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SCHUR'S THEOREM AND THE DRAZIN INVERSE

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It is shown that if $M = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$ is a square $2n \times 2n$ matrix over a ring R, such that $AC = CA \in R_{n\times n}$, and with the property that A and C possess Drazin inverses, then M is invertible in $R_{2n\times 2n}$ if and only if DA-BC is invertible in $R_{n\times n}$.

1. Introduction. In a recent paper [7], Herstein and Small extended the classic result of Schur [5, p. 46] to matrices over E-rings. These are rings for which every primitive image is artinian. This result states that for a square complex block matrix $M = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$, with A, B, C, D square of the same size such that $AC = CA$, then M is invertible exactly when $\Delta = DA - BC$ in invertible. This is a different but equivalent formulation of the problem as stated in [7].

The purpose of this note is to show that this result by Schur is basically a consequence of the *local* existence of the Drazin inverse [2] of the matrices A and C; that is, the strong-π-regularity of A and C [1] [4]. The proof of [7] was based on the fact that Schur's result for matrices over E-rings is really equivalent to the corresponding result for matrices over simple artinian rings (which may be taken to be division rings). Since artinian rings with unity are noetherian [8], p. 69, it follows that artinian rings with unity are strongly-π-regular, so that our local result extends the Schur theorem for artinian rings as proven in [7].

The Drazin inverse a^d of a ring element a, is the unique solution, if any, to the equations

$$(1) \qquad a^k x a = a^k,\ xax = x,\ ax = xa\ ,$$

for some $k \geqq 0$, while the group inverse $a^\#$ of a is the unique solution, if any, of these equations with $k = 0$, or 1. For example, if a is algebraic over some field $\mathscr{F}$ and $a^{n+1}b = a^n$, with $ab = ba$, then $a^d = a^n b^{n+1}$. The element a^d exists if and only if a is strongly-π-regular, that is, when both chains $\{a^i R\}$ and $\{Ra^i\}$ are ultimately stationary, [5, Theorem 4]. A ring element is called (von Neumann) *regular* if $aa^-a = a$ for some ring element a^-. If there exists such a^- that is invertible, a is called *unit*-regular.

We shall assume familiarity with the properties of these inverses [4] [2] [6] and in particular with the fact that $ac = ca \Rightarrow a^d c = ca^d$ [4, Theorem 1].

It is known that, unlike regularity and unit regularity, $R_{2\times 2}$ does *not* inherit strong-regularity from R [9] [11]. It is *not* known however, whether the strong-Π-regularity of R, or the related concept of finite regularity ($ab = 1 \Rightarrow ba = 1$) is inherited by $R_{2\times 2}$ [10].

We shall use the notation ${}^0S$ and S^0 to indicate the right and left annihilators of S respectively, e.g.,

$$S^0 = \{x \in R;\ xs = 0,\ \forall\ s \in S\}\ .$$

For notational convenience we shall state our results in terms of rings R with unity, with the translation to matrices over R, being self evident. In particular $aR + cR = R$ is equivalent to the 1×2 matrix $[a, c]$ having a right inverse.

2. Preliminaries.

The key to our main result are the following two lemmas.

LEMMA 1. *Let R be a ring with unity* 1, *and let e, f be commuting idempotents in R. If $g = e + f(1 - e)$ then*

(i) $g^2 = g$, (ii) $eR + fR = gR$, (iii) $Re + Rf = Rg$, (iv) $e^0 \cap f^0 = g^0$, (v) ${}^0e \cap {}^0f = {}^0g$, (vi) $eR + fR = R \Leftrightarrow g = 1 \Leftrightarrow Re + Rf = R \Leftrightarrow e^0 \cap f^0 = (0) \Leftrightarrow {}^0e \cap {}^0f = (0) \Leftrightarrow (1 - e)(1 - f) = 0$.

LEMMA 2. *Let R be a ring with unity* 1, *and let a, c be commuting elements of R. Then*

(i) $aR + cR = R \Leftrightarrow a^mR + c^nR = R$ *for some* $m, n \geqq 1 \Leftrightarrow a^mR + c^nR = R$ *for all* $m, n \geqq 1$.

(ii) ${}^0a \cap {}^0c = (0) \Leftrightarrow {}^0(a^m) \cap {}^0(c^n) = (0)$ *for some* $m, n \geqq 1 \Leftrightarrow {}^0(a^m) \cap {}^0(c^n) = (0)$ *for all* $m, n \geqq 1$.

(iii) $Ra + Rc = R \Leftrightarrow Ra^m + Rc^n = R$ *for some* $m, n \geqq 1 \Leftrightarrow Ra^m + Rc^n = R$ *for all* $m, n \geqq 1$.

