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ON JOINT NUMERICAL RANGES

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The joint numerical status of commuting bounded operators A_1 and A_2 on a Hilbert space is defined as $\{(\phi(A_1), \phi(A_2))$ such that ϕ is a state on the C^*-algebra generated by A_1 and $A_2\}$. It is shown that if A_1 and A_2 are commuting normal operators then their joint numerical status equals the closure of their joint numerical range. It is also shown that certain points in the boundary of the joint numerical range are joint approximate reducing eigenvalues.

The joint numerical range of A_1 and A_2 denoted by $w(A_1, A_2)$ is $\{((A_1x, x), (A_2x, x))$ such that $x \in H$ and $||x|| = 1\}$. Thus $w(A_1, A_2)$ is a bounded subset of C^2. It is not known whether this set is convex, Dash [4, 6]. In this note, we shall show that there is faithful $*$ representation of the C^*-algebra generated by A_1 and A_2, $C^*(A_1, A_2)$, under which the joint numerical range of A_1 and A_2 is convex. Following Berberian and Orland [1], we study the joint numerical status of A_1 and A_2, $\Sigma(A_1, A_2) = \{(\phi(A_1), \phi(A_2))$ such that ϕ is a state on $C^*(A_1, A_2)\}$. If A_1 and A_2 are commuting normal operators then $\Sigma(A_1, A_2) = \bar{w}(A_1, A_2)$. We also show that certain points in the boundary of $w(A_1, A_2)$ are joint approximate reducing eigenvalues.

For the sake of notational convenience, all the results are being stated for two commuting operators. However, the results hold for any finite family of commuting operators.

Let $B(H)$ denote the algebra of all bounded linear operators on the Hilbert space H. Let $C^*(A_1, A_2)$ denote the C^*-algebra generated by I, A_1, and A_2. Let Σ denote the set of all states on $C^*(A_1, A_2)$. Any state ϕ in Σ induces a representation Π_ϕ of $C^*(A_1, A_2)$ which acts on a Hilbert space H_ϕ and has a cannonical cyclic vector ξ_ϕ. Also any maximal left ideal of $C^*(A_1, A_2)$ is of the form $K(\psi) = \{A \in C^*(A_1, A_2)$ such that $\psi(A^*A) = 0\}$ for some pure state ψ on $C^*(A_1, A_2)$. For details concerning this the reader is referred to Dixmier [7]. The joint approximate point spectrum of A_1 and A_2, denoted by $a(A_1, A_2)$, is $\{(z_1, z_2)$ such that there exists a sequence $x_n \in H$, $||x_n|| = 1$ such that $||(A_1 - z_1)x_n|| \to 0$ and $||(A_2 - z_2)x_n|| \to 0\}$ which is the same as $\{(z_1, z_2)$ such that $B(H)(A_1 - z_1) + B(H)(A_2 - z_2) \neq B(H)\}$.

First we shall show that $a(A_1, A_2)$ depends only on the C^*-algebra generated by A_1 and A_2. Our proof is similar to Bunce [2].

PROPOSITION 1.

$a(A_1, A_2) = \{(\lambda_1, \lambda_2): C^*(A_1, A_2)(A_1-\lambda_1)+C^*(A_1, A_2)(A_2-\lambda_2)\neq C^*(A_1, A_2)\}$.

Proof. If $(\lambda_1, \lambda_2)\in a(A_1, A_2)$ then there exists $x_n \in H$, $||x_n|| = 1$ such that $||(A_1 - \lambda_1)x_n|| \to 0$ and $||(A_2 - \lambda_2)x_n|| \to 0$. Suppose there are B_1 and B_2 in $C^*(A_1, A_2)$ such that $B_1(A_1 - \lambda_1) + B_2(A_2 - \lambda_2) = I$ then $||x_n|| \leqq ||B_1(A_1 - \lambda_1)x_n|| + ||B_2(A_2 - \lambda_2)x_n||$ and hence $x_n \to 0$, which is a contradiction. Conversely, if $C^*(A_1, A_2)(A_1 - \lambda_1) + C^*(A_1, A_2)(A_2 - \lambda_2) \neq C^*(A_1, A_2)$, then $C^*(A_1, A_2)(A_1 - \lambda_1) + C^*(A_1, A_2)(A_2 - \lambda_2)$ being a proper left ideal of $C^*(A_1, A_2)$ is contained in $K(\phi)= \{B\in C^*(A_1, A_2): \phi(B^*B) = 0\}$ where ϕ is a pure state on $C^*(A_1, A_2)$ [7]. Thus, in particular $\phi(D^2) = 0$ where $D = (A_1 - \lambda_1)^*(A_1 - \lambda_1)+ (A_2 - \lambda_2)^*(A_2 - \lambda_2)$. If $(\lambda_1, \lambda_2) \notin a(A_1, A_2)$ then $0 \notin a(D)$ i.e., there exists $m > 0$ such that $(Dx, Dx) \geqq (m^2x, x)$ for all $x\in H$, i.e., $\phi(D^2) \geqq m^2I$ which is a contradiction.

