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A CLASS OF ISOTROPIC DISTRIBUTIONS IN R_n AND THEIR CHARACTERISTIC FUNCTIONS

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Let $r(t)$ be a characteristic function. Suppose that there is an integer $n \geqq 2$ such that $r((t_1^2 + \cdots + t_n^2)^{1/2})$ is, as a function of n variables, also the characteristic function of some distribution in R_n. Then, as is known, the distribution is necessarily rotationally invariant, and r has a canonical form as a certain Bessel transform of a bounded nondecreasing function. A certain subclass of the class of such characteristic functions was defined and studied by Mittal, who furnished an analytic characterization of functions in the subclass. The purposes of this paper are (i) to present an alternative probabilistic characterization of these functions, and (ii) to characterize, for this subclass, the bounded nondecreasing function appearing in the Bessel transform.

1. Introduction and summary. Let $h(x)$, $x \geqq 0$, be a nonnegative function. For a fixed integer $n \geqq 2$, let $\boldsymbol{x}$ be an element of R_n, and let $||\boldsymbol{x}||$ be its Euclidian norm. Consider the extension of h to a radial function on R_n: $h(x) = h(||\boldsymbol{x}||)$, where $x = ||\boldsymbol{x}||$. If $x^{n-1}h(x)$ is integrable over $x \geqq 0$, then $h(||\boldsymbol{x}||)$ is integrable over R_n, and

$$\int_{R_n} h(||\boldsymbol{x}||)d\boldsymbol{x} = S_n \int_0^\infty x^{n-1}h(x)dx ,$$

where S_n is the surface area of the n-dimensional unit sphere. If the integrals above have the value 1, then $h(||\boldsymbol{x}||)$ is a density function on R_n. Its characteristic function is, by definition,

$$\int_{R_n} \exp[i(\boldsymbol{t}, \boldsymbol{x})]h(||\boldsymbol{x}||)d\boldsymbol{x} , \tag{1.1}$$

and is a radial function, denoted $r(||\boldsymbol{t}||)$, $\boldsymbol{t} \in R_n$, where r is function of a real variable. According to Schoenberg's theorem [8], the function $r(t)$ is necessarily of the form

$$r(t) = \Gamma(n/2) \int_0^\infty \frac{J_{(n-2)/2}(ut)}{(ut/2)^{(n-2)/2}} dG(u) , \tag{1.2}$$

where J_ν is the Bessel function of order ν, and where G is a bounded, nondecreasing function of variation 1. The latter is directly related to h:

$$S_n h(x)x^{n-1}dx = dG(x) . \tag{1.3}$$

This is obtained by averaging the exponential factor in (1.1) over spheres to obtain the Bessel kernel in (1.2), as in [8], and then comparing the resulting integral to (1.2). Radial characteristic functions are commonly studied in the context of isotropic random fields where they are called covariances.

Mittal [6] recently introduced and studied a particular class V_n of radial characteristic functions in R_n. The construction of this class was motivated in part by Berman's representation of characteristic functions in R_1 [2]. Let $f(x)$, $x > 0$, be a density function such that $f(x)/x^{n-1}$ is nonincreasing, and define

$$g(x) = \frac{f(x)}{S_n x^{n-1}}\,, \quad x > 0\,. \tag{1.4}$$

Define the function $K(s, t)$ on $R_n \times R_n$ as

$$\begin{aligned} K(\boldsymbol{s}, \boldsymbol{t}) &= \text{volume of the } (n+1)\text{-dimensional set} \\ &\{(x_1, \cdots, x_n, z)\colon g(\|\boldsymbol{x} + \boldsymbol{s}\|) > z,\ g(\|\boldsymbol{x} + \boldsymbol{t}\|) > z\} \end{aligned} \tag{1.5}$$

where $\boldsymbol{x} \in R_n$ and $z > 0$. For each z and $\boldsymbol{t}$ the set in R_n

$$\{\boldsymbol{x}\colon g(\|\boldsymbol{x} + \boldsymbol{t}\|) > z\}$$

is a ball centered at $-\boldsymbol{t}$, so that, by the invariance of volume under translation and rotation, K is a radial function of $\boldsymbol{s} - \boldsymbol{t}$. Therefore, there exists a function r of a real variable such that

$$K(\boldsymbol{s}, \boldsymbol{t}) = r(\|\boldsymbol{t} - \boldsymbol{s}\|)\,.$$

