PACIFIC JOURNAL OF MATHEMATICS
Vol. 77, No. 2, 1978

A GENERAL COINCIDENCE THEORY

BENJAMIN HALPERN

Consider two topological spaces X and Y, two maps $f, g: X \to Y$, and a relation R on $Y, R \subset Y \times Y$. A Lefschetz-type theorem is established with regard to the existence of an $x \in X$ such that $g(x)Rf(x)$. Several ideas related to the Lefschetz Fixed Point Theorem, such as periodic point theorems, local fixed point index, asymptotic fixed point and periodic point theorems, are carried over to this more general situation.

1. Introduction. Consider two topological spaces X and Y, two maps $f, g: X \to Y$, and a relation R on $Y, R \subset Y \times Y$. Under what circumstances must there exist an $x \in X$ such that $g(x)Rf(x)$, i.e., $(g(x), f(x)) \in R$. In this paper we do three things with respect to this problem. First, we show how to obtain related homology maps from $H_*(Y)$ into itself. Each of these homology maps h is such that $\Lambda(h) = \sum_{n \geq 0} (-1)^n$ trace $h_n \neq 0$ implies $g(x)Rf(x)$ for some $x \in X$. Secondly, we find an interpretation in terms of f and g for the condition $\Lambda(h^r) \neq 0$, where $h^r = h \circ h \circ \cdots \circ h$, r times. We show that if $\Lambda(h^r) \neq 0$, then there exist points $x_1, x_2, \cdots, x_r \in X$ such that $g(x_{i+1})Rf(x_i)$ for $1 \leqq i < r$ and $g(x_1)Rf(x_r)$. Thirdly, we show that there are many situations in which one can assert that $\Lambda(h^r) \neq 0$ for some $r \leqq N$, where N is determined by the situation at hand.

One application is the following apparently new theorem about spheres. Theorem 4.5: If $f, g: S^{2n} \to S^{2n}$ are continuous maps such that $f(A) \neq g(A)$ for all $A \subset S^{2n}$ with cardinality of $A = 1$ or 2, then both f and g are null homotopic. Another application concerns lines in the complex projective plane as follows. Remark 4.9: If L is a line in CP^2 and f is a continuous map from L into the space of lines in CP^2, then there exists two points $a, b \in L$ such that $f(a)$ goes through b and $f(b)$ goes through a.

Section 2 contains some notation and conventions. In § 3 we establish our central theorems. Section 4 contains applications. In § 5 we consider extensions to more than two functions. We outline a local index theory in § 6. In § 7 we establish some asymptotic theorems. In the last section, § 8, we discuss a very general situation involving four spaces, two maps, and two relations.

Several authors have dealt with the problems of fixed points and coincidence points by similar methods. Fuller [8], Fadell [7], Brown [3] and [4], and Roitberg [15] deal with fixed points and coincidence points on manifolds. In [13] Lefschetz considers coincidence points for compact metric ANR's. Lefschetz obtains a theorem involving the Lefschetz number of certain cycle mappings which depend on choices of cycles in the graphs of extensions of the transformations. This result seems to be related to Theorem 8.1 below. But, the transformations considered in [13] are not assumed to be continuous, nor single-valued, nor defined for all points of the domain. Hence, no attempt is made in [13] to relate these cycle mappings to induced homology or cohomology maps of the transformations.

The author would like to thank Professor Robert Brown for his suggestion which led to this work.

2. Notation. We list here some notations and conventions we use throughout the paper. We use singular homology and cohomology. We use the notation and conventions of Spanier [16] for the various products in homology and cohomology. Unless explicitely noted otherwise, the coefficients are taken in a fixed field F. We denote the integers by $\boldsymbol{Z}$, the reals by $\boldsymbol{R}$, and the complex numbers by $\boldsymbol{C}$. By "map" we mean continuous function. $\# A$ denotes the cardinality of the set A.

3. Main theorems. We establish here the central result of this paper, Theorem 3.2. For purposes of exposition we find it convenient to prove a special case, Theorem 3.1, first. (Actually Theorem 3.1 is not strictly a special case of Theorem 3.2 because R is not assumed closed.)

Let X and Y be two topological spaces, and $f, g\colon X \to Y$ two continuous maps, and R a relation on Y, $R \subset Y \times Y$. Suppose n is a nonnegative integer. Let $a \in H_n(X)$, and $b \in H^n(Y \times Y)$ be such that $\varphi^*(b) = 0$ where $\varphi\colon Y \times Y - R \to Y \times Y$ is the inclusion map. Let $\bar{h}_i\colon H_i(Y) \to H_i(Y)$ be the composite

$$\begin{array}{ccc} H^{n-i}(Y) & \xrightarrow{g^*} & H^{n-i}(X) & & \\ \uparrow b/ & & \downarrow \cap a & & \\ H_i(Y) & & H_i(X) & \xrightarrow{f_*} & H_i(Y) \end{array}$$

where $b/$ and $\cap\, a$ are the maps such that $(b/)(z) = b/z$ for $z \in H_i(Y)$ and $(\cap\, a)(z) = z \cap a$ for $z \in H^{n-i}(x)$. Set $h_i = (-1)^{ni}\bar{h}_i$. Assume further that $H_i(Y)$ is a finite dimensional vector spaces over F for each $i \geqq 0$.

For such a, b define a Lefschetz R-number by $L(f, g) = \Lambda(h) =$

$\Sigma(-1)^i$ trace h_i.

THEOREM 3.1. *If some Lefschetz R-number* $L(f, g) \neq 0$, *then* $g(x)Rf(x)$ *for some* $x \in X$.

Proof. Consider the following diagram

$$\begin{array}{ccccc} X & \multicolumn{3}{c}{\overset{\lambda}{\dashrightarrow}} & Y \times Y - R \\ {\scriptstyle d}\downarrow & & & & \downarrow{\scriptstyle \varphi} \\ X \times X & \xrightarrow{f \times g} & Y \times Y & \xrightarrow{T} & Y \times Y \end{array}$$

where $d(x) = (x, x)$ is the diagonal map, and $T(y, y') = (y', y)$ is the interchange map. If there is no $x \in X$ such that $g(x)Rf(x)$, then we may define a map $\lambda: X \to Y \times Y - R$ by setting $\lambda(x) = (g(x), f(x))$, and the above diagram would commute. It would then follow that $b' = \lambda^* \circ \varphi^*(b) = 0$ since $\varphi^*(b) = 0$ by hypothesis. Hence, to prove the theorem it is sufficient to show that $\langle b', a \rangle = \Lambda(h)$.

Let $\{\alpha_i\}$, $\{\beta_i\}$, and $\{\gamma_i\}$ be bases for $H_*(X)$, $H_*(Y)$, and $H^*(X)$ respectively. Let $\{\hat{\beta}_i\}$ be the basis for $H^*(Y)$ dual to $\{\beta_i\}$, i.e., $\langle \hat{\beta}_i, \beta_j \rangle = \delta_{ij}$. Define f_{ij}, g_{ij}, a_{ij}, b_{ij}, and $\bar{b}_{ij}$ by requiring

$$\begin{aligned} f_*(\alpha_i) &= \textstyle\sum_j f_{ij}\beta_j \\ g^*(\hat{\beta}_i) &= \textstyle\sum_j g_{ij}\gamma_j \\ \gamma_i \cap a &= \textstyle\sum_j a_{ij}\alpha_j \\ b/\beta_i &= \textstyle\sum_j b_{ij}\hat{\beta}_j \\ b &= \textstyle\sum_{ij} \bar{b}_{ij}\hat{\beta}_i \times \hat{\beta}_j \,. \end{aligned}$$

We need one preliminary fact before we show $\langle b', a \rangle = \Lambda(h)$. We first show that $\bar{b}_{ij} = b_{ji}$. We simply calculate

$$\begin{aligned} \langle b/\beta_k, \beta_l \rangle &= \langle b, \beta_l \times \beta_k \rangle \\ &= \sum_{ij} \bar{b}_{ij} \langle \hat{\beta}_i \times \hat{\beta}_j, \beta_l \times \beta_k \rangle \\ &= \sum_{ij} \bar{b}_{ij}\delta_{il}\delta_{jk} = \bar{b}_{lk} \,. \end{aligned}$$

On the other hand

$$\begin{aligned} \langle b/\beta_k, \beta_l \rangle &= \sum_m b_{km} \langle \hat{\beta}_m, \beta_l \rangle \\ &= \sum_m b_{km}\delta_{ml} = b_{kl} \,. \end{aligned}$$

Hence $\bar{b}_{ij} = b_{ji}$ as claimed.