The joint numerical status of A_1 and A_2, $\Sigma(A_1, A_2)$ is defined to be $\{(\phi(A_1), \phi(A_2))$ such that ϕ is a state on $C^*(A_1, A_2)\}$. Since Σ is a weak $*$ compact convex subset of the dual space of $C^*(A_1, A_2)$, it can be easily shown that $\Sigma(A_1, A_2)$ is a compact, convex subset of C^2. Since each unit vector in H induces a state on $C^*(A_1, A_2)$, $w(A_1, A_2)\subset \Sigma(A_1, A_2)$. Also the extreme points of $\Sigma(A_1, A_2)$ correspond to the pure states in Σ. Let $\Pi_0: C^*(A_1, A_2) \to B(K)$, for some Hilbert space K, be the universal representation (Gelfand-Naimark representation) of $C^*(A_1, A_2)$. We shall show that $a(A_1, A_2) = a(\Pi_0(A_1), \Pi_0(A_2))$ and also the closure of $w(\Pi_0(A_1), \Pi_0(A_2)) = \Sigma(A_1, A_2)$.

LEMMA 2. *If $(\lambda_1, \lambda_2)\in a(A_1, A_2)$ then there exists a pure state ϕ on $C^*(A_1, A_2)$ such that $\Pi_\phi(A_i)\xi_\phi = \lambda_i\xi_\phi$ for $i = 1$ and 2, where Π_ϕ is the irreducible representation induced by ϕ with canonical cyclic vector ξ_ϕ.*

Proof. If $(\lambda_1, \lambda_2)\in a(A_1, A_2)$ then by Proposition 1 there exists a pure state ϕ on $C^*(A_1, A_2)$ such that $C^*(A_1, A_2)(A_1 - \lambda_1) + C^*(A_1, A_2)(A_2 - \lambda_2)$ is contained in $K(\phi) = \{B\in C^*(A_1, A_2): \phi(B^*B) = 0\}$. Thus $A_1 - \lambda_1 I\in K(\phi)$ and so $A_2 - \lambda_2 I \in K(\phi)$. Hence $\Pi_\phi(A_i) = \lambda_i I$ and hence $\Pi_\phi(A_i) = \lambda_i\xi_\phi$ for $i = 1$ and 2.

LEMMA 3. *If $(\lambda_1, \lambda_2)\notin a(A_1, A_2)$ then $(\lambda_1, \lambda_2)\notin a(\Pi(A_1), \Pi(A_2))$ for any $*$ representation Π of $C^*(A_1, A_2)$.*

Proof. If $(\lambda_1, \lambda_2)\notin a(A_1, A_2)$ then by Proposition 1, there exist D_1 and D_2 in $C^*(A_1, A_2)$ such that $D_1(A_1 - \lambda_1) + D_2(A_2 - \lambda_2) = I$. Therefore $\Pi(D_1)(\Pi(A_1) - \lambda_1) + \Pi(D_2)(\Pi(A_2) - \lambda_2) = I$ and hence $(\lambda_1, \lambda_2)\notin a(\Pi(A_1), \Pi(A_2))$.

PROPOSITION 4. $a(A_1, A_2) = a(\Pi_0(A_1), \Pi_0(A_2))$.

Proof. By Lemma 3, $a(\Pi_0(A_1), \Pi_0(A_2)) \subset a(A_1, A_2)$. On the other hand, Π_0 is a faithful $*$ representation, by Lemma 2, $a(A_1, A_2) \subset a(\Pi_0(A_1), \Pi_0(A_2))$.

PROPOSITION 5. *The closure of* $w(\Pi_0(A_1), \Pi_0(A_2)) = \Sigma(A_1, A_2)$.

Proof. If $\phi \in \Sigma$, then there exists $\xi_\phi \in K$ such that $\Pi_0(A_i)\xi_\phi = \lambda_i \xi_\phi$ for $i = 1$ and 2. Thus $\Sigma(A_1, A_2) \subset w(\Pi_0(A_1), \Pi_0(A_2))$. On the other hand, for any $x \in K$, $\|x\| = 1$, if $\lambda_i = \Pi_0(A_i)x$, we can define $\phi: C^*(A_1, A_2) \to C$ by $\phi(A) = (\Pi_0(A)x, x)$. Thus $\phi \in \Sigma$ and $\phi(A_i) = \lambda_i$ for $i = 1$ and 2.

PROPOSITION 6. *If A_1 and A_2 are commuting normal operators then* $\Sigma(A_1, A_2) = \bar{w}(A_1, A_2) =$ *convex hull* $a(A_1, A_2)$.

