PACIFIC JOURNAL OF MATHEMATICS
Vol. 78, No. 2, 1978

SUBMANIFOLDS WITH L-FLAT NORMAL CONNECTION OF THE COMPLEX PROJECTIVE SPACE

MASAFUMI OKUMURA

Real submanifolds with L-flat normal connection of the complex projective space are studied. As a special case "a complex submanifold with L-flat normal connection of the complex projective space is necessarily totally geodesic" is proved.

Introduction. As is well known an odd-dimensional sphere S^{2n+1} is a principal circle bundle over a complex projective space $P^n(C)$. The Riemannian structure on $P^n(C)$ is given by the submersion $\pi: S^{2n+1} \to P^n(C)$ which is defined by the Hopf-fibration. If we construct a circle bundle over a real submanifold of $P^n(C)$ in such a way that it is compatible with the Hopf-fibration, the circle bundle is a submanifold of the odd-dimensional sphere. Thus when we want to study submanifolds of the complex projective space it is useful to study the circle bundle over the submanifold. From this point of view, H. B. Lawson, Jr. [2] and the present author [3, 4, 5] have studied real submanifolds of the complex projective space. In the previous paper [5], the author studied relatious between the normal connection of a submanifold of $P^n(C)$ and that of the circle bundle over the submanifold and established the notion of L-flatness for the normal connection of a real submanifold of $P^n(C)$.

The purpose of the present paper is to study submanifolds with L-flat normal connection of $P^n(C)$. The main result is the following.

THEOREM 1. *The totally geodesic complex projective linear subspaces $P^n(C)$ are the only complex submanifolds with L-flat normal connection of $P^{n+p}(C)$.*

In § 1 we state some formulas for real submanifolds of a Kaehlerian manifold and in § 2 we discuss the case when the ambient manifold is the complex projective space. There we explain L-flatness of the normal connection. In § 3, we calculate the Laplacian for a function which is defined on the submanifold and prove some theorems including Theorem 1.

1. Real submanifolds of a Kaehlerian manifold. Let M' be a real $(n + p)$-dimensional Kaehlerian manifold with Kaehlerian structure (J, G), that is, J is the endomorphism of the tangent bundle $T(M')$ satisfying $J^2 = -$ identity and G the Riemannian metric of M' satisfying the Hermitian condition $G(JX', JY') = G(X', Y')$ for any $X', Y' \in T(M')$.

Let M be an immersed submanifold of M' and i be the immersion. Then the tangent bundle $T(M)$ is identified with a subbundle of $T(M')$ and a Riemannian metric g of M is induced from the Riemannian metric G of M' in such a way that $g(X, Y) = G(iX, iY)$, where $X, Y \in T(M)$. The normal bundle $N(M)$ is the subbundle of $T(M')$ consisting of all $X' \in T(M')$ which are orthogonal to $T(M)$ with respect to G. At each point of M, we choose orthonormal normal vectors $n_1, n_2, \cdots, n_p$ to M and extend them respectively to $N_1, N_2, \cdots, N_p$ in such a way that they belong to $N(M)$.

For any $X \in T(M)$ and for N_A, $A = 1, 2, \cdots, p$, the transforms JiX and JN_A are respectively written in the following forms:

$$JiX = iFX + \sum_{A=1}^{p} u^A(X) N_A \,, \tag{1.1}$$

$$JN_A = -iU_A + \sum_{B=1}^{p} F'^B_A N_B \,, \tag{1.2}$$

where F, F', U_A, and u^A define respectively an endomorphism of $T(M)$, that of $N(M)$, local tangent vector fields and local 1-forms on M. They satisfy the relations $u^A(X) = g(U_A, X)$, $F'^B_A = -F'^A_B$. Applying J to both side members of (1.1) and (1.2), we find