We now observe that

$$\begin{aligned}\langle b', a\rangle &= \langle d^*\circ(f\times g)^*\circ T^* \sum_{ij} b_{ji}\hat{\beta}_i \times \hat{\beta}_j, a\rangle \\ &= \sum_{ij} \sigma(i) b_{ji} \langle f^*(\hat{\beta}_j) \cup g^*(\hat{\beta}_i), a\rangle\end{aligned}$$

where $\sigma(i) = (-1)^{\dim\hat{\beta}_i \cdot \dim\hat{\beta}_j}$. Hence,

$$\begin{aligned}\langle b', a\rangle &= \sum_{ij} \sigma(i) b_{ji} \langle f^*(\hat{\beta}_j) \cup \sum_k g_{ik}\gamma_k, a\rangle \\ &= \sum_{ijk} \sigma(i) b_{ji} g_{ik} \langle f^*(\hat{\beta}_j)\,, \quad \gamma_k \cap a\rangle \\ &= \sum_{ijk} \sigma(i) b_{ji} g_{ik} \langle f^*(\hat{\beta}_j)\,, \quad \sum_l a_{kl}\alpha_l\rangle \\ &= \sum_{ijkl} \sigma(i) b_{ji} g_{ik} a_{kl} \langle \hat{\beta}_j, f_*\alpha_l\rangle \\ &= \sum_{ijklp} \sigma(i) b_{ji} g_{ik} a_{kl} f_{lp} \langle \hat{\beta}_j, \beta_p\rangle \\ &= \sum_{ijkl} \sigma(i) b_{ji} g_{ik} a_{kl} f_{lj} \\ &= \sum_p \sum_{\substack{j \text{ such that} \\ \dim \beta_j = p}} \sum_{ijk} \sigma(i) b_{ji} g_{ik} a_{kl} f_{lj}\,.\end{aligned}$$

Note that if $\dim \beta_j = p$ and $\dim \hat{\beta}_i \neq n - p$, then $b_{ji} = 0$. Hence, in the above expression for $\langle b', a\rangle$ we may replace

$$\sigma(i) \equiv (-1)^{\dim\hat{\beta}_i \cdot \dim\hat{\beta}_j}$$

by $(-1)^{(n-p)p}$. Thus,

$$\begin{aligned}\langle b', a\rangle &= \sum_p (-1)^{(n-p)p} \sum_{\{j \mid \dim\beta_j = p\}} \sum_{ijk} b_{ji} g_{ik} a_{kl} f_{lj} \\ &= \sum_p (-1)^{(n-p)p} \operatorname{trace} \bar{h}_p \\ &= \sum_p (-1)^{(n-p)p} (-1)^{np} \operatorname{trace} h_p \\ &= \Lambda(h)\,.\end{aligned}$$

Next we will be concerned with two sequences of functions. In order to facilitate our notation we will index the spaces and functions by elements of Z_m, the cyclic group of order m. We consider z_m to consist of the integers $1, 2, \cdots, m$, so that $m = 0$ in Z_m. Let $n \geqq 0$ be a fixed integer. Suppose that for each $i \in Z_m$, X_i and Y_i are topological spaces, $f^i: X_i \to Y_i$ and $g^i: X_i \to Y_{i-1}$ are continuous functions, $R_i \subset Y_i \times Y_i$ is a relation on Y_i, $a^i \in H_n(X_i)$, and $b^i \in H^n(Y_i \times Y_i)$ with $\varphi^{i*}(b^i) = 0$ where $\varphi^i: Y_i \times Y_i - R_i \to Y_i \times Y_i$ is the inclusion map. Let $\bar{h}_j: H_j(Y_1) \to H_j(Y_1)$ be the composite

$$\begin{array}{ccccccc} H^{n-j}(Y_1) \xrightarrow{g^{2*}} H^{n-j}(X_2) & & H^{n-j}(Y_m) \xrightarrow{g^{1*}} H^{n-j}(X_1) \\ b^1/\uparrow \qquad\qquad \downarrow \cap a^2 & & b^m/\uparrow \qquad\qquad \downarrow \cap a^1 \\ H_j(Y_1) \qquad H_j(X_2) \xrightarrow{f_*^2} \cdots \xrightarrow{f_*^m} H_j(Y_m) & & H_j(X_1) \xrightarrow{f_*^1} H_j(Y_1) . \end{array}$$

Set $h_j = (-1)^{nmj}\bar{h}_j$.

Also assume $\dim H_j(Y_i) < \infty$ for all $j \geqq 0$ and $i \in Z_m$.

We will be using the cohomology cross product for the Cartesian products of several spaces. In order to insure that the excesiveness conditions (necessary for the definition and properties of the cross product) are always satisfied, we will assume that R_i is closed in $Y_i \times Y_i$ for each $i \in Z_m$.

Under these conditions we define a generalized Lefschetz R-number by

$$L(\{f^i\}, \{g^i\}) = \Lambda(h) = \Sigma(-1)^j \text{ trace } h_j .$$

THEOREM 3.2. *If some generalized Lefschetz R-number* $L(\{f^i\}, \{g^i\}) \neq 0$, *then there are points* $x_i \in X_i$ *such that* $g^{i+1}(x_{i+1})R_i f^i(x_i)$ *for* $i \in \boldsymbol{Z}_m$.

Proof. First we establish some notation. If (A, A') and (B, B') are two pairs of spaces then $(A, A') \times (B, B') = (A \times B, A \times B' \cup A' \times B)$, and $\operatorname{rel}(A, A') = A'$. If a collection of objects $\{A_i\}$ are indexed by the integers $i = 1, 2, \cdots, q$, then $\prod A_i = A_q \times A_{q-1} \times \cdots \times A_2 \times A_1$. Note the order. This notation is used for the various interpretations of $\times$, e.g., multiplication in F, Cartesian product of spaces or pairs of spaces, and cross product in homology and cohomology. Consider the following diagram

$$\begin{array}{ccc} \prod X_i & \xrightarrow{\lambda} & \operatorname{rel} \prod (Y_i \times Y_i, Y_i \times Y_i - R_i) \\ \prod d^i \downarrow & & \downarrow \varphi \\ \prod X_i \times X_i \xrightarrow{\prod f^i \times g^i} \prod Y_i \times Y_{i-1} & \xrightarrow{T} & \prod Y_i \times Y_i \\ & & \downarrow \psi \\ & & \prod (Y_i \times Y_i, Y_i \times Y_i - R_i) \end{array}$$

where $d^i: X_i \to X_i \times X_i$ is the diagonal map, $d^i(x) = (x, x)$ for $x \in X_i$, $T(\prod y_i \times y'_{i-1}) = \prod y'_i \times y_i \in \prod Y_i \times Y_i$ for $\prod y_i \times y'_{i-1} \in \prod Y_i \times Y_{i-1}$, and φ and ψ are inclusion maps. Set $b = \prod b^i \in H^{nm}(\prod Y_i \times Y_i)$. We claim that $\varphi^*(b) = 0$. It is sufficient to show that $b = \psi^*(c)$ for some $c \in H^{nm} \prod (Y_i \times Y_i, Y_i \times Y_i - R_i)$. Since $\varphi^{i*}(b^i) = 0$, there exists a $c^i \in H^n(Y_i \times Y_i, Y_i \times Y_i - R_i)$ such that $b^i = \psi^{i*}(c^i)$ where $\psi^i: Y_i \times Y_i \to (Y_i \times Y_i, Y_i \times Y_i - R_i)$ is the inclusion map. Since

$\psi = \prod \psi^i$, we may set $c = \prod c^i$ and then $\psi^*(c) = \prod \psi^{i*}(c^i) = \prod b^i = b$. This establishes the claim that $\varphi^*(b) = 0$.

Now we reason as in the proof of Theorem 3.1. Consider the element $b' = (\prod d^i)^* \circ (\prod f^i \times g^i)^* \circ T^*(b) \in H^{nm}(\prod X_i)$. If there is no sequence $x_i \in X_i$, $i \in Z_m$, such that $g(x_{i+1}) R_i f(x_i)$ for $i \in Z_m$, then we may define a map $\lambda\colon \prod X_i \to \operatorname{rel} \prod (Y_i \times Y_i, Y_i \times Y_i - R_i)$ by setting $\lambda(\prod x_i) = \prod g^{i+1}(x_{i+1}) \times f^i(x_i)$. The above diagram would then commute. It would then follow that $b' = \lambda^* \circ \varphi^*(b) = 0$ since $\varphi^*(b) = 0$. Hence, to prove the theorem it is sufficient to show that $(-1)^{n(m-1)}\langle b', a\rangle = \Lambda(h)$, where $a = \prod a^i \in H_{nm}(\prod X_i)$.

