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MAPS AND h-NORMAL SPACES

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Further consequences of hard sets are explored in this paper, and some new relations between a space X and its extension δX are shown. A generalization of perfect maps, called δ-perfect maps, is introduced. It is found that among the WZ-maps, these are precisely the ones which pull hard sets back to hard sets. Applications to δX are made. Maps which carry hard sets to closed sets and maps which carry hard sets to hard sets are considered, and it is seen that the image of a realcompact space under a closed map is realcompact if and only if the map carries hard sets to hard sets.

The last part of the paper introduces a generalization of normality, called h-normal, in which disjoint hard sets are completely separated. It is found that X is h-normal whenever υX is normal. The hereditary and productive properties of h-normal spaces are investigated, and the h-normal spaces are characterized in terms of δ-perfect WZ-maps. Finally as an analogue of closed maps on normal spaces, a necessary and sufficient condition is found that the image of an h-normal space under a δ-perfect WZ-map be h-normal.

1. Introduction. All spaces discussed in this paper are assumed Tychonov (completely regular and Hausdorff) and the word *map* means a continuous surjection. The notation of [2] is used throughout. In particular, βX is the Stone-Čech compactification and υX is the Hewitt realcompactification of X.

The following facts concerning hard sets will be used here. They are found in [8] and [9].

DEFINITION 1. For any space X, let $cl_{\beta X}(\upsilon X - X) = K(=K_X)$. A set $H \subseteqq X$ is called *hard* (in X) if H is closed as a subset of $X \cup K$. (A characterization of hard sets internal to X is given in [8].) Let δX be the subspace of βX given by $\delta X = \beta X - (K - X)$. Thus $X \subseteqq \delta X \subseteqq \beta X$.

PROPOSITION 2. *A subset H of space X is hard if and only if there is a compact subset of δX whose restriction to X is H.*

PROPOSITION 3. *Every compact set in X is hard, but every hard set is compact if and only if $X = \delta X$. (Note every pseudocompact space is of this type.)*

PROPOSITION 4. *Every hard set of X is closed, but every closed* *set is hard if and only if X is realcompact.*

PROPOSITION 5. *A closed subset of a hard set is hard.*

It follows immediately from the definition that X is realcompact if and only if δX is compact. We conclude this section with some new results.

LEMMA 6. *The set of points at which δX fails to be locally compact is precisely the set of points at which X fails to be locally realcompact.*

Proof. Let $R(\delta X)$ be the set of points at which δX fails to be locally compact. By ([5], 2.10), the set of points at which X fails to be locally realcompact is $X \cap K$. But $\beta(\delta X) - \delta X = \beta X - \delta X = K - X$. Thus $cl_{\beta X}(\beta X - \delta X) = cl_{\beta X}(K - X) = K$. So $R(\delta X) = \delta X \cap cl_{\beta X}(\beta\delta X - \delta X) = \delta X \cap K = X \cap K$.

COROLLARY 7. *X is locally realcompact if and only if δX is locally compact.*

COROLLARY 8. *Let X be locally realcompact. The hard zero sets form a base for the hard sets.*

Proof. In δX as in any locally compact space, the compact zero sets form a base for the compact sets.

COROLLARY 9. *X is locally realcompact if and only if every hard set of X is contained in the interior of a regular-hard (i.e., hard and regular-closed) set of X.*

Proof. Let H be a hard set of X. Then $cl_{\delta X}H$ is compact in the locally compact space δX, so it is contained in the interior of a regular compact set B of δX. Restrict B to X.

THEOREM 10. *For any X, δX is the union of the βX-closures of the hard sets of X.*

Proof. Let $p \in \delta X - X$. By Lemma 6, there is a compact set F such that $x \in \text{int}_{\delta X}(F) \subseteqq F \subseteqq \delta X$. Let $G = X \cap \text{int}_{\delta X}(F)$, then $cl_{\delta X}(G) = cl_{\delta X}\,\text{int}_{\delta X}(F)$. Let $H = cl_X(G)$, so H is a hard set of X and $p \in cl_{\delta X}(H) = cl_{\beta X}(H)$.

II. **δ-perfect maps.** Let $f\colon X \to Y$ be any map and $f_\beta\colon \beta X \to \beta Y$

be its Stone extension. Henriksen and Isbell [3] have studied those maps, now called *perfect*, which are closed and pull compact sets back to compact sets.