$$F^2X = -X + \sum_{A=1}^{p} u^A(X) U_A \,, \tag{1.3}$$

$$u^A(FX) = -\sum_{B=1}^{p} F'^A_B u^B(X) \,, \tag{1.4}$$

$$FU_A = -\sum_{B=1}^{p} F'^B_A U_B \,, \tag{1.5}$$

$$\sum_{C=1}^{p} F'^C_A F'^B_C = -\delta^B_A + u^B(U_A) \,. \tag{1.6}$$

If at any point of M, $FU_A = 0$ are valid for $A = 1, 2, \cdots, p$, we know that F satisfies $F^3 + F = 0$ and that the rank of $F \geqq n - p$. In this case the submanifold M is called an *F-submanifold* of rank $\geqq n - p$ [7]. We denote by ∇ and D the Riemannian connection of M and M' respectively and by D^N the induced normal connection from D to $N(M)$. Then they are related by the following Gauss and Weingarten equations.

(1.7) $$D_{iX}iY = i\nabla_X Y + h(X, Y),\ h(X, Y) = \sum_{A=1}^{p} h^A(X, Y)N_A ,$$

(1.8) $$D_{iX}N_A = -iH_AX + D_X^N N_A,\ D_X^N N_A = \sum_{B=1}^{p} L_A{}^B(X)N_B ,$$

where $h(X, Y)$ is the second fundamental form and H_A's are symmetric linear transformations of $T(M)$ which are called the Weingarten maps for the normal N_A. The last two equations show that $h^A(X, Y) = g(H_AX, Y)$. The mean curvature vector field μ of M is defined by

(1.9) $$\mu = \left(\sum_{A=1}^{p} (\mathrm{trace}\, H_A)N_A\right)/n$$

and it is well known that μ is independent of the choice of N_A's. If the mean curvature vector field vanishes identically on M, M is called a minimal submanifold of M'. By definition M is minimal if and only if $\mathrm{trace}\, H_A = 0$, $A = 1, 2, \cdots, p$ at each point of M. Differentiating (1.1) and (1.2) covariantly and making use of the fact that the Riemannian connection D of M' leaves the almost complex structure J invariant, we have

(1.10) $$(\nabla_Y F)X = \sum_{A=1}^{p} \{u^A(X)H_AY - g(H_AX, Y)U_A\} ,$$

(1.11) $$\nabla_X U_A = FH_AX + \sum_{B=1}^{p} \{L_A{}^B(X)U_B - F'^B_A H_BX\} ,$$

(1.12) $$D_X^N F'^B_A = g(H_BU_A - H_AU_B, X) .$$

Differentiating (1.9) covariantly, we have

$$nD_X^N\mu = \sum_{A=1}^{p} \left\{X(\mathrm{trace}\, H_A)N_A + \sum_{B=1}^{p} (\mathrm{trace}\, H_A)L_A{}^B(X)N_B\right\} ,$$

from which we know that the mean curvature vector field is parallel with respect to the normal connection if and only if

(1.13) $$X(\mathrm{trace}\, H_A) = \sum_{B=1}^{p} L_A{}^B(X)\,\mathrm{trace}\, H_B ,$$

because of the fact that $L_A{}^B(X) = -L_A{}^B(X)$ for any $X \in T(M)$.

2. Real submanifolds of the complex projective space. Let the ambient manifold M' be the complex projective space $P^{(n+p)/2}(C)$ with the Fubini-Study metric of constant holomorphic sectional curvature 4. Since the curvature tensor $R'(X', Y')Z'$ of the ambient manifold satisfies

$$(2.1)\quad R'(X', Y')Z' = G(Y', Z')X' - G(X', Z')Y' + G(JY', Z')JX' - G(JX', Z')JY' - 2G(JX', Y')JZ'$$

the Codazzi, Ricci equations become respectively

$$(2.2)\qquad (\nabla_X H_A)Y - (\nabla_Y H_A)X = \sum_{B=1}^{p} \{L_A{}^B(X)H_B Y - L_A{}^B(Y)H_B X\} - u^A(Y)FX + u^A(X)FY - 2g(FX, Y)U_A ,$$