For each $i \in Z_m$, let $\{\alpha_j^i\}$, $\{\beta_j^i\}$, and $\{\gamma_j^i\}$ be bases for $H_*(X_i)$, $H_*(Y_i)$, and $H^*(X_i)$ respectively. Let $\{\hat\beta_j^i\}$ be the basis for $H^*(Y_i)$ dual to $\{\beta_j^i\}$, i.e., $\langle \hat\beta_j^i, \beta_k^i\rangle = \delta_{jk}$. Define f_{jk}^i, g_{jk}^i, a_{jk}^i, b_{jk}^i, and $\bar b_{jk}^i$ by requiring

$$f_*^i(\alpha_j^i) = \sum_k f_{jk}^i \beta_k^i$$

$$g^{i*}(\hat\beta_j^{i-1}) = \sum_k g_{jk}^i \gamma_k^i$$

$$\gamma_j^i \cap a^i = \sum_k a_{jk}^i \alpha_k$$

$$b^i/\beta_j^i = \sum_k b_{jk}^i \hat\beta_k^i$$

$$b^i = \sum_{jk} \bar b_{jk}^i \hat\beta_j^i \times \hat\beta_k^i\ .$$

We have already established in the proof of Theorem 1 that $\bar b_{jk}^i = b_{kj}^i$.

Using the associative and commutative laws for the cross product we get

$$(\prod b_{k_i j_i}^i) T^* \prod \hat\beta_{j_i}^i \times \hat\beta_{k_i}^i = (\prod b_{k_i j_i}^i)\sigma(\dim \hat\beta_{j_m}^m) \prod \hat\beta_{k_i}^i \times \hat\beta_{j_{i-1}}^{i-1}$$

where $\sigma(p) = (-1)^{p[n-p+n(m-1)]}$. Here we have made use of the fact that $(\prod b_{k_i j_i}^i) \neq 0$ implies $\dim \hat\beta_{j_i}^i + \dim \hat\beta_{k_i}^i = n$ for all $i \in Z_m$. T^* simply takes $\hat\beta_{j_m}^m$ from the far left position to the far right position, commuting past $\hat\beta_{k_m}^m \times \prod_1^{m-1} \hat\beta_{j_i}^i \times \hat\beta_{k_i}^i$ of homological dimension $n - \dim \hat\beta_{j_m}^m + n(m-1)$.

We now proceed as in the proof of Theorem 3.1.

$$\begin{aligned}\langle b', a\rangle &= \langle (\prod d^i)^* \circ (\prod f^i \times g^i)^* \circ T^*(\prod \sum_{k_i, j_i} b_{k_i, j_i}^i \hat\beta_{j_i}^i \times \hat\beta_{k_i}^i)\,,\quad \prod a^i\rangle \\ &= \sum_{\{k_i, j_i, l_i, p_i \mid i \in Z_m\}} \prod \sigma(\dim \hat\beta_{j_m}^m) b_{k_i j_i}^i g_{j_{i-1} l_i}^i a_{l_i p_i}^i f_{p_i k_i}^i \\ &= \sum_{\{k_i, j_i, l_i, p_i \mid i \in Z_m\}} \prod \sigma(n - \dim \beta_{k_1}^1) b_{k_i j_i}^i g_{j_i l_{i+1}}^{i+1} a_{l_{i+1} p_{i+1}}^{i+1} f_{p_{i+1} k_{i+1}}^{i+1}\ .\end{aligned}$$

In the last equality we have used the fact that if $\dim \hat\beta_{j_m}^m \neq$

dim $\gamma^1_{l_1}$, then $g^1_{j_m l_1} = 0$, and if dim $\gamma^1_{l_1} \neq n$-dim $\alpha^1_{p_1}$, then $a^1_{l_1 p_1} = 0$, and if dim $\alpha^1_{p_1} \neq$ dim $\beta^1_{k_1}$, then $f^1_{p_1 k_1} = 0$. Hence, by summing over k_1 with dim $\beta^1_{k_1} = r$ first, we find

$$\langle b', a\rangle = \sum_r (-1)^{(n-r)(r+n(m-1))} \operatorname{trace} \bar{h}_r ,$$

and so

$$(-1)^{n(m-1)}\langle b', a\rangle = \sum_r (-1)^r \operatorname{trace} h_r = \Lambda(h) .$$

4. Applications. First we note some immediate specializations of Theorem 3.1. Suppose X and Y are compact, closed, oriented n-dimensional topological manifolds, and $a \in H_n(X)$ is the fundamental class for X, and $b \in H^n(Y \times Y)$ is the image of the Thom class of Y under the map $\psi^*\colon H^n(Y \times Y,\ Y \times Y - \Delta Y) \to H^n(Y \times Y)$ where ΔY is the diagonal of Y and $\psi\colon Y \times Y \to (Y \times Y,\ Y \times Y - \Delta Y)$ is the inclusion. Then $\Lambda(h) = \Lambda(f_*\circ(\cap a)\circ g^*\circ(b/))$ is the Lefschetz number for f and g and Theorem 1 with $R = \Delta Y$ reduces to the Lefschetz's coincidence point theorem. If we specialize further and take $Y = X$ and $g = 1_x$, then we obtain the Lefschetz fixed point theorem for oriented manifolds.

We now discuss some consequences of Theorem 3.2. Suppose in Theorem 3.2 we have $m = pq$, $X_{i+q} = X_i$, $Y_{i+q} = Y_i$, $f^{i+q} = f^i$, $g^{i+q} = g^i$, $a^{i+q} = a^i$, and $b^{i+q} = b^i$ for all $i \in Z_m$. Now let $\bar{k}_r\colon H_r(Y_1) \to H_r(Y_1)$ be the composite

$$\begin{array}{ccccccc}
H^{n-r}(Y_1) & \xrightarrow{g^{2*}} & H^{n-r}(X_2) & & H^{n-r}(Y_q) & \longrightarrow & H^{n-r}(X_{q+1}) \\
{\scriptstyle b^1/}\uparrow & & \downarrow{\scriptstyle \cap a^2} & & {\scriptstyle b^q/}\uparrow & & \downarrow{\scriptstyle \cap a^{q+1}} \\
H_r(Y_1) & & H_r(X_2) \xrightarrow{f_*^2} \cdots & \xrightarrow{f_*^q} & H_r(Y_q) & & H_r(X_{q+1}) \xrightarrow{f_*^{q+1}} H_r(Y_{q+1}) = H_r(Y_1) .
\end{array}$$

Set $k_r = (-1)^{nqr}\bar{k}_r$. Clearly, for the $\bar{h}_r$ of Theorem 2 we have $\bar{h}_r = (\bar{k}_r)^p = \bar{k}_r\circ\bar{k}_r\circ\cdots\circ\bar{k}_r$, p times. Hence, $h_r = (-1)^{npqr}\bar{h}_r = ((-1)^{nqr}\bar{k}_r)^p = k_r^p$, and so $h = k^p\colon H_*(Y_1) \to H_*(Y_1)$. Now we may apply the theory developed in [9], [10], and [12] for dealing with the Lefschetz's numbers of iterates. We will use Theorem 4 of]10] to illustrate how most of the theorems of [10] may be carried over to the present context. In algebraic terms, Theorem 4 of [10] is essentially the following result. Let ch $\mathscr{F}$ denote the characteristic of $\mathscr{F}$.

THEOREM 4.1. *Given a finite sequence* $V = \{V_1, \cdots, V_N\}$ *of finite dimensional vector spaces over* $\mathscr{F}$, *and* $T\colon V \to V$ *a sequence of linear maps* $T_i\colon V_i \to V_i$. *If*

(a) T_i *has a nonzero eigenvalue for some even* i, *and*

(b) $T_i = 0$ *for all odd* i,

(c) *ch* $\mathscr{F} \notin \{2, \cdots, n\}$, *where* $n = \sum_{i \text{ even}} \dim V_i$

then $\Lambda(T^j) = \sum_i (-1)^i$ *trace* $T_i^j \neq 0$ *for some* j, $1 \leqq j \leqq \sum_{i \text{ even}} \dim V_i$.

THEOREM 4.2. *Given a topological space* X *and two maps* $f, g: X \to S^n$ *where* $n \geqq 2$ *is even. Either*

(a) *both* f^* *and* $g^*: H^n(S^n: \mathscr{F}) \to H^n(X; \mathscr{F})$ *are zero for all fields* $\mathscr{F}$ *such that* $\mathrm{ch}\, \mathscr{F} \neq 1$ *or* 2, *or*

(b) $f(X') = g(X')$ *for some* $X' \subset X$ *with* $\# X' = 1$ *or* 2.

Proof. Without loss of generality we may assume X is connected. Suppose that $g^*: H^n(S^n) \to H^n(X)$ is not zero with coefficients in a field $\mathscr{F}$. Set $b_1 = j^*(\tau)$ where $\tau \in H^n(S^n \times S^n, S^n \times S^n - \Delta)$ is the Thom class of S^n and $j: S^n \times S^n \to (S^n \times S^n, S^n \times S^n - \Delta)$ is the inclusion map. Let s_0 be the canonical generator of $H_0(S^n) \cong \boldsymbol{Z}$. Then b_1/s_0 is a generator of $H^n(S^n) \cong \boldsymbol{Z}$. Consequently $g^*(b_1/s_0) \in H^n(X)$ is not zero. Hence $\langle g^*(b_1/s_0), a\rangle \neq 0$ for some $a \in H_n(X)$. Let a_1 be such an a. Let x_0 be the cannonical generator of $H_0(X) \cong \boldsymbol{Z}$ and $\hat{x}_0$ the dual generator of $H^0(X)$. Now we have $0 \neq \langle g^*(b_1/s_0), a_1\rangle = \langle \hat{x}_0 \cup g^*(b_1/s_0)\rangle = \langle \hat{x}_0, (g^*(b_1/s_0)) \cap a_1\rangle$. Hence $(g^*(b_1/s_0)) \cap a_1 = tx_0$ with $t \neq 0$.