(iv) $a^0 \cap c^0 = (0) \Leftrightarrow (a^m)^0 \cap (c^n)^0 = (0)$ *for some* $m, n \geqq 1 \Leftrightarrow (a^m)^0 \cap (c^n)^0 = (0)$ *for all* $m, n \geqq 1$.

If in addition, the Drazin inverses a^d and c^d exists, these conditions are all equivalent to

(v) $(1 - aa^d)(1 - cc^d) = 0$.

Proof. The proof of (i)-(iv) follows by induction. Now suppose that a^d and c^d exist and that index $(a) = k$, index $(c) = l$. Then for all $m \geqq k$, $a^mR = a^kR = a^dR = a^daR$. And so, taking $m \geqq k$, $n \geqq l$, we see that (i) is equivalent to

$$R = a^mR + c^nR = a^kR + c^lR = a^dR + c^dR = a^daR + c^dcR\ ,$$

which by Lemma 1 is equivalent to

$$(1 - aa^d)(1 - cc^d) = 0\ . \qquad (3)$$

Left-right symmetry now shows that (iii) is also equivalent to (v). Lastly, since for $m \geqq k$, $(a^m)^0 = (a^k)^0 = (a^d)^0 = (a^d a)^0$, it follows that with $m \geqq k$, $n \geqq l$, (iv) is equivalent to $(a^d a)^0 \cap (c^d c)^0 = (0)$, which again by Lemma 1 is equivalent to (v). Symmetry again yields the remaining equivalence.

Before proceeding with our theorem we remark that:

1. It is not necessary for a^d and c^d to exist in order for

$$Ra + Rc = R \Longleftrightarrow a^0 \cap c^0 = (0)$$

to be valid. It would suffice if a, c and $c(1 - a^- a)$ were regular.

2. The equivalence of (iv) and (v) has uses in the theory of differential equations, [2] Lemma 1. The above furnishes a short and purely algebraic proof of this useful result.

3. Main results.

THEOREM 1. *Let R be a ring with unity 1 and let $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R_{2\times 2}$ with $ac = ca$. Suppose further that a^d and $[(1 - aa^d)c]^d$ exist. If $\Delta = da - bc$, then:*

(i) *Δ is left invertible $\Leftrightarrow$ M is left invertible.*

(ii) *M is right invertible $\Leftrightarrow$ Δ is right invertible.*

(iii) *M is invertible $\Leftrightarrow$ Δ is invertible.*

Proof. Consider the matrix

$$N = \begin{bmatrix} a & u \\ b & z \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 1 & -a^d c \\ 0 & 1 \end{bmatrix}, \tag{4}$$

where $u = (1 - aa^d)c$ and $z = d - ba^d c$. Since a, a^d and c commute it follows that

$$za - bu = (d - ba^d c)a - b(1 - aa^d)c = da - bc = \Delta\,. \tag{5}$$

Now because $a^d u = 0 = ua^d = a^d u^d = u^d a^d$, we may construct the matrices:

$$\begin{bmatrix} a & u \\ b & z \end{bmatrix} \begin{bmatrix} a^d & -u \\ u^d & a \end{bmatrix} = \begin{bmatrix} aa^d + uu^d & 0 \\ t & \Delta \end{bmatrix} = T \tag{6}$$

and

$$\begin{aligned} \begin{bmatrix} a & u \\ -u^d & a^d \end{bmatrix} \begin{bmatrix} a^d & -u \\ u^d & a \end{bmatrix} &= \begin{bmatrix} aa^d + uu^d & 0 \\ 0 & aa^d + uu^d \end{bmatrix} \\ &= \begin{bmatrix} a^d & -u \\ u^d & a \end{bmatrix} \begin{bmatrix} a & u \\ -u^d & a^d \end{bmatrix}. \end{aligned} \tag{7}$$

In general however, $au^d \neq 0$ unless index $(a) \leqq 1$. Suppose now that Δ has a left inverse Δ^-, then by (5),

$$R = Ra + Rc = Ra + Ru \,. \tag{8}$$

By Lemma 2, applid to a and u, we see that

$$(1 - aa^d)(1 - uu^d) = 0$$

or equivalently

$$aa^d + uu^d = 1 \,. \tag{9}$$

Hence, by (7), it follows that the matrix $P = \begin{bmatrix} a^d & -u \\ u^d & a \end{bmatrix}$ is invertible. Now since

$$\begin{bmatrix} 1 & 0 \\ -\Delta^- t & \Delta^- \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & \Delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

it follows that M has a left inverse M^- and that

$$R = Ra + Rb = Rc + Rd \,.$$

If in addition $\Delta\Delta^- = 1$, then

$$\begin{bmatrix} 1 & 0 \\ t & \Delta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\Delta^- t & \Delta^- \end{bmatrix} = I_2$$

and consequently M is also invertible.