PROPOSITION 11. *A map $f: X \to Y$ is perfect if and only if for each $y \in Y$, $f_\beta^{\leftarrow}(y) \subseteqq X$.*

Proof. This follows from the characterization in [3] that f is perfect if and only if $f_\beta[\beta X - X] = \beta Y - Y$.

DEFINITION 12. The map $f: X \to Y$ is *δ-perfect* if for each $y \in Y$, $f_\beta^{\leftarrow}(y) \subseteqq \delta X$.

Clearly every perfect map is δ-perfect. Yet if Y is compact and X is realcompact and not compact, there are no perfect maps from X onto Y, but every map is δ-perfect since $\delta X = \beta X$.

LEMMA 13. *A map $f: X \to Y$ is δ-perfect if and only if $f_\beta^{\leftarrow}[\delta Y] \subseteqq \delta X$.*

Proof. One direction is trivial. For the other, note $\upsilon X \subseteqq f_\beta^{\leftarrow}[\upsilon Y] = f_\beta^{\leftarrow}[Y] \cup f_\beta^{\leftarrow}[\upsilon Y - Y]$. By hypothesis, $f_\beta^{\leftarrow}[y] \cap (\upsilon X - X) = \varnothing$. Thus $\upsilon X - X \subseteqq f_\beta^{\leftarrow}[\upsilon Y - Y] \subseteqq f_\beta^{\leftarrow}[K_Y]$ which is a compact set. Whence $cl_{\beta X}(\upsilon X - X) \subseteqq f_\beta^{\leftarrow}[K_Y]$, so $X \cup K_X \subseteqq f_\beta^{\leftarrow}[Y \cup K_Y]$. Therefore $f_\beta^{\leftarrow}[\delta Y] \subseteqq \delta X$.

COROLLARY 14. *The composition of δ-perfect maps is δ-perfect.*

In [4], Isiwata introduced the concept of a *WZ-map* as a map $f: X \to Y$ such that for each $y \in Y$, $f_\beta^{\leftarrow}(y) = cl_{\beta X} f^{\leftarrow}(y)$. He showed that every Z-map (i.e., a map which carries zero sets to closed sets) is a WZ-map. Clearly every closed map is a Z-map, and every perfect map is a WZ-map. We shall see (Lemma 19 and Corollary 21) that δ-perfect maps and WZ-maps are independent concepts; but those maps which are both δ-perfect and WZ are of particular interest.

LEMMA 15. *A map $f: X \to Y$ is a δ-perfect WZ-map if and only if for all $y \in Y$, $f_\beta^{\leftarrow}(y) = cl_{\delta X} f^{\leftarrow}(y)$.*

Proof. $cl_{\delta X} f^{\leftarrow}(y) \subseteqq cl_{\beta X} f^{\leftarrow}(y) \subseteqq f_\beta^{\leftarrow}(y)$.

In [8], we showed that a perfect map pulls hard sets back to hard sets. More generally, THEOREM 16. *Let $f: X \to Y$ be a map. Each of the following conditions implies the next one.*

(a) *f is δ-perfect.*

(b) *f pulls hard sets back to hard sets.*

(c) *f pulls points back to hard sets.*

Moreover if f is a WZ-map, they are all equivalent.

Proof. (a) implies (b) since a set H in Y is hard if and only if $cl_{\beta Y}H \subseteqq \delta Y$. Thus $f_{\beta}^{\leftarrow}[cl_{\beta Y}H]$ is compact and contained in δX, whence $X \cap f_{\beta}^{\leftarrow}[cl_{\beta Y}H]$ is hard in X and contains the closed set $f^{\leftarrow}[H]$. But a closed subset of a hard set is hard. (b) implies (c) since every compact set is hard.

Finally, suppose f is a WZ-map satisfying (c). Then for every $y \in Y$, $f_{\beta}^{\leftarrow}(y) = cl_{\beta X}f^{\leftarrow}(y) = cl_{\delta X}f^{\leftarrow}(y)$. Whence by Lemma 15, f is δ-perfect.

COROLLARY 17. *If $X = \delta X$ and $f: X \to Y$ is a δ-perfect map, then $Y = \delta Y$.*

Proof. Let H be a hard set in Y. Then $f^{\leftarrow}[H]$ is hard, hence compact in δX. Thus $H = f \circ f^{\leftarrow}[H]$ is compact in Y. Therefore, by Proposition 3, $Y = \delta Y$.