Let k be defined as in the paragraph preceding Theorem 4.1. Then, $k_0(s_0) = f_* \circ (\cap a_1) \circ g^* \circ (b_1/)(s_0) = f_*(tx_0) = ts_0$ and hence $k_0 \neq 0$.

It is clear now that Theorem 4.1 applies with $V_i = H_i(S^n)$ and $T = k$. It follows that either $\Lambda(k) \neq 0$ or $\Lambda(k^2) \neq 0$. Hence (b) holds.

Similarly, if $f^*: H^n(S^n) \to H^n(X)$ is not zero for some field of coefficients, then (b) holds.

We would like to use Hopf's theorem on homotopy classes of maps into S^n to interpret geometrically condition (a) of Theorem 2.4.

REMARK 4.3. We will need the following elementary algebraic fact whose proof we leave to the reader.

If an Abelian group G, homomorphism $f: \boldsymbol{Z} \to G$, and integer $n \geqq 2$ satisfy

(a) G has no element of order (prime)^2,

(b) G has no element of order 2 or 3 or $\cdots$ or n,

(c) $f \otimes 1: \boldsymbol{Z} \otimes \mathscr{F} \to G \otimes \mathscr{F}$ is zero for all fields $\mathscr{F}$ which satisfy $\mathrm{ch}\, \mathscr{F} \notin \{2, 3, \cdots, n\}$,

then f is zero.

THEOREM 4.4. *Given an even integer* $n \geqq 2$ *and a topological space* X *which has the same homotopy type as an* n*-dimensional CW-complex, and two maps* $f, g: X \to S^n$. *Assume* $H^n(X; \boldsymbol{Z})$ *has no* *element of order (prime)^2, nor of order 2. Either*

(a) *both f and g are null homotopic*

or

(b) $f(X') = g(X')$ *for some* $X' \subset X$ *with* $\sharp X' = 1$ *or* 2.

Proof. Assume (b) does not hold.

We apply Theorem 4.2 and conclude that f^* and $g^*\colon H^n(S^n; \mathscr{F}) \to H^n(X; \mathscr{F})$ is zero for all fields $\mathscr{F}$ such that $\operatorname{ch} \mathscr{F} \neq 2$. Since $H^{n+1}(X; \boldsymbol{Z}) = 0$, the universal coefficient theorem which expresses $H^*(\cdot\,; \mathscr{F})$ in terms of $H^*(\cdot\,; \boldsymbol{Z})$ implies $f_Z^* \otimes 1$, $g_Z^* \otimes 1\colon H^n(S^n; \boldsymbol{Z}) \otimes \mathscr{F} \to H^n(X; \boldsymbol{Z}) \otimes \mathscr{F}$ are zero for all fields $\mathscr{F}$ such that $\operatorname{ch} \mathscr{F} \neq 2$ where f_Z^*, $g_Z^*\colon H^n(S^n; \boldsymbol{Z}) \to H^n(X; \boldsymbol{Z})$ are the induced cohomology map with integer coefficients. Now from Remark 4.3 we see that f_Z^* and g_Z^* are zero. Finally, both f and g are null homotopic by Hopf's theorem.

COROLLARY 4.5. *If $f, g\colon S^{2n} \to S^{2n}$ are continuous maps such that $f(A) \neq g(A)$ for all $A \subset S^{2n}$ with $\sharp A = 1$ or 2, then both f and g are null homotopic.*

Proof. Set $X = S^{2n}$ in Theorem 4.4 and consider the case $n=0$ separately.

REMARK. Responding to a preprint of this paper Professor Dold has given an elegant short proof of Corollary 4.5 relying only on properties of the cross product and intersection number.

Next we present an application of Theorem 3.2 where the relation R is not taken to be equality. Let $Y = \boldsymbol{C}P^2$ = the complex projective plane. We consider $\boldsymbol{C}P^2$ to be defined as the quotient of $\boldsymbol{C}^3 - \{0\}$ by the equivalence relation $\sim$ which sets $(u, v, w) \sim (\lambda u, \lambda v, \lambda w)$ for all $(u, v, w) \in \boldsymbol{C}^3 - \{0\}$ and $\lambda \in \boldsymbol{C} - \{0\}$. Denote the equivalence class of (u, v, w) by $[u, v, w]$. We say $U = [u, v, w]$ is perpendicular to $U' = [u', v', w']$, (notations: $U \perp U'$), provided $uu' + vv' + ww' = 0$. This notion is well defined, i.e., it does not depend on the choices of (u, v, w) and (u', v', w') representing U and U'.

THEOREM 4.6. *Given a topological space X and two maps $f, g\colon X \to \boldsymbol{C}P^2$. Either*

(a) *both $f^*, g^*\colon H^2(\boldsymbol{C}P^2; \mathscr{F}) \to H^2(X; \mathscr{F})$ are zero for all fields $\mathscr{F}$ such that $\operatorname{ch} \mathscr{F} \neq 2$,*

or

(b) $f(x) \perp g(x)$ *for some* $x \in X$,

or

(c) *$f(x) \perp g(x')$ and $f(x') \perp g(x)$ for some $x, x' \in X$.*

Proof. By considering f and g restricted to the components of X it is readily seen that the theorem is reduced to the case where X is connected. So assume X is connected. We wish to apply Theorem 3.2 as we did in the proof of Theorem 4.2. Here we have $R = \{(U, U') \in Y \times Y \mid U \perp U'\}$, where $Y = \boldsymbol{CP}^2$. The only significant difference between the present proof and the proof of Theorem 4.2 is in finding an appropriate $b_1 \in H^2(Y \times Y)$. We consider this point and leave the rest of the proof to the reader. What we need is a $b_1 \in H^2(Y \times Y)$ such that

(i) $b_1/s_0 = s^2$ where s_0 is the cannonical generator of $H_0(Y) \cong \mathscr{F}$ and s^2 is a generator of $H^2(Y) \cong \mathscr{F}$.

(ii) $i^*(b_1) = 0$ where $i: Y \times Y - R \to Y \times Y$ is the inclusion map.

It is well known that $\boldsymbol{CP}^2$ is a complex analytic manifold of complex dimension 2. It is easily verified that $R \subset Y \times Y$ is a compact connected complex analytic submanifold of complex dimension 3. Hence $Y \times Y$ and R are orientable smooth manifolds of (real) dimensions 8 and 6 respectively. Pick orientations for Y and R so that now $Y \times Y$ and R are oriented manifolds. Let $V = Y \times Y$. The Thom Isomorphism Theorem and the Tubular Neighborhood Theorem give a generator $r \in H_2(V, V - R; \mathscr{F}) \cong H_0(R; \mathscr{F}) \cong \mathscr{F}$ with the following property. [See Milnor [14] pages 67 – 69 for definitions and proofs. Though Milnor uses homology with $\boldsymbol{Z}$ coefficients in [14], the relavent part, pages 67 – 69, may be done using homology with $\mathscr{F}$ coefficients.] If M is a compact oriented submanifold of V of (real dimension 2 which intersects R transversally, and $[M] \in H_2(M; \mathscr{F})$ is its fundamental class, and $h: M \to V$ and $j: V \to (V, V - R)$ are the inclusion maps, then $j_* \circ h_*[M] = R \cdot Mr$, where $R \cdot M$ is the intersection number of R and M.

We apply the above formula to the case $M = S^2 \times p_0$, where we consider $S^2 = \{[u, v, w] \in \boldsymbol{CP}^2 \mid w = 0\}$ and $p_0 = \{[u, v, w] \in \boldsymbol{CP}^2 \mid v = w = 0\}$. By a straight forward calculation one may verify that $S^2 \times p_0$ intersects R in exactly one point, $([0, 1, 0], [1, 0, 0]) \in Y \times Y$, and this intersection is transverse. Hence $R \cdot (S^2 \times p_0) = \pm 1$ depending on the orientation we choose for $S^2 \times p_0$. Choose the orientation of $S^2 = S^2 \times p_0$ such that $R \cdot (S^2 \times p_0) = 1$. Now we have $j_* \circ h_*[S^2 \times p_0] = r$. Let $\hat{r}$ be the generator of $H^2(V, V - R; \mathscr{F}) \cong \mathscr{F}$ dual to r. Set $b_1 = j^*(\hat{r}) \in H^2(Y \times Y)$. Condition (ii) for b_1 follows from the exact sequence for the pair $(Y \times Y, Y \times Y - R)$.