It is shown in [6] that K arises as the covariance of a certain isotropic Gaussian random field over R_n, so that r is of the form (1.2). It will be shown in §3 below that the spectral distribution function corresponding to K is absolutely continuous, and the density is characterized.

The purpose of this paper is to extend the work of Mittal in two directions:

(1) Here the class V_n is characterized in a simpler and more probabilistic way than was done by Mittal.

(2) We show that if r belongs to V_n, then the function G in the Schoenberg representation (1.2) is absolutely continuous, and its derivative has a canonical form as an integral transform of the function g (in (1.5)) with a squared Bessel kernel.

2. A new characterization of V_n.

THEOREM 2.1. *The function $r(t)$ belongs to the class V_n if and only if $1 - r(2t)$, $t \geqq 0$, is the distribution function of the product* *of independent positive random variables Y and U, where Y has a density function $f(x)$ such that $f(x)/x^{n-1}$ is nonincreasing, and where U has the density function*

$$\begin{aligned}(2.1)\qquad \psi_n(u) &= \frac{\Gamma(n/2)}{\Gamma((n-1)/2)\Gamma(1/2)}(1-u^2)^{(n-3)/2} \qquad 0 < u < 1\,, \\ &= 0\,, \quad \textit{elsewhere}\,.\end{aligned}$$

Proof. By a direct calculation of the volume (1.5), it is shown in [6] that

$$(2.2)\qquad r(t) = \frac{2}{c}\int_{t/2}^{\infty}\left(\int_0^{\cos^{-1}(t/2x)} \sin^{n-2}\alpha d\alpha\right)f(x)dx\,,$$

where

$$c = \int_0^{\pi}\sin^{n-2}\alpha d\alpha\,.$$

Now change the variable of integration from α to $u = \cos\alpha$, and then invert the order of integration:

$$r(t) = (2/c)\int_0^1\left(\int_{t/2u}^{\infty} f(x)dx\right)(1-u^2)^{(n-3)/2}du\,.$$

We also find that

$$c/2 = \frac{\Gamma((n-1)/2)\Gamma(1/2)}{\Gamma(n/2)}\,.$$

It follows immediately that

$$(2.3)\qquad 1 - r(2t) = \int_0^1\left(\int_0^{t/u} f(x)dx\right)\psi_n(u)du\,,$$

where ψ_n is defined in (2.1). The statement of Theorem 2.1 is now evident from (2.3).

We can think of the relation (2.2) as defining an integral transform of f. Mittal actually derived an inversion formula for this transform, and characterized r in terms of the operations used in the inversion [6]. However, the inversion formula is quite complicated; furthermore, it has different forms for even and odd n. We now deduce a simpler and different inversion formula from Theorem 2.1:

COROLLARY 2.1. *r belongs to V_n if and only if $1 - r(2e^{-t})$, $-\infty < t < \infty$, is a distribution function whose characteristic function factors into a product*

$$(2.4)\qquad \int_{-\infty}^{\infty} e^{iux}f(e^{-x})e^{-x}dx \cdot \int_0^{\infty} e^{iux}\psi_n(e^{-x})e^{-x}dx\,,$$

where f is a density on the positive axis such that $f(x)/x^{n-1}$ is nonincreasing, and where ψ_n is defined by (2.1).