COROLLARY 18. *If X is compact and $X \times Y = \delta(X \times Y)$, then $Y = \delta Y$.*

Zenor [11] constructed a useful map: let A be a closed subset of space X and define φ_A to be the natural function taking X onto $Y = X/A$. Topologize Y with the finest completely regular topology making φ_A continuous. Zenor shows that φ_A is always a WZ-map.

LEMMA 19. *φ_A is δ-perfect if and only if A is hard in X.*

Proof. By Theorem 16, φ_A is δ-perfect if and only if the pre-image of every point is hard. The pre-image of every point other than $\varphi_A(A)$ is itself, and compact sets are always hard. But $A = \varphi_A^{\leftarrow} \circ \varphi_A(A)$.

THEOREM 20. *A space X is realcompact if and only if every map on X (to a Tychonov space) is δ-perfect.*

Proof. We have already observed one direction. Conversely, let A be an arbitrary nonempty closed set of X. The Zenor's map

$\varphi_A\colon X \to Y = X/A$ is δ-perfect. Whence by Lemma 19, A is hard. The result follows from Proposition 4.

COROLLARY 21. *Any nonclosed map on a normal realcompact space is δ-perfect and not WZ.*

Proof. Isiwata ([4], 1.3) has shown that every WZ-map on a normal space is closed.

THEOREM 22. *$X = \delta X$ if and only if every δ-perfect WZ-map on X (to a Tychonov space) is perfect.*

Proof. (*If*). Let A be an arbitrary hard set of X and $\varphi_A\colon X \to Y = X/A$ be the Zenor map. By Lemma 19, φ_A is a δ-perfect WZ-map, so it is perfect. Hence the pre-image of every point is compact. In particular, the pre-image A of the point $\varphi_A(A)$ is compact. But $X = \delta X$ precisely when every hard set is compact. (Only if). For each $y \in Y$, $f_\beta^{\leftarrow}(y) = cl_{\delta X} f^{\leftarrow}(y) = cl_X f^{\leftarrow}(y) = f^{\leftarrow}(y) \subseteqq X$.

DEFINITION 23. A map $f\colon X \to Y$ is an *H-map* if the image of each hard set in X is a closed set of Y. If f carries hard sets to hard sets, we shall call f a *hard* map.

Clearly closed maps and hard maps are H-maps. If X is realcompact, then every closed set is hard, so every H-map on a realcompact space is closed. If $X = \delta X$, then every hard set is compact, so every map on X is a hard map. Isiwata ([4], 3.6) has constructed an example of a map on a pseudocompact space which is not a WZ-map. Thus an H-map need not be WZ. However,

LEMMA 24. *If $f\colon X \to Y$ is a δ-perfect H-map, then f is a WZ-map.*

Proof. Let $y \in Y$. Since $f_\beta^{\leftarrow}(y) \subseteqq \delta X$, we see that $cl_{\beta X} f^{\leftarrow}(y) = cl_{\delta X} f^{\leftarrow}(y)$.

Suppose $x \in f_\beta^{\leftarrow}(y) - cl_{\delta X} f^{\leftarrow}(y)$. Since $x \in \delta X - X$ by Lemma 6 there is a δX-open set N such that $x \in N \subseteqq cl_{\beta X} N \subseteqq \delta X - cl_{\delta X} f^{\leftarrow}(y)$. Let $M = cl_X(N \cap X)$. Since X is dense in δX, $cl_{\beta X}(M) = cl_{\delta X}(N)$, and M is a nonempty hard set of X disjoint from $f^{\leftarrow}(y)$. Thus y is not in $f(M)$, and since f is an H-map, $f(M) = cl_Y f(M)$. But $y = f_\beta(x) \in f_\beta[cl_{\beta X} M] \cap Y = cl_{\beta Y}[f_\beta(M)] \cap Y = cl_{\beta Y}[f(M)] \cap Y = cl_Y f(M) = f(M)$, contradiction.

LEMMA 25. *Let $f\colon X \to Y$ be a hard map. Then $\delta X \subseteqq f_\beta^{\leftarrow}[\delta Y]$.*

Proof. By Theorem 10, $\delta X = \cup \{cl_{\beta X} H\colon H \text{ is hard in } X\}$. For each hard set H of X, $f[H]$ is hard in Y and $cl_{\beta X}H \subseteq f_\beta^{\leftarrow}[cl_{\beta Y}f(H)]$.