Next note that

$$\langle b_1, h_*[S^2 \times p_0]\rangle = \langle j^*(\hat{r}), h_*[S^2 \times p_0]\rangle = \langle \hat{r}, j_* \circ h_*[S^2 \times p_0]\rangle$$
$$= \langle \hat{r}, r\rangle = 1 .$$

Since CP^2 can be written as $CP^2 = S^2 \cup$ (a 4-cell) and $h: S^2 \times p_0 \to CP^2 \times CP^2$ is the inclusion, we have $h_*[S^2 \times p_0] = s_2 \times s_0 \in H_2(Y \times Y; \mathscr{F})$ where s_2 is a generator of $H_2(Y; \mathscr{F}) \cong \mathscr{F}$. Let s^2 be the generator of $H^2(Y; \mathscr{F})$ dual to s_2. Then, $\langle b_1/s_0, s_2\rangle = \langle b_1, s_2 \times s_0\rangle = \langle b_1, h_*[S^2 \times p_0]\rangle = 1$. This shows that $b_1/s_0 = s^2$. We have thus verified condition (i) for b_1. The rest of the proof follows closely the proof of Theorem 4.2 and is left to the reader.

Next we prove an analog of Theorem 4.4.

THEOREM 4.7. *Given a topological space X which has the same homotopy type as a 2-dimensional CW-complex, and two maps $f, g: X \to CP^2$. Assume $H^n(X; Z)$ has no element of order (prime)^2, nor of order 2.*

Either

(a) both f and g are null homotopic

or

(b) $f(x) \perp g(x)$ for some $x \in X$

or

(c) $f(x) \perp g(x')$ and $f(x') \perp g(x)$ for some $x, x' \in X$.

Proof. Assume both (b) and (c) fail to hold.

Just as in the proof of Theorem 4.4 we can conclude that $f^*, g^*: H^2(CP^2; Z) \to H^2(X)$ are zero. Write $CP^2 = S^2 \cup$ (a 4-cell) and let $\psi: S^2 \to CP^2$ be the corresponding inclusion map. Then $\psi^*: H^2(CP^2; Z) \to H^2(S^2; Z)$ is an isomorphism. By cellular approximation $f \sim \psi \circ f'$ and $g \sim \psi \circ g'$ for some $f', g': X \to S^2$. Then $0 = f^* = f'^* \circ \psi^*$ and $0 = g^* = g'^* \circ \psi^*$ and so $f'^* = g'^* = 0$. Thus f' and g' are null homotopic and consequently f and g are also null homotopic.

COROLLARY 4.8. *If CP^1 is any complex projective line in CP^2 and $f: CP^1 \to CP^2$ any continuous function, then there exist $a, b \in CP^1$ such that $f(a) \perp b$ and $f(b) \perp a$.*

Proof. Note that (1) $CP^1 \cong S^2$, and (2) the inclusion map $g: CP^1 \to CP^2$ induces isomorphism on $H^2(CP^2; Z) \cong Z$ and consequently g is not null homotopic. Theorem 4.7 now gives the desired conclusion.

REMARK 4.9. One can identify CP^2 with the space of lines in

CP^2 and then $a \perp b$ can be interpreted as saying that the line a goes through the point b. Now Corollary 4.8 becomes: If L is a line in CP^2 and f is a continuous map from L into the space of lines in CP^2, then there exists two points $a, b \in L$ such that $f(a)$ goes through b and $f(b)$ goes through a.

5. **Higher order coincidence problems.** Consider two sets X and Y and a finite sequence of functions $f_i\colon X \to Y$, $i = 1, 2, \cdots, N$. A point $x \in X$ such that

$$f_1(x) = f_2(x) = \cdots = f_N(x)$$

is called a coincidence point for the sequence f_i, $1 \leqq i \leqq N$. In Theorem 3.1 (with R taken as equality) we considered the case $N=2$, and we call x in this case a simple coincidence point. When $N \geqq 3$ we will refer to x as a higher order coincidence point. The adjectives "simple" and "higher order" will be applied also to the problem of finding or proving the existence of coincident points (or iterative analogs as in Theorem 3.2).

The purpose of this section is to make the observation that a higher order coincidence problem may be reduced in a useful manner to a simple coincidence problem. The reduction is done by considering $\bar{Y} = Y^{N-1}$ and $f, g\colon X \to \bar{Y}$ defined by $f(x) = (f_1(x), f_2(x), \cdots, f_{N-1}(x))$ and $g(x) = (f_2(x), f_3(x), \cdots, f_N(x))$ for all $x \in X$. Then $x \in X$ is a simple coincidence point for f and g iff x is a higher order coincidence point for the sequence $f_1, f_2, \cdots, f_N$. One may now apply the preceeding results to f and g. In applying §§ 3 and 4 to f and g one sometimes finds that there exists an $m \geqq 1$ and $x_j \in X$, $j \in Z_m$, such that $f(x_j) = g(x_{j+1})$ for $j \in Z_m$. Expressed in terms of the f_i's the condition $f(x_j) = g(x_{j+1})$ for $j \in Z_m$ means $f_1(x_j) = f_2(x_{j+1}) = f_3(x_{j+2}) = \cdots = f_N(x_{j+N-1})$ for $j \in Z_m$. The hypotheses which one encounters in §§ 3 and 4 are usually expressed in terms of $f_*\colon H_*(X; \mathscr{F}) \to H_*(\bar{Y}; \mathscr{F})$ and $g^*\colon H^*(\bar{Y}; \mathscr{F}) \to H^*(X; \mathscr{F})$. The homomorphism f_* may be expressed in terms of $f_{j*}\colon H_*(X; \mathscr{F}) \to H_*(Y; \mathscr{F})$, $j = 1, 2, \cdots, N-1$ as follows. Set $\bar{X} = X^{N-1}$ and use the Künneth formula to obtain $H_*(\bar{X}; \mathscr{F}) = \bigotimes_{j=1}^{N-1} H_*(X_j; \mathscr{F})$ and $H_*(\bar{Y}; \mathscr{F}) = \bigotimes_{j=1}^{N-1} H_*(Y_j; \mathscr{F})$, where $X_j = X$ and $Y_j = Y$ for all j. Let $d\colon X \to \bar{X}$ be the diagonal map given by $d(x) = (x, x, \cdots, x) \in X^{N-1} = \bar{X}$ for all $x \in X$. Now we may write

$$f_* = (f_{1*} \otimes f_{2*} \otimes \cdots \otimes f_{N-1*}) \circ d_* \ .$$

This is the desired formula. Reasoning similarly for g^* we find

$$g^* = d^* \circ (f_2^* \otimes f_3^* \otimes \cdots \otimes f_N^*) \ .$$

Hence $g^*(c_2 \times c_3 \times \cdots \times c_N) = f_2^*(c_2) \cup f_3^*(c_3) \cup \cdots \cup f_N^*(c_N)$ where $c_i \in H^*(Y; \mathscr{F})$ for $i = 2, 3, \cdots, N$.

We consider one example. Let $X = S^{2n} \times S^{2n}$ and $Y = S^{2n}$ with $n \geqq 1$, and suppose $f_1, f_2, f_3: X \to Y$ are three maps. The bidegree of f_i is the pair of integers (α_i, β_i) where $\alpha_i =$ degree $f_i \mid S^{2n} \times q$ and $\beta_i =$ degree $f_i \mid p \times S^{2n}$ for any $p, q \in S^{2n}$.

THEOREM 5.1. *If at least one of the numbers $\alpha_1\beta_2 + \beta_1\alpha_2$, $\alpha_2\beta_3 + \beta_2\alpha_3$, $\alpha_3\beta_1 + \beta_3\alpha_1$ does not vanish, then there exists $A \subset S^{2n} \times S^{2n}$ such that $\sharp A \leqq 4$ and $f_1(A) = f_2(A) = f_3(A)$.*

Proof. Let f and g be defined as in the discussion above. Set the coefficient field F equal to the rationals. Let a be a generator for $H_{2n}S^{2n}$ and a_0 the canonical generator for H_0S^{2n}, and let $\hat{a}$ and $\hat{a}_0$ be the corresponding dual generators of $H^{2n}(S^{2n})$ and $H^0(S^{2n})$. Let $\tau \in H^{2n}(S^{2n} \times S^{2n}, S^{2n} \times S^{2n} - \Delta)$ be the Thom class for S^{2n} corresponding to a. Set $b = j^*(\tau) \in H^{2n}(S^{2n} \times S^{2n})$ where $j: S^{2n} \times S^{2n} \to (S^{2n} \times S^{2n}, S^{2n} \times S^{2n} - \Delta)$ is the inclusion map. Then $b/a_0 = \hat{a}$. Set $a_1 = a \times a \in H_{4n}(X)$ and $b_1 = b \times b \in H^{4n}(\bar{Y} \times \bar{Y})$, where $\bar{Y} = Y^2 = S^{2n} \times S^{2n}$. It is easy to check that $f_i^*(\hat{a}) = \alpha_i \hat{a} \times \hat{a}_0 + \beta_i \hat{a}_0 \times \hat{a} \in H^{2n}(X) = H^{2n}(S^{2n} \times S^{2n})$. Next, a straight forward calculation similar to the calculation made in the proof of Theorem 4.2 yields $k_0(a_0 \times a_0) = (\alpha_2\beta_3 + \beta_2\alpha_3)a_0 \times a_0$ where $k_0 = f_* \circ (\cap a_1) \circ g^* \circ (b_1/): H_0(\bar{Y}) \to H_0(\bar{Y})$. If $\alpha_2\beta_3 + \beta_2\alpha_3 \neq 0$, then Theorems 3.2 and 4.1 give the desired conclusion. This takes care of one case. The other two cases follow by permuting the functions f_1, f_2, f_3.