Proof. According to Theorem 2.1, r belongs to V_n if and only if $1 - r(2e^{-t})$ is the distribution function of the sum of independent random variables $-\log Y$ and $-\log U$. The factors in (2.4) are the characteristic functions of these random variables.

According to this corollary, if r belongs to V_n, then the ratio

$$\frac{\int_{-\infty}^{\infty} e^{iut} d[1 - r(2e^{-t})]}{\int_0^{\infty} e^{iut} \psi_n(e^{-x}) e^{-x} dx}$$

is the Fourier transform of the density $f(e^{-x})e^{-x}$, $-\infty < x < \infty$, which can be determined by the classical inversion formula.

We remark that Theorem 2.1 can be viewed as an extension to $n \geqq 2$ of a known result for Polya characteristic functions on R_1 [7]. Indeed, every such characteristic function is of the form

$$r(t) = \int_t^{\infty} f(x) dx \,, \qquad t \geqq 0 \,,$$

where $f(x)$, $x \geqq 0$, is a nonincreasing density function (see, for example, [2]). Thus r is a Polya characteristic function if and only if $1 - r(t)$, or equivalently, $1 - r(2t)$, is the distribution function of a positive random variable Y having a nonincreasing density function. Mittal also observed that her characterization of V_n implied the same extension to $n \geqq 2$.

3. Absolute continuity of G and the canonical form of its derivative.

THEOREM 3.1. *If r is of the class V_n, then the function G in the representation* (1.2) *is absolutely continuous, with the derivative determined by* (1.3). *The function h is then of the form*

$$h(x) = -\int_0^{\infty} z^n J_{n/2}^2(xz) x^{-n} dg(z) \,, \tag{3.1}$$

where J is the Bessel function, and where g is the monotone function (1.4) *which defines r in* (1.5).

The proof will be completed in a series of lemmas.

LEMMA 3.1. *If $f(x)$, $x > 0$, is a density function such that $f(x)/x^{n-1}$ is nonincreasing, then*

$$\int_0^\infty x^n d(f(x)/x^{n-1}) > -\infty .$$

Proof. First we analyze the improper integral at the upper limit. By integration by parts:

$$\int_1^x t^n d(f(t)/t^{n-1}) = xf(x) - f(1) - n\int_1^x f(t)dt . \tag{3.2}$$

Since $f(t)/t^{n-1}$ is nonincreasing, the left hand side of the equation above is negative and nonincreasing in x; therefore, it approaches a limit, finite or negatively infinite. The last term on the right hand side converges to a finite limit because f is a density. It follows that the term $xf(x)$ also converges to a limit; hence, its average,

$$\frac{1}{x-1}\int_1^x yf(y)dy ,$$

also converges to the same limit for $x \to \infty$. But the latter limit is finite: indeed, for $x > 2$,

$$\frac{1}{x-1}\int_1^x yf(y)dy \leqq 2\int_1^x f(y)dy \uparrow 2\int_1^\infty f(y)dy \leqq 2 .$$

Therefore, the left hand side of (3.2) converges to a finite limit for $x \to \infty$.

By the same kind of calculation, we can show that the improper integral in the statement of the lemma converges to a finite limit at the lower limit of integration. This completes the proof.

As the first consequence of this lemma we note that if h is defined by (3.1), then

$$\int_0^\infty h(x)x^{n-1}dx < \infty .$$

Indeed, the integral above, after the inversion of the order of integration and a change of variable of integration, is equal to

$$-\int_0^\infty J_{n/2}^2(x)x^{-1}dx \cdot \int_0^\infty z^n dg(z) .$$

The first factor is, by a classical formula, [9], p. 405, formula (1), equal to $1/n$. The second factor is finite according to Lemma 3.1. It follows that $h(\|\boldsymbol{x}\|)$, properly normalized, is a density function over R_n. (See the opening remarks of §1.)