THEOREM 26. *$f: X \to Y$ is a hard map if and only if f is an H-map and $\delta X \subseteq f_\beta^{\leftarrow}[\delta Y]$.*

Proof. Every hard map is an H-map, so one direction follows from Lemma 25. Conversely, let H be a hard set of X. Then $f_\beta[cl_{\beta X}H] \subseteq \delta Y$. But $cl_{\beta X}H$ is compact, so $f_\beta[cl_{\beta X}H]$ is compact. Since $f(H) \subseteq cl_{\beta Y}f(H) \subseteq f_\beta(cl_{\beta X}H)$, we have $Y \cap cl_{\beta Y}f(H) = cl_Y f(H)$ is hard in Y. Since f is an H-map, $f(H) = cl_Y f(H)$.

COROLLARY 27. *Let $f: X \to Y$ be a δ-perfect H-map. Then f is a hard map if and only if $\delta X = f_\beta^{\leftarrow}[\delta Y]$.*

Proof. Theorem 26 and Lemma 13.

COROLLARY 28. *Let X be realcompact and $f: X \to Y$ be a closed map. Then Y is realcompact if and only if f is a hard map.*

Proof. Since X is realcompact, $\delta X = \beta X$ and $f_\beta[\delta X] = \beta Y$. Every map on a realcompact space is δ-perfect, so by Corollary 27, f is a hard map if and only if $\beta Y = \delta Y$, i.e., Y is realcompact.

In a private communication, John Mack states that he has investigated a class of maps $f: X \to Y$, which he calls *R-perfect* maps, satisfying the condition that the graph of f, $\mathscr{G}(f)$, is closed in $(\upsilon X) \times Y$. Since these results are not reproduced elsewhere, the author has Mack's permission to include them here.

LEMMA 29 (Mack). *Let $f: X \to Y$ be a map and $f_\upsilon: \upsilon X \to \upsilon Y$ be its Hewitt extension. The following are equivalent:*

(a) *f is R-perfect.*

(b) *$\mathscr{G}(f) = \mathscr{G}(f_\upsilon) \cap (\upsilon X \times Y)$.*

(c) *$f_\upsilon^{\leftarrow}(\upsilon Y - Y) = \upsilon X - X$.*

Proof. (a) implies (b). For any map, $\mathscr{G}(f_\upsilon)$ is the closure of $\mathscr{G}(f)$ in $\upsilon X \times \upsilon Y$. So if f is R-perfect, then $\mathscr{G}(f)$ is the intersection of $\upsilon X \times Y$ with the $\upsilon X \times \upsilon Y$-closure of $\mathscr{G}(f)$, which is $(\upsilon X \times Y) \cap \mathscr{G}(f_\upsilon)$. (b) implies (c). By (b), we have $f_\upsilon^{\leftarrow}(Y) = f^{\leftarrow}(Y) = X$, whence $f_\upsilon^{\leftarrow}(\upsilon Y - Y) = \upsilon X - X$.

(c) implies (a) $f_\upsilon^{\leftarrow}(\upsilon Y - Y) = \upsilon X - X$ implies $\mathscr{G}(f) = (\upsilon X \times Y) \cap \mathscr{G}(f_\upsilon)$, which is the intersection of $\upsilon X \times Y$ with the $\upsilon X \times \upsilon Y$-closure of $\mathscr{G}(f)$. Thus $\mathscr{G}(f)$ is closed in $\upsilon X \times Y$.

THEOREM 30 (Mack). *Let $f: X \to Y$ be an R-perfect map.*
(a) *If $F \subseteq Y$ is realcompact, then $f^{\leftarrow}(F)$ is realcompact.*
(b) *If Y is locally realcompact, then X is locally realcompact.*

Proof. (a) $\upsilon X \times F$ is realcompact. Since the graph $\mathscr{G}(f)$ is closed in $\upsilon X \times Y$, then $\mathscr{G}(f) \cap (X \times F) = \mathscr{G}(f_\nu) \cap (\upsilon X \times F)$ is realcompact. But $f^{\leftarrow}[F]$ is homeomorphic to $\mathscr{G}(f) \cap (X \times F)$.

(b) $\upsilon X - X = f_\nu^{\leftarrow}(\upsilon Y - Y)$. But Y is locally realcompact if and only if $\upsilon Y - Y$ is closed in υY. Whence X is open in υX.