6. Local index. Consider two topological spaces X and Y, a relation R in Y, $R \subset Y \times Y$, and elements $a \in H_n(X)$ and $b' \in H^n(Y \times Y, Y \times Y - R)$. For each open set $V \subset X$, and pair of maps $f, g: V \to Y$ such that $C = \{x \in V \mid g(x)Rf(x)\}$ is a closed subset of X we can define a local index $I(V, f, g, a, b', R)$ as follows. Let $\varphi: X \to (X, X - C)$ and $\psi(V, V - C) \to (X, X - C)$ be the inclusion maps. Since C is closed in X, $\psi_*: H_n(V, V - C) \to H_n(X, X - C)$ is an isomorphism by excision. Set $a_{V,C} = \psi_*^{-1} \circ \varphi_*(a) \in H_n(V, V - C)$. Let $(g, f): (V, V - C) \to (Y \times Y, Y \times Y - R)$ be the map defined by $(g, f)(x) = (g(x), f(x))$ for all $x \in V$. Now set

$$I(V, f, g, a, b', R) = \langle (g, f)^*(b'), a_{V,C} \rangle$$

where $(g, f)^*: H^n(Y \times Y, Y \times Y - R) \to H^n(V, V - C)$.

This approach to defining a local index seems to be well known to the experts for the case $R =$ equality. See [1] page 558, and [5]. Consequently we discuss only briefly some of its important properties. It is immediate that I is linear in $a \in H_n(X)$ and in $b' \in H^n(Y \times Y, Y \times Y - R)$. Also, the following naturality property in R is clear. If $R \subset R'$ and $k: (Y \times Y, Y \times Y - R') \to (Y \times Y, Y \times Y - R)$ is the inclusion map, then $I(V, f, g, a, b', R) = I(V, f, g, a, k^*(b'), R')$ where $k^*: H^n(Y \times Y, Y \times Y - R) \to H^n(Y \times Y, Y \times Y - R')$.

It is an elementary exercise in diagram chasing to verify that if $V, V' \subset X$ are open and $C \subset V$ and $C' \subset V'$ are closed in X, and $V \cap V' = \varnothing$, then

$$a_{V \cup V', C \cup C'} = j_*(a_{V,C}) + j'_*(a_{V',C'}) \in H_n(V \cup V; V \cup V' - C \cup C'),$$

where $j: (V, C) \to (V \cup V', C \cup C')$ and $j': (V', C') \to (V \cup V', C \cup C')$ are the inclusion maps. It follows that if $I(V, f, g, a, b', R)$ and $I(V', f', g', a, b', R)$ are defined and $V \cap V' = \varnothing$, then $I(V \cup V', f \cup f', g \cup g', a, b', R) = I(V, f, g, a, b', R) + I(V', f', g', a, b', R)$ where the maps $f \cup f': V \cup V' \to Y$ and $g \cup g': V \cup V' \to Y$ are induced from f and f', and g and g' respectively.

The following properties follow easily from the definition. If $V, V' \subset X$ are open subsets with $V' \subset V$, and $\{x \in V \mid g(x) R f(x)\} \subset V'$, then $I(V, f, g, a, b', R) = I(V', f \mid V', g \mid V', a, b', R)$. If $F: V \times I \to Y$ and $G: V \times I \to Y$ are a homotopies from f to f' and g to g' respectively such that $\{x \in V \mid G(x, t) R F(x, t)$ for some $t \in I\}$ is closed in X, then $I(V, f, g, a, b', R) = I(V, f', g', a, b', R)$. Finally, I is "normalized" as follows. If $V = X$, then $I(X, f, g, a, b', R) = \Lambda(f_* \circ (\cap a) \circ g^* \circ (b/)) =$ the Lefschetz R-number $L(f, g)$ based on a and $b = j^*(b') \in H^n(Y \times Y)$, where $j: Y \times Y \to (Y \times Y, Y \times Y - R)$ is the inclusion map.

Now consider the situation in Theorem 3.2 and the higher order coincidence situation. Both these situations reduce to the simple coincidence situation. This is implicit in the proof of Theorem 3.2 and explicit in the discussion of higher order coincidence points. In this way the local index described above carries over to these other situations.

7. **Asymptotic theorems.** An asymptotic fixed point theorem for a function $f: X \to X$ is a theorem asserting the existence of a fixed point for f under hypotheses on the iterates f^n of f, especially for n large. An asymptotic periodic point theorem uses hypotheses on f^n for n large to conclude that f^n has a fixed point for n rather small. In this section we will show how the asymptotic fixed point theorems of Browder [2] and asymptotic periodic point theorems of Halpern [11] have analogs in coincidence theory.

The key observation for our study of asymptotic coincidence point theorems is that $x \in X$ is a coincidence point for the functions

$f, g: X \to Y$ and the relation $R \subset Y \times Y$, i.e., $g(x)Rf(x)$, iff x is a fixed point for the set-valued map $g^{-1} \circ R \circ f: X \to 2^X$, i.e., $x \in g^{-1}(R(f(x)))$, where $R(y) = \{y' \in Y \mid y'Ry\}$ for all $y \in Y$. In general, a set-valued function F from a set A into a set B is an ordinary function from A into 2^B = the set of all subsets of B, (notation $F: A \to 2^B$). If $h: A \to B$ is an ordinary function, then $h^{-1}: B \to 2^A$ denotes the set-valued function which assigns to each $b \in B$ the set $h^{-1}(b) \subset A$. If $T \subset A \times B$ is a relation, then T can also be thought of as a set-valued function $T: B \to 2^A$ defined by $T(b) = \{a \in A \mid aRb\}$ for all $b \in B$. If $F: A \to 2^B$ and $G: B \to 2^C$ are two set-valued functions, then their composite $G \circ F: A \to 2^C$ is defined by $G \circ F(a) = \bigcup_{b \in F(a)} G(b)$ for all $a \in A$. Also, if $A' \subset A$ and $F: A \to 2^B$, then set $F(A') = \bigcup_{a \in A'} F(a)$.

Now suppose $f, g: X \to Y$, $R \subset Y \times Y$, and $x_i \in X$ for $i \in Z_m$ with $m \geqq 1$. The points x_i satisfy $g(x_{i+1})Rf(x_i)$ for $i \in Z_m$ iff x_1 is a fixed point for the mth iterate of the set-valued function $g^{-1} \circ R \circ f$, i.e., $x_1 \in (g^{-1} \circ R \circ f)^m(x_1)$.

The following lemma will allow us to translate the results of [2] and [11] into coincidence theory.

LEMMA 7.1. *Given topological spaces X, Y, and Z and maps $f: X \to Z$ and $g: Y \to Z$, and a subset $R \subset Z \times Z$, and elements $a \in H_n(Y)$ and $b \in H^n(Z \times Z)$ such that $i^*(b) = 0$ where $i: Z \times Z - R \to Z \times Z$ is the inclusion map. Suppose $\operatorname{cl} g^{-1}(R(f(X))) \subset Y' \subset Y$, where Y' is open. Then $(\cap a) \circ g^* \circ (b/) \circ f_*(H_*(X)) \subset k_*(H_*(Y'))$, where $k: Y' \to Y$ is the inclusion map.*

