)$ and an $\varepsilon > 0$ such that for each λ, $-\varepsilon \leqq \lambda \leqq \varepsilon$,*

$$f_\lambda(z) = z + e^{i\lambda}\left(\sum_{n=2}^{N} a_n e^{i\alpha_n} z^n + \sum_{n=N+1}^{\infty} a_n z^n\right) \notin LS .$$

Proof. The corollary is established upon applying Lemma 1 to Theorem 2.

COROLLARY 2. *If $\sum_{k=2}^{N} n_k |a_{n_k}| > 1$, then there exist real numbers $\alpha_2, \cdots, \alpha_N$ $(-\pi < \alpha_j \leqq \pi)$ and $\varepsilon > 0$ such that for each λ, $-\varepsilon \leqq \lambda \leqq \varepsilon$,*

$$f_\lambda(z) = z + e^{i\lambda}\left(\sum_{k=2}^{N} a_{n_k} e^{i\alpha_k} z^{n_k} + \sum_{\substack{n=2 \\ n \neq n_k}}^{\infty} a_n z^n\right) \notin LS .$$

Proof. The proof is the same as that of Corollary 1 except for a rearrangement of terms.

3. Extremal examples. One might ask for conditions under which $\sum_{n=2}^{\infty} n |a_n| > 1$ guarantees the existence of a real number λ

such that $f_\lambda(z) = z + e^{i\lambda} \sum_{n=2}^{\infty} a_n z^n \notin S$. Before answering this question, we need

LEMMA 2. *For $n \geqq 3$ there exist real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $|\sum_{k=1}^{n} e^{i\alpha_k} z^k| < n$ $(|z| \leqq 1)$.*

Proof. We have $|\sum_{k=1}^{n} e^{i\alpha_k} e^{ik\theta}| = n$ if and only if there is a θ for which

$$\alpha_1 + \theta = \alpha_k + k\theta \pmod{2\pi}, \qquad k = 2, \cdots, n .$$

Clearly the α_k's can be chosen to preclude the existence of such a θ.

THEOREM 3. (i) *If $\sum_{k=2}^{3} n_k |a_{n_k}| > 1$, then $f_\lambda(z) = z + e^{i\lambda}(a_{n_2} z^{n_2} + a_{n_3} z^{n_3}) \notin LS$ for some λ.*

(ii) *If $N > 3$, then we can find $\{a_{n_k} | k = 2, 3, \cdots, N\}$ such that $\sum_{k=2}^{N} n_k |a_{n_k}| > 1$ and $f_\lambda(z) = z + e^{i\lambda} \sum_{k=2}^{N} a_{n_k} z^{n_k} \in S$ for all λ.*

Proof. To prove (i), assume $\arg a_{n_2} = \alpha_2 + \pi$ and $\arg a_{n_3} = \alpha_3 + \pi$. Then for $\theta = (\alpha_2 - \alpha_3)/(n_3 - n_2)$, we have

$$f'_\lambda(re^{i\theta}) = 1 - e^{i\lambda}(n_2 |a_{n_2}| r^{n_2 - 1} + n_3 |a_{n_3}| r^{n_3 - 1}) e^{i\beta} ,$$

where

$$\beta = \alpha_2 + \frac{(n_2 - 1)(\alpha_2 - \alpha_3)}{n_3 - n_2} .$$

Hence $f_\lambda(z) \notin LS$ for $\lambda = -\beta$.

To prove (ii), we choose $\{\alpha_k\}$ so that

$$\left| \sum_{k=2}^{N} e^{i\alpha_k} z^{n_k - 1} \right| \leqq A < N - 1 \qquad (|z| \leqq 1) .$$

For $1 < B \leqq (N-1)/A$, set

$$f_\lambda(z) = z + e^{i\lambda} \sum_{k=2}^{N} \frac{B e^{i\alpha_k}}{(N-1) n_k} z^{n_k} .$$

Then

$$f'_\lambda(z) = 1 + \frac{B}{N-1} e^{i\lambda} \sum_{k=2}^{N-1} e^{i\alpha_k} z^{n_k - 1} ,$$

so that

$$\operatorname{Re} f'_\lambda(z) \geqq 1 - \frac{B}{N-1} \left| \sum_{k=2}^{N-1} e^{i\alpha_k} z^{n_k - 1} \right| > 1 - \frac{AB}{N-1} \geqq 0 \qquad (z \in U) .$$

By a criterion of Kaplan [3], $f_\lambda(z) \in S$ for all λ.