Notice that it follows from Lemma 13 and 29(c) that every δ-perfect map is R-perfect. The converse is false.

EXAMPLE 31 of an R-perfect map which is not δ-perfect. Let W be the ordinals less than the first uncountable ordinal ω_1, and let T be the free union of countably infinitely many copies of W. Then υT is the free union of the one point compactifications of the W's, so K_T is homeomorphic to βN (where N is the discrete space of positive integers). Let $p \in K_T - \upsilon T$ and define $X = T \cup \{p\}$ as a subspace of βT. Then $\upsilon X = \upsilon T \cup \{p\}$ and $X \cup K_X = T \cup K_T$. Now let Y be the quotient space of $T \cup (K_T - \upsilon T)$ obtained by factoring the compact set $K_T - \upsilon T$ to a point k. It is not difficult to see that Y is Tychonov, and $\upsilon Y = \upsilon T \cup \{k\} = Y \cup K_T$. Note $K_Y \cap Y = \{k\}$. Let $f: X \to Y$ be the restriction of the quotient map, so $f(p) = k$ and $f(x) = x$ otherwise. Moreover f_ν extends f by being the identity map on $\upsilon T - T$, so $f_\nu^{\leftarrow}(\upsilon Y - Y) = \upsilon X - X$ and f is an R-perfect map. But $k \in Y$ and $f_\beta^{\leftarrow}(k) = K_T - \upsilon T \supseteq K_X - \upsilon X \neq \varnothing$. So f is not δ-perfect.

THEOREM 32. *Let $f: X \to Y$ be an R-perfect map. If Y is locally realcompact, then f is δ-perfect.*

Proof. Since f is R-perfect, $\upsilon X - X = f_\nu^{\leftarrow}(\upsilon Y - Y) \subseteq f_\beta^{\leftarrow}(\upsilon Y - Y) \subseteq f_\beta^{\leftarrow}(K_Y)$, which is compact. Hence $K_X \subseteq f_\beta^{\leftarrow}(K_Y)$. Since Y is locally realcompact, $\delta Y = \beta Y - K_Y$. So $f_\beta^{\leftarrow}(\delta Y) = f_\beta^{\leftarrow}(\beta Y - K_Y) = \beta X - f_\beta^{\leftarrow}(K_Y) \subseteq \beta X - K_X = \delta X$, by Theorem 30(b).

III. h-normal spaces.

DEFINITION 33. Let $X \subseteq T \subseteq \beta X$. A set $H \subseteq X$ is *T-hard* if H is closed in $X \cup cl_{\beta X}(T - X)$. We shall call X a *T-normal* space if disjoint T-hard sets of X are completely separated in X. Notice that for any T, every normal space is always T-normal. If $T = \nu X$, the T-hard sets of X are simply the hard sets, and we shall use the term *h-normal* space in this case.

It follows from Proposition 4 that a realcompact space is h-normal if and only if it is normal. Similarly by Proposition 3, for any X we have that δX is an h-normal space. In particular, every pseudocompact space is h-normal. Thus the Tychonov plank is an h-normal space which is not normal.

THEOREM 34. *Let $X \subseteqq T \subseteqq \beta X$. The following are equivalent:*
(a) *X is T-normal.*
(b) *There is a Y, $X \subseteqq Y \subseteqq T$ and Y is T-normal.*
(c) *$X \cup cl_{\beta X}(T - X)$ is normal.*
(d) *Each closed subset of X is completely separated from every disjoint T-hard set.*