Conversely, suppose that $MM^- = I$. Then because N also has a right inverse, it follows that

$$aR + uR = R \,.$$

Again by Lemma 2, applied to a and u, we may conclude that (9) holds so that P is invertible. Hence $T = \begin{bmatrix} 1 & 0 \\ t & \Delta \end{bmatrix}$ has a right inverse $T^- = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$. Now $TT^- = I \Rightarrow \gamma = 0 \Rightarrow \Delta\delta = 1$, and so Δ has a right inverse. If in addition, $M^-M = I$, then $T^-T = I$ and hence again as $\gamma = 0$, $\delta\Delta = 1$, completing the proof.

COROLLARY 1. *If R is a ring with unity and $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R_{2\times 2}$ with $ac = ca$ such that a^d and c^d exist, then M is invertible if and only if $\Delta = da - bc$ is invertible.*

Proof. Note that $ac = ca$ implies that $aa^d c = caa^d$, so that $u^d = (1 - aa^d)c^d$. Again because square matrices over artinian ring with unity possess Drazin inverses, this result includes the second part of Theorem 2 of [7].

COROLLARY 2. *Let R be a ring with unity 1, and let $a, c \in R$ such that $ac = ca$ and a^d, $[(1 - aa^d)c]^d$ exist. Then if $R = Ra + Rc$ there exists $d \in R$ so that $\begin{bmatrix} a & c \\ c & d \end{bmatrix}$ is invertible.*

Proof. From Theorem 1, it suffices to select $d \in R$ such that $\Delta = da - c^2$ is invertible. One such choice is given by $d = a^d + c^2a^d$, because then $\Delta = aa^d - u^2$ which has inverse $aa^d - u^du^d$. Indeed, if $R = Ra + Rc = Ra + Ru$, then $aa^d + uu^d = 1$ which coupled with the fact $a^du^d = 0$, yields the desired result.

We conclude this note with several remarks.

1. If $a^\#$ exists we could also select $d = a + c^2a^\#$ in the last corollary, for then $\Delta = a^2 - u^2$ has as inverse $(a^\#)^2 - uu^d$ since now $au = 0$. Moreover, in this case

$$\begin{bmatrix} a & c \\ c & a + c^2a \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a^\#c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & u^d \\ u^d & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -ca & 1 \end{bmatrix}$$
$$= \begin{bmatrix} a - (a^\#)^3c^2 & u^d - c(a^\#)^2 \\ u^d - c(a^\#)^2 & a^\# \end{bmatrix}.$$

2. The fact that: "If $ac = ca$, and a^d, u^d exist, then $R = Ra + Rc$ ensures that $a^da + u^du = 1$", should be compared with the corresponding results for Moore-Penrose inverses [6]. Namely, if $a^\dagger$ and $v^\dagger = [c(1 - a^\dagger a)]^\dagger$ exists, then

$$R = Ra + Rc \Longrightarrow 1 = a^\dagger a + v^\dagger v .$$

3. If a is *unit*-regular, that is $aa^= a = a$ for some unit $a^=$, then under suitable conditions $aR + cR = R \Rightarrow Ra + Rc = R$. Indeed if $u = (1 - aa^=)c$ is regular and $c^\#$ exists, then

$$aR + cR = R \Longrightarrow aa^= + (1 - a^=)cu^-(1 - aa^=) = 1 .$$

Thus $aa^=[(1 - aa^=cu^-(1 - aa^=] + cu^-(1 - aa^=) = 1$, which on multiplying through by

$$p = [1 + aa^=cu^-(1 - aa^=)](a^=)^{-1}$$

yields:

$$a + ct = p = \text{unit, where} \quad t = u^-(1 - aa^=)(a^=)^{-1} .$$

Now if in addition, $ac = ca$ *and* $a^=c = ca^=$ then we may take $u^- = c^\#$. Hence

$$a + (1 - aa^=)(a^=)^{-1}c^\#c = p$$

implying that $Ra + Rc = R$.

4. It is now clear how to extend this to the following: If a^k is unit regular for some $k \geqq 1$, say $a^k(a^k)^= a^k = a^k$, where $(a^k)^=$ is a unit, and if c^d exist, such that cc^d commutes with $a^k(a^k)^=$ and $(a^k)^=$ then

$$R = aR + cR \Longrightarrow R = Ra + Rc \ .$$

The case where a^d exists and $ac = ca$ easily follows from this example because then $(a^k)^\#$ exist, for some $k \geqq 1$ and one may then take $(a^k)^= = (a^k)^\# + (1 - aa^d)$.

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