4. **A radius of starlikeness theorem.** Denote by F functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic in U and satisfy $\operatorname{Re} f(z)/z > 1/2$. MacGregor has shown [4] that the radius of starlikeness of F is $1/\sqrt{2}$. In this section we generalize this result. It is known [1] that the family F is the closed convex hull of convex functions, and that a function $f(z)$ is in F if and only if it can be expressed as

$$f(z) = \int_X \frac{z}{1 - xz} d\mu(x) \tag{5}$$

for some probability measure μ defined on the unit circle X.

In [2] Campbell proves

LEMMA A. *Let M be a class of starlike functions with $b(r) = \max\{\arg f(z)/z\colon |z| = r, f \in M\}$. If $|z(f'(z)/f(z)) - a(r)| \leqq d(r)$ for $f \in M$, where $a(r)$ and $d(r)$ are continuous functions of r satisfying $\lim_{r \to 1} a(r)/d(r) > 0$, then the radius of starlikeness of the closed convex hull of M is at least as large as the first positive root of $a(r) - d(r) \sec b(r) = 0$.*

In the sequel, we let

$$H_\varepsilon = \left\{z + e^{i\lambda} \sum_{n=2}^{\infty} z^n \,\middle|\, |e^{i\lambda} - 1| \leqq \varepsilon < 1\right\} \tag{6}$$

and

$$G_\varepsilon = \left\{\frac{h(xz)}{x} \,\middle|\, h \in H_\varepsilon, |x| = 1\right\}. \tag{7}$$

We shall also need

LEMMA 3. *Set $a(r) = 1/(1 - r^2) - \varepsilon^2 r^2/(1 - \varepsilon^2 r^2)$ and $d(r) = r/(1 - r^2) + \varepsilon r/(1 - \varepsilon^2 r^2)$. For $g \in G_\varepsilon$, the values for $(zg'(z)/g(z))$, $|z| \leqq r$, lie in a disk centered at $a(r)$ and having radius $d(r)$.*

Proof. It suffices to prove the result for $h \in H_\varepsilon$ since the class G_ε consists of rotations of these functions. Writing $h(z) = (z + (e^{i\lambda} - 1)z^2)/(1 - z)$ for $h \in H_\varepsilon$, we see that $zh'(z)/h(z) = 1/(1 - z) + ((e^{i\lambda} - 1)z)/(1 + (e^{i\lambda} - 1)z)$. Since

$$\frac{-\varepsilon r}{1 - \varepsilon r} \leqq \operatorname{Re} \frac{(e^{i\lambda} - 1)z}{1 + (e^{i\lambda} - 1)z} \leqq \frac{\varepsilon r}{1 + \varepsilon r} \qquad (|z| \leqq r),$$

it follows that

$$\frac{1}{1+r} - \frac{\varepsilon r}{1-\varepsilon r} \leqq \operatorname{Re}\frac{zh'(z)}{h(z)} \leqq \frac{1}{1-r} + \frac{\varepsilon r}{1+\varepsilon r}\,. \tag{8}$$

Thus the values for $zh'(z)/h(z)$ are contained in a disk centered at

$$\frac{1}{2}\left[\left(\frac{1}{1-r} + \frac{\varepsilon r}{1+\varepsilon r}\right) + \left(\frac{1}{1+r} - \frac{\varepsilon r}{1-\varepsilon r}\right)\right] = a(r)$$

whose radius is

$$\frac{1}{2}\left[\left(\frac{1}{1-r} + \frac{\varepsilon r}{1+\varepsilon r}\right) - \left(\frac{1}{1+r} - \frac{\varepsilon r}{1-\varepsilon r}\right)\right] = d(r)\,.$$