$$(2.3)\qquad R^N(X, Y)N_A = \sum_{B=1}^{p} \{g((H_A H_B - H_B H_A)X, Y) + u^A(Y)u^B(X) - u^A(X)u^B(Y) - 2g(FX, Y)F'^B_A\}N_B ,$$

where $R^N(X, Y)$ denotes the normal curvature of M'. If we choose an orthonormal basis $\{E_1, \cdots, E_n\}$ of $T_x(M)$ at a point $x \in M$, it follows from (1.4) and (2.3) that

$$(2.4)\quad \sum_{i=1}^{n} R^N(FE_i, E_i)N_A = \sum_{B=1}^{p} \{\text{trace}\,(H_A H_B - H_B H_A)F + 2g(FU_A, U_B) - 2\,(\text{trace}\, F^2)F'^B_A\}N_B .$$

Now recall the fact that an $n + p + 1$-dimensional sphere of radius 1 is a principal circle bundle over the complex projective space and let π be the Hopf-fibration. We construct the circle bundle over the submanifold M in such a way that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(M) & \longrightarrow & S^{n+p+1} \\ \downarrow & & \downarrow \\ M & \longrightarrow & P^{(n+p)/2}(C) . \end{array}$$

In the previous paper [5] the author proved that the normal connection of $\pi^{-1}(M)$ in S^{n+p+1} is flat if and only if the following two conditions are satisfied on M:

$$(2.5)\qquad R^N(X, Y)N_A = -2g(FX, Y)\sum_{B=1}^{p} F'^B_A N_B ,$$

$$(2.6)\qquad D^N_X F'^B_A = 0 .$$

Moreover the author proved that if $n > p + 2$, (2.5) implies (2.6). Thus we call the normal connection of M in $P^{(n+p)/2}(C)$ *lift-flat* normal connection or briefly *L-flat* normal connection if it satisfies (2.5) and (2.6). It is easily checked that the totally geodesic complex submanifold $P^{n/2}(C)$ of $P^{(n+p)/2}(C)$ is a submanifold with L-flat normal connection. We can also check the fact that if the submanifold M is a complex submanifold, the normal connection is L-flat if and only if the Weingarten maps H_A and H_B commute for any pair of A and $B = 1, 2, \cdots, p$.

3. Submanifolds with L-flat normal connection. We put

$$f = \sum_{A=1}^{p} u^A(U_A) \; . \tag{3.1}$$

Then from (1.3) we have

$$f = \text{trace } F^2 + n \;, \tag{3.2}$$

which shows that f is a globally defined function on the submanifold M. Using the formulas which are stated in § 1, we now calculate the Laplacian Δf.

By means of (1.10) and (3.2), it follows that

$$\frac{1}{2} Yf = \frac{1}{2} Y(\text{trace } F^2) = \text{trace}\,(\nabla_Y F)F = 2\sum_{A=1}^{p} g(FH_A Y, U_A) \;, \tag{3.3}$$

from which

$$\begin{aligned} \frac{1}{4}(\nabla_X\nabla_Y f - \nabla_{\nabla_X Y} f) &= \frac{1}{4}(\nabla_X(Yf) - (\nabla_X Y)f) \\ &= \sum_{A=1}^{n} \{g((\nabla_X F)H_A Y, U_A) + g(F(\nabla_X H_A)Y, U_A) \\ &\quad + g(FH_A Y, \nabla_X U_A)\} \\ &= \sum_{A,B=1}^{p} \{g(H_A U_B, Y)g(H_B U_A, X) - g(H_A H_B X, Y)g(U_A, U_B) \\ &\quad - g((\nabla_X H_A)FU_A, Y) - g(F^2 H_A Y, H_A X) \\ &\quad - L_A{}^B(X)g(H_A FU_B, Y) - F'^B_A g(X, H_B F H_A Y)\} \;, \end{aligned} \tag{3.4}$$

because of (1.10) and (1.11).