LEMMA 3.2. *$H(x)$ be a nonnegative continuous function, and $g(x)$ a nonnegative nonincreasing function. Put*

$$g^{-1}(y) = \sup(x: g(x) > y) ;$$

then for all continuity points $\underline{a}$ *and* $\underline{b}$ *of* $\underline{g}$, *with* $\underline{a} < \underline{b}$,

$$\int_{g(b)}^{g(a)} H(g^{-1}(y))dy = -\int_a^b H(x)dg(x) . \tag{3.3}$$

Proof. The equation above is directly verified in the case where g is a step function with a finite number of jumps. It is extended to the general function g by approximation by step functions and application of the Helly-Bray theorem.

As a consequence of Lemma 3.2, we have:

$$\int_{g(\infty)}^{g(0+)} \text{volume}\,(\boldsymbol{x}: g(\|\boldsymbol{x}\|) > z)dz = -B_n\int_0^\infty x^n dg(x) , \tag{3.4}$$

where B_n is the volume of the n-dimensional unit ball. For the proof note that the set in the integrand on the left hand side of (3.4) is a ball of radius $g^{-1}(z)$, so that its volume is $B_n(g^{-1}(z))^n$. Then apply Lemma 3.2 with $H(x) = x^n$, and let $\underline{a} \downarrow 0$ and $\underline{b} \to \infty$ in (3.3).

LEMMA 3.3. *If* r *belongs to* V_n, *and is determined by* g *according to the formula* (1.5), *then* G *in* (1.2) *is absolutely continuous, and the function* h *in* (1.3) *is necessarily of the form*

$$h(u) = (2\pi)^{-n}\int_0^\infty \left|\int_{\{x\ g(\|x\|)>z\}} e^{i(x,u)}d\boldsymbol{x}\right|^2 dz , \tag{3.5}$$

where $u = \|\boldsymbol{u}\|$, *and* $\boldsymbol{u} \in R_n$.

Proof. We are going to express the function K in (1.5) in a form which exhibits $r(\|\boldsymbol{t}-\boldsymbol{s}\|) = K(\boldsymbol{s}, \boldsymbol{t})$ as the characteristic function of an absolutely continuous distribution, and then show that the density is appropriately related to h. For $\boldsymbol{x} \in R_n$ and $z > 0$, let W be the function defined as

$$\begin{aligned} W(\boldsymbol{x}, z) &= 1 \quad \text{if} \quad g(\|\boldsymbol{x}\|) > z \\ &= 0 \quad \text{elsewhere;} \end{aligned}$$

then K in (1.5) may be represented as

$$\int_0^\infty \int_{R_n} W(\boldsymbol{x}+\boldsymbol{s}, z)W(\boldsymbol{x}+\boldsymbol{t}, z)d\boldsymbol{x}dz . \tag{3.6}$$

For fixed $z > 0$, the set $\{\boldsymbol{x}: g(\|\boldsymbol{x}\|) > z\}$ is a ball centered at the origin and of finite radius $g^{-1}(z)$; hence, for each $z > 0$, W belongs to $L_2(R_n)$, and so its Fourier transform $\hat{W}(\boldsymbol{u}, z)$, $\boldsymbol{u} \in R_n$, is well defined and is equal to

(3.7)
$$\hat{W}(\boldsymbol{u}, z) = \int_{\{\boldsymbol{x}: g(||\boldsymbol{x}||>z\}} e^{i(\boldsymbol{u},\boldsymbol{x})} d\boldsymbol{x} \ .$$

When the inner integral in (3.6) is transformed according to Parseval's theorem, the integral (3.6) becomes

(3.8)
$$(2\pi)^{-n} \int_0^\infty \int_{R_n} \left| \hat{W}(\boldsymbol{u}, z) \right|^2 e^{i(\boldsymbol{u}, \boldsymbol{t}-\boldsymbol{s})} d\boldsymbol{u} dz \ .$$

Let us now show that we may invert the order of integration in (3.8). Put $\boldsymbol{s} = \boldsymbol{t} = \boldsymbol{0}$ in (3.6); then the integral is equal to

$$\int_0^\infty \int_{R_n} W(\boldsymbol{x}, z) d\boldsymbol{x} dz \ ,$$

which, by Lemma 3.1 and the formula (3.4), is finite. Therefore W^2 is integrable over $\boldsymbol{x}$ and z, and so, by Parseval's theorem, $\hat{W}^2$ is also integrable over $\boldsymbol{u}$ and z.