LEMMA 4. *A function $g \in G_\varepsilon$ defined by* (7) *is starlike if and only if $\varepsilon \leqq 1/3$.*

Proof. It suffices to consider $h \in H_\varepsilon$ defined by (6). Letting $r \to 1$ in the left-hand side of (8), we see that $\operatorname{Re} zh'(z)/h(z) \geqq 0$ when $\varepsilon \leqq 1/3$. For $e^{i\lambda} - 1 = \varepsilon e^{i\sigma}$ and $\beta = \pi - \sigma$, we have $e^{i\beta}h'(e^{i\beta})/h(e^{i\beta}) = 1/2 - \varepsilon/(1-\varepsilon) < 0$ when $\varepsilon > 1/3$.

REMARK. Since the Pólya-Schoenberg conjecture is true [5], we know that for f convex and $h \in H_\varepsilon$ ($\varepsilon \leqq 1/3$), the Hadamard product $h * f$ is starlike. However if f is only required to be starlike, then $h_\lambda * f$ need not be in S for any h_λ defined by (4), $\lambda \neq 0$. To see this, observe that $h_\lambda * z/((1-z)^2) = z + e^{i\lambda}\sum_{n=2}^\infty nz^n$ is not in S for $\lambda \neq 0$ because $f(z) = z + \sum_{n=2}^\infty a_n z^n$, $|a_2| = 2$, is in S only if $f(z) = z/((1-xz)^2)$, $|x| = 1$.

We now give a bound for the radius of starlikeness for $h * f$, $f \in F$, $h \in H_\varepsilon$ with $\varepsilon \leqq 1/3$.

THEOREM 4. *Suppose $h(z) = z + e^{i\lambda}\sum_{n=2}^\infty z^n \in H_\varepsilon$, $\varepsilon \leqq 1/3$, and $f \in F$, with $a(r)$ and $d(r)$ defined in Lemma* 3. *Then $h * f$ is starlike in a disk $|z| < r_0$, where r_0 is the first positive root of*

$$a(r) - \frac{d(r)}{\sqrt{(1-\varepsilon^2 r^2)(1-r^2)} - \varepsilon r^2} = 0\,. \tag{9}$$

Proof. In view of (5),

$$(h * f)(z) = \int_X h * \frac{z}{1-xz}d\mu(x) = \int_X \frac{h(xz)}{x}d\mu(x)\,, \qquad |x| = 1\,.$$

By Lemma 4, the kernel functions are all starlike. An application of Lemma 3 to Lemma A shows that $h * f$ is starlike in a disk whose radius is at least as large as the first positive root of

$$a(r) - d(r) \sec b(r) = 0 \; , \tag{10}$$

where

$$b(r) = \max_{|z|=r} \left\{ \arg \frac{g(z)}{z} \,\middle|\, g \in G_\varepsilon \right\} .$$

Since

$$\begin{aligned} b(r) &\leqq \max_{|z|=r} |\arg (1 + \varepsilon z)| + \max_{|z|=r} |\arg (1 - z)| \\ &= \sin^{-1}(\varepsilon r) + \sin^{-1}(r) = t(r) \; , \end{aligned}$$

$$\sec b(r) \leqq \sec t(r) = \frac{1}{\cos t(r)} = \frac{1}{\sqrt{(1 - \varepsilon^2 r^2)(1 - r^2)} - \varepsilon r^2} . \tag{11}$$

A substitution of the right-hand side of (11) into (10) yields the desired result.

REMARK. If $\varepsilon = 0$ then (9) reduces to $1/(1 - r^2) - r/((1 - r^2)^{3/2}) = 0$, whose smallest positive root is $1/\sqrt{2}$. This coincides with a sharp result of MacGregor [4].

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