Proof. Set $R' = g^{-1} \circ R \circ f(X)$. Let $\alpha: Y - R' \to Y$, $\beta: Y \to (Y, Y - R')$, and $j: Z \times Z \to (Z \times Z, Z \times Z - R)$ be the inclusion maps. The following diagram commutes

$$\begin{array}{ccc}
(Y \times X, (Y - R') \times X) & & \\
\| & & \\
(Y, Y - R') \times X & \xrightarrow{(g \times f)'} & (Z \times Z, Z \times Z - R) \\
\uparrow{\scriptstyle \beta \times 1} & & \uparrow{\scriptstyle j} \\
Y \times X & \xrightarrow{g \times f} & Z \times Z
\end{array}$$

where $(g \times f)'$ is induced by $g \times f$. Since $i^*(b) = 0$, $b = j^*(\bar{b})$ for some $\bar{b} \in H^n(Z \times Z, Z \times Z - R)$. Let $v \in H_p(X)$ and set $t = g^* \circ (b/) \circ f_*(v) \in H^{n-p}(Y)$. Now we will show that $\langle \alpha^*(t), s \rangle = 0$ for all $s \in H_{n-p}(Y - R')$. Indeed,

$$\begin{aligned}
\langle \alpha^*(t), s \rangle &= \langle t, \alpha_*(s) \rangle = \langle g^* \circ (b/) \circ f_*(v), \alpha_*(s) \rangle = \langle (b/) \circ f_*(v), g_* \circ \alpha_*(s) \rangle \\
&= \langle b, g_*(\alpha_*(s)) \times f_*(v) \rangle
\end{aligned}$$

$$\begin{aligned}
&= \langle j^*(\bar{b}), (g \times f)_*(\alpha_*(s) \times v)\rangle \\
&= \langle \bar{b}, j_* \circ (g \times f)_*(\alpha_*(s) \times v)\rangle \\
&= \langle \bar{b}, (g \times f)'_* \circ (\beta \times 1)_*(\alpha_*(s) \times v)\rangle \\
&= \langle \bar{b}, (g \times f)'_*(\beta_*(\alpha_*(s)) \times v)\rangle \\
&= 0
\end{aligned}$$

because $\beta_*(\alpha_*(s)) = 0$ by exactness. Hence $\alpha^*(t) = 0$ and so $t = \beta^*(\bar{t})$ for some $\bar{t} \in H^{n-p}(Y, Y-R')$. The cap product pairs $H^{n-p}(Y, Y-R')$ and $H_n(Y, (Y-R') \cup Y')$ with $H_p(Y, Y')$. Our assumption that $Y' \supset \operatorname{cl} R'$ guaranties that $(Y - R', Y')$ is an excisive couple. The naturality of the cap product now gives:

$$\psi_*(t \cap a) = \psi_*(\beta^*(\bar{t}) \cap a) = \bar{t} \cap \gamma_*(a)$$

where $\psi\colon Y \to (Y, Y')$ and $\gamma\colon Y \to (Y, (Y-R') \cup Y')$ are the inclusion maps. But $(Y - R') \cup Y' = Y$ and hence $H_n(Y, (Y - R') \cup Y') = H_n(Y, Y) \cong 0$. Thus $\psi_*(t \cap a) = 0$ and so $t \cap a = (\cap a) \circ g^* \circ (b/) \circ f_*(v) \in k_* H_p(Y')$ by exactness.

REMARK. Lemma 7.1 is related to the properties of the transfer homomorphisms (Umkehr-homomorphisms). See [6] pages 308-314.

THEOREM 7.2. *Let X and Y be two compact, path connected Hausdorff spaces, $f, g\colon X \to Y$ two maps, and $R \subset Y \times Y$ a closed subset. Assume that there exists a $b \in H^n(Y \times Y)$ such that $\varphi^*(b) = 0$ and $g^*(b/y_0) \neq 0$, where $\varphi\colon Y \times Y - R \to Y \times Y$ is the inclusion map and y_0 is the canonical generator of $H_0(Y)$. Assume also that $H_*(X)$ and $H_*(Y)$ are of finite type, and $\operatorname{ch} F = 0$, where F is the field of coefficients. Set $\psi = g^{-1} \circ R \circ f\colon X \to 2^X$. If there exists an N and an open set $X' \subset X$ such that $\psi^N(X) \subset X'$ and $\gamma_{*i}\colon H_i(X') \to H_i(X)$ is zero for i odd, where $\gamma\colon X' \to X$ is the inclusion map, then there exists an $m \leqq \sum_{i \text{ even}} \operatorname{rank} \gamma_{*i}$ and elements $x_j \in X$ for $j \in Z_m$ such that $g(x_{j+1}) R f(x_j)$ for $j \in Z_m$.*

Proof. Since $g^*(b/y_0) \in H^n(X)$ is not zero, there is an $a \in H_n(X)$ such that $\langle g^*(b/y_0), a\rangle \neq 0$. Hence

$$\begin{aligned}
0 &\neq \langle g^*(b/y_0), a\rangle \\
&= \langle \hat{x}_0 \cup g^*(b/y_0), a\rangle \\
&= \langle \hat{x}_0, (g^*(b/y_0)) \cap a\rangle
\end{aligned}$$

where $\hat{x}_0$ is the canonical generator of $H^0(X)$. Therefore $(\cap a) \circ g^* \circ (b/)(y_0) \neq 0$ and hence $f_* \circ (\cap a) \circ g^* \circ (b/)(y_0) = \alpha y_0$ with $\alpha \neq 0$. Let $\bar{k}_i\colon H_i(Y) \to H_i(Y)$ and $\bar{k}'_i\colon H_i(X) \to H_i(X)$ be the composites $\bar{k}_i = f_* \circ (\cap a) \circ g^* \circ (b/)$ and $\bar{k}'_i = (\cap a) \circ g^* \circ (b/) \circ f_*$ in the following diagram

$$\begin{array}{ccc} & H^{n-i}(Y) \xrightarrow{\ g^*\ } & H^{n-i}(X) \\ & \uparrow b/ & \downarrow \cap a \\ H_i(X) \xrightarrow{f_*} H_i(Y) & & H_i(X) \xrightarrow{f_*} H_i(Y)\,. \end{array}$$

Set $k_i = (-1)^{ni}\bar{k}_i$ and $k'_i = (-1)^{ni}\bar{k}'_i$. From the commutative property of the trace function, trace $AB =$ trace BA, we conclude that $\Lambda(k^p) = \Lambda(k'^p)$ for all $p \geqq 1$. From this observation, Theorem 3.2, and the proof of Theorem 1 of [11] it follows that it is sufficient to show that rank $k_i'^N \leqq$ rank γ_{*i} for all i. We accomplish this by repeated application of Lemma 7.1.

We will construct a sequence of open sets $X_i \subset X$, $0 \leqq i \leqq N$, such that

(1) $X_0 = X'$

(2) $\psi^{N-j}(X) \subset X_j$

(3) $\psi(\mathrm{cl}\, X_{j+1}) \subset X_j$ for $0 \leqq j < N$.

We will construct the X_j's by induction. Set $X_0 = X'$. Then conditions (1), (2), and (3) hold for those j's for which they make sense. Assume now that $X_0, X_1, \cdots, X_j$ have been defined and satisfy (1), (2), and (3) where applicable. Set $R^{-1} = \{(y, y') \in Y \times Y \mid (y', y) \in R\}$. Using the compactness of X and Y it is easy to see that $\psi^{N-j-1}(X)$ is closed and $X'_{j+1} = f^{-1}(Y - R^{-1}(g(X - X_j)))$ is open. From $\psi^{N-j}(X) \subset X_j$ it follows that $\psi^{N-j-1}(X) \subset X'_{j+1}$. So by normality there exists an open set $X_{j+1} \subset X$ such that

$$\psi^{N-i-1}(X) \subset X_{j+1} \subset \mathrm{cl}\, X_{j+1} \subset X'_{j+1}\,.$$

It is easy to verify that $\psi(\mathrm{cl}\, X_{j+1}) \subset \psi(X'_{j+1}) \subset X_j$, and so the induction step is complete.

Now we can show inductively that

$$A_j\colon k'^j(H_*(X)) \subset \gamma_{N-j*}(H_*(X_{N-j})) \subset H_*(X)$$

where $\gamma_{N-j}\colon X_{N-j} \to X$ is the inclusion map.

Since $X = \psi^0(X) \subset X_N \subset X$, statement A_0 holds trivially. Assume statement A_j. Let $\bar{\gamma}_i\colon \mathrm{cl}\, X_i \to X$ be the inclusion map. Then

$$\begin{aligned} k'^{j+1}(H_*(X)) &= k'(k'^j(H_*(X))) \\ &\subset k'(\gamma_{N-j*}(H_*(X_{N-j}))) \\ &\subset k'(\bar{\gamma}_{N-j*}(H_*(\mathrm{cl}\, X_{N-j}))) \\ &= (\cap a)\circ g^*\circ(b/)\circ f_*\circ\bar{\gamma}_{N-j*}(H_*(\mathrm{cl}\, X_{N-j})) \\ &= (\cap a)\circ g^*\circ(b/)\circ(f\circ\bar{\gamma}_{N-j})_*(H_*(\mathrm{cl}\, X_{N-j})) \\ &\subset \gamma_{N-j-1*}(H_*(X_{N-j-1})) \end{aligned}$$

by property (3) and Lemma 7.1. (Note that $\psi(\mathrm{cl}\, X_{N-j})$ is closed.)

Hence by induction we get

$$k'^N(H_*(X)) \subset \gamma_{0*}(H_*(X_0)) = \gamma_*(H_*(X')) .$$

It now follows that rank $k_i'^N \leqq$ rank γ_{*i} for all i, as we wished to show. This completes the proof.