After inversion of the order of integration in (3.8), we recognize this integral as the Fourier transform of the radial function of u,

$$(2\pi)^{-n} \int_0^\infty \left| \hat{W}(\boldsymbol{u}, z) \right|^2 dz \ ,$$

and so, by the Schoenberg representation (1.2), the integral (3.8) is representable in the form (1.2) with h of the form (3.5) and related to G through (1.3).

The last step in the proof of Theorem 3.1 is showing that the function h in (3.5) may be transformed into the expression (3.1). The inner multiple integral in (3.5) has as its domain of integration the ball in R_n centered at the origin and of radius $q = g^{-1}(z)$, and so may be expressed in the form

$$\int_{\{\boldsymbol{x}: ||\boldsymbol{x}|| \leq q\}} e^{i(\boldsymbol{u},\boldsymbol{x})} d\boldsymbol{x} \ ,$$

which, upon transformation to spherical coordinates in R_n, may be reduced to the single integral

$$\frac{\pi^{(n-1)/2}}{\Gamma((n+1)/2)} \int_{-q}^{q} (q^2 - x^2)^{(n-1)/2} e^{iux} dx \ ,$$

where $u = ||\boldsymbol{u}||$. By a classical formula, [9], p. 48, the latter is equal to

$$\pi^{n/2} q^{n/2} J_{n/2}(uq)/(u/2)^{n/2} \ ,$$

so that (3.5) becomes

$$h(u) = \int_0^{g(0)} (g^{-1}(z))^n J_{n/2}^2(ug^{-1}(z)) u^{-n} dz \ ,$$

where $g(0)$ may be finite or infinite. Lemma 3.2 implies that the latter is equal to (3.1) because z_n is integrable with respect to $dg(z)$ (Lemma 3.1) and the Bessel function is bounded on the real line.

4. Related work. A different but related class of radial characteristic functions was introduced by Askey [1]. This represents another generalization of Polya characteristic functions to R_n. Askey's Theorem 2 states that if $r(t) \to 0$ for $t \to \infty$, and if $(-1)^{[n/2]} r^{[n/2]}(t)$ is convex for $t \geqq 0$, then $r(\|t\|)$, $t \in R_n$, is a characteristic function. This condition on r is much simpler but different from Mittal's hypothesis on r which characterizes the class V_n. It is also of interest to compare the explicit form of Askey's functions with the form (2.2) of the class V_n. Under Askey's hypothesis there exists a nonnegative, nonincreasing function $g(s)$, $s \geqq 0$, such that

$$(-1)^m r^{(m)}(t) = \int_t^\infty g(s)ds,\ t \geqq 0,\ m = [n/2]\ .$$

By repeated integration we obtain

$$r(t) = \int_t^\infty \frac{(s-t)^m}{m!} g(s)ds\ , \quad t \geqq 0\ . \tag{4.1}$$

A comparable form of a function in V_n is obtained by the change of variable $u = \cos\alpha$ in (2.2):

$$r(t) = (2/c)\int_{t/2}^\infty \left\{\int_{t/2x}^1 (1-u^2)^{(n-3)/2} du\right\} f(x)dx\ . \tag{4.2}$$

I have found no direct relation between the forms (4.1) and (4.2).

The conjectures stated by Askey [1], upon which his theorem rests, have been proven by Gasper [4, 5] and Fields and Ismail [3]. I am indebted to the referee for bringing to my attention the references mentioned in this section.

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