On the other hand, from (2.2), it follows that

$$\begin{aligned} (\nabla_X H_A)FU_A = (\nabla_{FU_A} H_A)X &+ \sum_{B=1}^{p} \{L_A{}^B(X)H_B FU_A - L_A{}^B(FU_A)H_B X\} \\ &+ u^A(X)F^2 U_A - 2g(FX, FU_A)U_A \;. \end{aligned}$$

Substituting this into (3.4), we find

$$\begin{aligned} \frac{1}{4}(\nabla_X\nabla_Y f - \nabla_{\nabla_X Y} f) = \sum_{A'B=1}^{p} &\{g(H_A U_B, Y)g(H_B U_A, X) \\ &- g(H_A H_B X, Y)g(U_A, U_B) - g((\nabla_{FU_A} H_A)X, Y) \\ &+ L_A{}^B(FU_A)g(H_B X, Y) - g(U_A, X)g(F^2 U_A, Y) \\ &- 2g(F^2 U_A, X)g(U_A, Y) + g(H_A^2 Y, X) \\ &- g(H_A U_B, Y)g(H_A U_B, X) - F'^B_A g(X, H_B F H_A Y)\} \;, \end{aligned}$$

from which

$$\frac{1}{4}\Delta f = \sum_{A,B=1}^{p} \{g(H_A U_B,\ H_B U_A - H_A U_B) - \operatorname{trace}(H_A H_B) g(U_A,\ U_B)$$
$$- (\nabla_{FU_A}(\operatorname{trace} H_A) - L_A{}^B(FU_A)\operatorname{trace} H_B)$$
$$+ 3g(FU_A,\ FU_A) + \operatorname{trace} H_A^2 - F_A'^B \operatorname{trace} H_A H_B F\}\ . \tag{3.5}$$

Now we rewrite (3.5), using the normal curvature R^N. By means of (2.4) and the fact that F is a skew-symmetric linear transformation, we have

$$\frac{1}{4}\Delta f = \sum_{A,B=1}^{p} \Big\{g(H_A U_B,\ H_B U_A - H_A U_B) - \operatorname{trace}(H_A H_B) g(U_A,\ U_B)$$
$$- (\nabla_{FU_A}(\operatorname{trace} H_A) - L_A{}^B(FU_A)\operatorname{trace} H_B)$$
$$+ 3g(FU_A,\ FU_A) + \operatorname{trace} H_A^2 \tag{3.6}$$
$$- F_A'^B\Big(\frac{1}{2}\sum_{i=1}^{n} G(R^N(FE_i,\ E_i)N_A,\ N_B) - g(FU_A,\ U_B)$$
$$+ (\operatorname{trace} F^2)F_A'^B\Big)\Big\}\ .$$

Before we prove Theorem 1, we prove the following more general result.

THEOREM 2. *Let M be a real submanifold with L-flat normal connection of a complex projective space. If the mean curvature vector field is parallel with respect to the normal connection, then the submanifold M is an F-submanifold of rank $\geqq n-p$. Particularly, if F is of almost everywhere rank n, M is a totally geodesic complex submanifold and consequently a complex projective linear subspace.*

We begin with the following

LEMMA. *When the normal connection of M in $P^{(n+p)/2}(C)$ is L-flat, the function f is constant.*

Proof. It follows from (1.5) and (3.3) that

$$\frac{1}{4}Yf = -\sum_{A=1}^{p} g(H_A Y,\ FU_A) = \sum_{A,B=1}^{p} F_A'^B g(H_A U_B,\ Y)$$
$$= \frac{1}{2}\sum_{A,B=1}^{p} F_A'^B g(H_A U_B - H_B U_A,\ Y)\ .$$

Thus, (1.12) and (2.6) imply that f is constant.