REMARK. It is not hard to see that under the hypothesis of Theorem 7.2 $\{x \in X | x \in \psi^m(x)$ for some $m \geqq 1\} \subset X'$. Indeed, if $x \in \psi^m(x)$, then $x \in \psi^m \subset \psi^m\psi^m(x) \subset \psi^m\psi^m\psi^m(x) \subset \cdots \subset \psi^{Nm}(x)$. Also $\psi^N(x) \subset X'$ implies $\psi^{N+j}(X) = \psi^N\psi^j(X) \subset \psi^N(X) \subset X'$. Hence $x \in \psi^{Nm}(x) \subset \psi^{Nm}(X) \subset X'$. Thus the x_j, $j \in Z_m$ in the conclusion of Theorem 7.2 satisfy $x_j \in X'$ for $j \in Z_m$.

REMARK. An alternate theory of asymptotic periodic coincidence point theorems can be developed by starting from the observation that there exists a coincidence point $x \in X$ for the functions f, g: $X \to Y$ and the relation $R \subset Y \times Y$, i.e., $g(x)Rf(x)$, iff there exists a fixed point $y \in Y$ for the set-valued map $f \circ g^{-1} \circ R$: $Y \to 2^Y$, i.e., $y \in f \circ g^{-1} \circ R(y)$.

8. A very general coincidence problem. Consider four sets X, X', Y, Y', two functions $f: X \to Y$, $g: X' \to Y'$, and two relations $S \subset X \times X'$ and $R \subset Y' \times Y$. A coincidence pair for this situation is a pair $(x, x') \in X \times X'$ such that xSx' and $g(x')Rf(x)$. If we take $X = X'$, $Y = Y'$ and $S =$ equality, this problem specializes to the type of problem already considered. In this section we will show how all the preceeding results can be extended to this more general coincidence situation. To this end we define a slant product pairing $H^p(X')$ and $H_n(X \times X')$ with $H_{n-p}(X)$, which will have the same role as the cap product did in the preceding development. We use a fixed field $\mathscr{F}$ for coefficients. Given topological spaces X and X' such that $H_*(X)$ and $H_*(X')$ are of finite type, and elements $c' \in H^p(X')$ and $h \in H_n(X \times X')$, define $c' \backslash h \in H_{n-p}(X)$ by requiring $\langle c, c' \backslash h\rangle = \langle c \times c', h\rangle$ for all $c \in H^{n-p}(X)$. This product is to be compared with the cap product as follows. Consider the case where $X = X'$. The equation

(a) $\langle c, c' \backslash h\rangle = \langle c \times c', h\rangle$ is formally the same as

(b) $\langle c, c' \cap a\rangle = \langle c \cup c', a\rangle$ where $a \in H_n(X)$. The equivalence between (a) and (b) becomes more than formal when $h = d_*(a)$, where $d: X \to X \times X$ is the diagonal map. For then

$$\begin{aligned} \langle c \times c', h\rangle &= \langle c \times c', d_*(a)\rangle \\ &= \langle d^*(c \times c'), a\rangle \\ &= \langle c \cup c', a\rangle . \end{aligned}$$

Hence $c'\backslash d_*(a) = c' \cap a$ for all $a \in H_n(X)$ and $c' \in H^p(X)$. In the extension from the special case $X = X'$, $Y = Y'$, $S =$ the diagonal $= \{(x, x) | x \in X\} \subset X \times X$, to the general case, we replace $c' \cap a = c'\backslash d_*(a)$ where $c' \in H^p(X)$ and $a \in H_n(X)$, by $c'\backslash \psi_*(a)$ where $c' \in H^p(X')$, $a \in H_n(S)$, and $\psi: S \to X \times X'$ is the inclusion map.

We illustrate how the results of §§ 3-7 can be extended to this more general coincidence situation by proving the following analog of Theorem 3.1.

THEOREM 8.1. *Let* X, X', Y, Y' *be four topological spaces,* $f: X \to Y$, $g: X' \to Y'$ *two maps, and* $R \subset Y' \times Y$ *and* $S \subset X \times X'$ *two relations. Suppose* n *is a nonnagative integer. Let* $a \in H_n(X \times X')$ *and* $b \in H^n(Y' \times Y)$ *be such that* $\eta_*(a) = 0$ *and* $\varphi^*(b) = 0$, *where* $\eta: X \times X' \to (X \times X', S)$ *and* $\varphi: Y' \times Y - R \to Y' \times Y$ *are the inclusion maps. Assume* $H_*(X)$, $H_*(X')$, $H_*(Y)$, *and* $H_*(Y')$ *are of finite type. Let* $\bar{h}_i: H_i(Y) \to H_i(Y)$ *be the composite*

$$\begin{array}{ccccc} H^{n-i}(Y') & \xrightarrow{g^*} & H^{n-i}(X') & & \\ \uparrow{\scriptstyle b/} & & \downarrow{\scriptstyle \backslash a} & & \\ H_i(Y) & & H_i(X) & \xrightarrow{f^*} & H_i(Y) \end{array}$$

wnere $b/$ *and* $\backslash a$ *are the maps such that* $(b/)(z) = b/z$ *for* $z \in H_i(Y)$, *and* $(\backslash a)(t) = t\backslash a$ *for* $t \in H^{n-i}(X')$. *Set* $h_i = (-1)^{ni}\bar{h}_i$. *If* $\Lambda(h) = \Sigma(-1)^i$ *trace* $h_i \neq 0$, *then there exists a pair* $(x, x') \in X \times X'$ *such that* xSx' *and* $g(x')Rf(x)$.

Proof. Consider the following diagram

$$\begin{array}{ccccc} S & \cdots\cdots\cdots\overset{\lambda}{\cdots}\cdots\cdots\cdots & & & Y' \times Y - R \\ \downarrow{\scriptstyle \psi} & & & & \downarrow{\scriptstyle \varphi} \\ X \times X' & \xrightarrow{f \times g} & Y \times Y' & \xrightarrow{T} & Y' \times Y \end{array}$$

where $T(y, y') = (y', y)$ is the interchange map. Set $b' = \psi^* \circ (f \times g)^* \circ T^*(b) \in H^n(S)$. If there is no ordered pair $(x, x') \in X \times X'$ such that xSx' and $g(x')Rf(x)$, then we may define a map $\lambda: S \to Y' \times Y - R$ be setting $\lambda(x, x') = (g(x'), f(x))$, and the above diagram would commute. It would then follow that $b' = \lambda^* \circ \varphi^*(b) = 0$ since $\varphi^*(b) = 0$ by hypothesis. We have also assumed that $\eta_*(a) = 0$. Hence $a = \psi_*(a')$ for some $a' \in H_n(S)$. To prove the theorem it is clearly sufficient to show that $\langle b', a' \rangle = \Lambda(h)$. Note that

$$\begin{aligned} \langle b', a' \rangle &= \langle \psi^* \circ (f \times g)^* \circ T^*(b), a' \rangle \\ &= \langle (f \times g)^* \circ T^*(b), \psi_*(a') \rangle \\ &= \langle (f \times g)^* \circ T^*(b), a \rangle . \end{aligned}$$

Now the calculation showing that $\langle b', a'\rangle = \Lambda(h)$ is essentially the same as the calculation in the proof of Theorem 3.1. Just replace $\cap a$ by $\backslash a$.

Similar adjustments can be made with the other results of the preceding sections.

The analogous problem to the problem considered in Theorem 3.2 will be refered to below as the "general composite problem." This problem involves two sequences of functions $f_n: X_n \to Y_n$ and $g_n: X'_n \to Y'_{n-1'}$ for $n \in Z_m$, and two sequences of relations, $R_n \subset Y'_n \times Y_n$ and $S_n \subset X_n \times X'_n$, for $n \in Z_m$, and asks for an $x \in X_1$, such that $x \in S_1 \circ g_1^{-1} \circ R_m \circ f_m \circ \cdots \circ f_2 \circ S_2 \circ g_2^{-1} \circ R_1 \circ f_1$. A natural question is: Is there a nontrivial generalization of the general composite problem? We will observe here that what appears to be a "most general" coincidence problem involving functions and binary relations can be viewed as a special case of the general composite problem.

Suppose $T_n, n \in Z_m$ is a sequence of spaces and for each $n \in Z_m$ either (a) $F_n: T_n \to T_{n+1}$ is a function from T_n into T_{n+1}, or (b) $F_n = g^{-1}$ where $g: T_{n+1} \to T_n$ is a function from T_{n+1} into T_n, or (c) $F_n \subset T_{n+1} \times T_n$ is a relation.

Problem Q. Find an $x \in T_1$ such that $x \in (\prod_n F_n)(x)$. To reduce problem Q to a general composite problem we simply set $F_n = f_n$ or g_n or R_n depending on whether case (a) or (b) or (c) applies to F_n, and set all the other functions and relations in the general composite problem equal to the appropriate identity maps or relations.

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Received August 7, 1973, and in revised form February 14, 1975.

INDIANA UNIVERSITY
BLOOMINGTON, IN 47401