Proof. If A_1 and A_2 are commuting normal operators, $a(A_1, A_2) = \{(\phi(A_1), \phi(A_2))$ such that ϕ is a character on $C^*(A_1, A_2)\}$. Let $(\lambda_1, \lambda_2) \in \Sigma(A_1, A_2)$ be any extreme point of $\Sigma(A_1, A_2)$, thus there is a pure state $\phi \in \Sigma$ such that $\phi(A_i) = \lambda_i$ for $i = 1$ and 2. Also $C^*(A_1, A_2)$ is commutative C^*-algebra and hence ϕ is a character on $C^*(A_1, A_2)$ and thus $(\lambda_1, \lambda_2) \in a(A_1, A_2) \subset \bar{w}(A_1, A_2)$. Since in this case $w(A_1, A_2)$ is convex (Dash [6]), the result is proved.

Juneja [8] showed that if (z_1, z_2) is an extreme point of $w(A_1, A_2)$ where A_1 and A_2 are commuting normal operators then (z_1, z_2) is a reducing approximate eigenvalue. We generalize this result to the case of arbitrary commuting operators.

LEMMA 7. *Let S_1 and S_2 be convex bounded subsets of C. Let $(z_1, z_2) \in (\partial S_1) \times (\partial S_2)$ where ∂S_i denotes the boundary of S_i. Then there exist λ_i in the complement of the closure of S_i, $i = 1$ and 2, such that* $|z_i - \lambda_i| = \text{dist.}(\lambda_i, S_i)$ *and* $|(z_1, z_2) - (\lambda_1, \lambda_2)| = \text{dist.}((\lambda_1, \lambda_2), S_1 \times S_2)$.

LEMMA 8. *Let $A \in B(H)$. If there exists a sequence of unit vectors x_n in H such that* $\|[(A - \lambda)^{-1} - (\mu - \lambda)^{-1}]x_n\| \to 0$ *and* $\|[(A^* - \bar{\lambda})^{-1} - (\bar{\mu} - \bar{\lambda})^{-1}]x_n\| \to 0$ *then* $\|(A - \mu)x_n\| \to 0$ *and* $\|(A^* - \bar{\mu})x_n\| \to 0$.

PROPOSITION 9. *If* $(z_1, z_2) \in [\partial w(A_1) \times \partial w(A_2)] \cap \bar{w}(A_1, A_2) \cap \sigma(A_1, A_2)$, *then there exists a sequence x_n of unit vectors in H such that* $\|(A_i - z_i)x_n\| \to 0$ *and* $\|(A_i^* - \bar{z}_i)x_n\| \to 0$ *for $i = 1$ and 2.*

Proof. Since $\sigma(A_1, A_2) \subset w(A_1) \times w(A_2)$, by Lemma 7, there exists $\lambda = (\lambda_1, \lambda_2) \notin w(A_1) \times w(A_2)$ such that $|\lambda_i - z_i| = \text{dist.}(\lambda_i, w(A_i)) = \text{dist.}(\lambda_i, \sigma(A_i))$ and $|\lambda - z| = \text{dist.}(\lambda, w(A_1) \times w(A_2)) = \text{dist.}(\lambda, w(A_1, A_2)) = \text{dist.}(\lambda, \sigma(A_1, A_2))$, where $z = (z_1, z_2)$. Thus $\|(A_i - \lambda_i)^{-1}\| = 1/|\lambda_i - z_i|$ and $1/(z_i - \lambda_i) \in \sigma((A_i - \lambda_i)^{-1})$ for $i = 1$ and 2. Now using Proposition 3.3 of Dash [5], there exists a sequence x_n of unit vectors in H such that $\|[(A_i - \lambda_i)^{-1} - (z_i - \lambda_i)^{-1}]x_n\| \to 0$ and $\|[(A_i^* - \bar{\lambda}_i)^{-1} - (\bar{z}_i - \bar{\lambda}_i)^{-1}]x_n\| \to 0$. The result follows from Lemma8.

COROLLARY 10. *If (z_1, z_2) is an extreme point of $w(A_1, A_2)$ where A_1 and A_2 are commuting normal operators then there exists a sequence of unit vectors x_n in H such that $\|(A_i - z_i)x_n\| \to 0$ and $\|(A_i^* - \bar{z}_i)x_n\| \to 0$.*

Proof. Since (z_1, z_2) is an extreme point of $w(A_1, A_2)$, A_1 and A_2 are commuting normal operators, as in the proof of Proposition 6, there exists a character ϕ on $C^*(A_1, A_2)$ such that $\phi(A_i) = z_i$ for $i = 1$ and 2. Also ϕ restricted $C^*(A_1)$ is also a character and hence $\phi(A_1)$ is an extreme point of $w(A_1)$. Similarly $\phi(A_2)$ is an extreme point of $w(A_2)$. Thus $(\lambda_1, \lambda_2) \in (\partial w(A_1) \times \partial w(A_2)) \cap \bar{w}(A_1, A_2) \cap \sigma(A_1, A_2)$.

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