Proof of Theorem 3. Since the $p \times p$ matrix (trace $H_A H_B$) is symmetric and can be assumed to be diagonal for a suitable choice of $N_1, N_2, \cdots, N_p$,

$$\sum_{A,B=1}^{p} \operatorname{trace}(H_A H_B) g(U_A, U_B) = \sum_{A=1}^{p} (\operatorname{trace} H_A^2) g(U_A, U_A) .$$

Moreover the conditions of the theorem, (1.13), (2.5), (2.6), and (3.6) imply that

$$\nabla_{FU_A} (\operatorname{trace} H_A) = \sum_{B=1}^{p} L_A{}^B(FU_A) \operatorname{trace} H_B ,$$

$$\sum_{i=1}^{n} G(R^N(FE_i, E_i)N_A, N_B) = -2 (\operatorname{trace} F^2) F'^B_A ,$$

and

$$\sum_{B=1}^{p} F'^B_A g(FU_A, U_B) = -g(FU_A, FU_A) .$$

Thus we have

$$\frac{1}{4}\Delta f = \sum_{A=1}^{p} \{\operatorname{trace} H_A^2 - (\operatorname{trace} H_A^2) g(U_A, U_A) + 2g(FU_A, FU_A)\} = 0 , \tag{3.7}$$

from which $FU_A = 0$, $A = 1, 2, \cdots, p$, because of $g(U_A, U_A) \leqq 1$. Thus M is an F-submanifold of rank $\geqq n - p$. To prove the last part of the theorem we assume that the rank of F is almost everywhere n. Then we have $U_A = 0$ for $A = 1, 2, \cdots, p$. This means that the submanifold is a complex submanifold. Again we use (3.7) and get that M is totally geodesic. This completes the proof.

Proof of Theorem 1. It is well known [6] that a complex submanifold is a minimal submanifold and the induced complex structure F is of rank n. Hence a complex submanifold with L-flat normal connection satisfies the conditions of Theorem 3. This completes the proof of Theorem 1.

Finally we point out that, as an application of our discussions the formula (3.6) gives another proof of the following known [1].

THEOREM 3. *There does not exist a complex submanifold with flat normal connection of a complex projective space.*

Proof. Since a complex submanifold is invariant under J it follows that $U_A = 0$, $u^A = 0$ for $A = 1, 2, \cdots, p$. Hence (3.6) becomes

$$\begin{aligned} \frac{1}{4}\Delta f &= \sum_{A=1}^{p} \operatorname{trace} H_A^2 - \sum_{A,B=1}^{p} (\operatorname{trace} F^2) F'^B_A F'^B_A \\ &= \sum_{A=1}^{p} \operatorname{trace} H_A^2 + np > 0 , \end{aligned}$$

because of (1.6). On the other hand, (3.3) implies that the function f is constant and consequently $\Delta f = 0$. This is contradiction. This completes the proof.

REFERENCES

1. B. Y. Chen and K. Ogiue, *Some extrinsic results for Kaehler submanifolds*, Tamkang J. Math., **4** (1973), 207-213.
2. H. B. Lawson Jr., *Rigidity theorems in rank*-1 *symmetric spaces*, J. of Differential Geometry, **4** (1970), 349-357.
3. M. Okumura, *On some real hypersurfaces of a complex projective spaces*, Trans. Amer. Math. Soc., **212** (1975), 355-364.
4. M. Okumura, *Submanifolds of real codimension of a complex projective space*, Atti della Academia Nazionale dei Lincei, **58** (1975), 544-555.
5. ———, *Normal curvature and real submanifold of the complex projective space*, Geometriae Dedicata.
6. J. A. Schouten and K. Yano, *On invariant subspaces in the almost complex* X_{2n}, Indagatione Math., **17** (1955), 261-269.
7. K. Yano and S. Ishihara, *The f-structure induced on submanifolds of complex and almost complex spaces*, Kōdai Math. Sem. Rep., **18** (1966), 120-160.

Received August 9, 1977 and in revised form March 26, 1978.

SAITAMA UNIVERSITY
URAWA, JAPAN