Proof. That (c) implies (d) and (d) implies (a) are easy exercises. It sufficies to show (b) implies (c). Let A_1 and A_2 be disjoint and closed in $X \cup cl_{\beta X}(T - X)$. Let $B_i = A_i \cap cl_{\beta X}(T - X)$, $i = 1, 2$. Then B_1 and B_2 are compact. By ([2], 3.11a), there are zero sets Z_j, $j = 1, 2, 3, 4$, of $X \cup cl_{\beta X}(T - X)$ such that
(i) $A_1 \subseteqq \text{int}(Z_1)$, $B_2 \subseteqq \text{int}(Z_2)$ and $Z_1 \cap Z_2 = \emptyset$, and
(ii) $A_2 \subseteqq \text{int}(Z_3)$, $B_1 \subseteqq \text{int}(Z_4)$ and $Z_3 \cap Z_4 = \emptyset$.
Let $H_1 = A_1 - \text{int}(Z_4)$ and $H_2 = A_2 - \text{int}(Z_2)$. If either H_1 or H_2 is empty, we have disjoint open neighborhoods of A_1 and A_2, so we are done. Otherwise H_1 and H_2 are nonempty, disjoint T-hard sets of X, hence of Y. Thus there are functions f and g in $C^*(Y)$ such that $H_1 \subseteqq \text{int}_Y Z(f)$, $H_2 \subseteqq \text{int}_Y Z(g)$ and $Z(f) \cap Z(g) = \emptyset$. Since $\beta Y = \beta X$ and disjoint zero sets of Y have disjoint closures in βY, we have that the $X \cup cl_{\beta X}(T - X)$-closures $Z'(f)$ and $Z'(g)$ are disjoint. Let G_1 and G_2 be the $X \cup cl_{\beta X}(T - X)$-interiors of $Z'(f)$ and $Z'(g)$ respectively. Note that $H_1 \subseteqq G_1$ and $H_2 \subseteqq G_2$. Let $F_1 = [\text{int}(Z_4) \cup G_1] \cap \text{int}(Z_1)$ and $F_2 = [\text{int}(Z_2) \cup G_2] \cap \text{int}(Z_3)$. Then $A_1 \subseteqq F_1$, $A_2 \subseteqq F_2$ and F_1, F_2 are disjoint sets open in $X \cup cl_{\beta X}(T - X)$.

COROLLARY 35. *X is h-normal if and only if $X \cup K$ is normal.*

Let X be a locally realcompact and not realcompact space. Then K is a nonempty compact set disjoint from X. In [5], it was shown that factoring K to a single point gave a one-point realcompactification $^*X$ of X. Moreover $^*X$ is maximal among the one-point realcompactifications of X in the sense that if $X \cup \{p\}$ is any other, then there is a map from $^*X$ onto $X \cup \{p\}$ which is the identity on X.

COROLLARY 36. *If X is locally realcompact and not realcompact, then X is h-normal if and only if $^*X$ is normal.*

COROLLARY 37. *If υX is normal, then every C-embedded subset of X is h-normal.*

Proof. Let A be C-embedded in X. Then $\upsilon A = cl_{\upsilon X}(A)$ ([2], 8.10a), and a closed subset of a normal space is normal. Hence A is h-normal by Theorem 34(b).

COROLLARY 38. *If υX is normal, then X is h-normal. Moreover for any space X for which $\upsilon X - X$ is closed in $\beta X - X$, υX is normal if and only if X is h-normal.*

Proof. If $\upsilon X - X$ is closed in $\beta X - X$, then $\upsilon X = X \cup K$.

EXAMPLE 39. Corson's space X ([2], p 272) is normal, hence h-normal, but υX is not normal.

EXAMPLE 40. Realcompact spaces and pseudocompact spaces trivially satisfy the condition $\upsilon X - X$ is closed in $\beta X - X$. In general, let Y by any Tychonov space and define $X = (Y \cup K_Y) - (\upsilon Y - Y)$. Then $Y \subseteq X \subseteq \beta Y$, so $\beta X = \beta Y$, $\upsilon X = Y \cup K_Y$, $\upsilon X - X = \upsilon Y - Y$ and $K_X \cap X = K_Y - \upsilon Y$. Thus $\upsilon X - X$ is closed in $\beta X - X$. By construction, $X = Y$ if and only if $\upsilon Y - Y$ is closed in $\beta Y - Y$, so this technique generates all the spaces with the desired property. Notice the generated space X is realcompact if and only if Y is realcompact, and X is pseudocampact if and only if $\upsilon X = \beta X$ which (since $\beta X = \beta Y$) is equivalent to $Y \cup K_Y = \beta Y$, which is true if and only if $Y = \delta Y$. Hence if Y is a nonrealcompact space for which $Y \neq \delta Y$, then X is neither realcompact nor pseudocompact, yet $\upsilon X - X$ is closed in $\beta X - X$. E.g., let $Y = W \times N$, where W is the usual space of ordinals with countable predecessors and N is the discrete space of positive integers. The author does not have any internal characterizations for the spaces X for which $\upsilon X - X$ is closed in $\beta X - X$.

DEFINITION 41. A subset of a space X will be called an H_σ-set if it is the union of a countable family of hard sets. Every σ-compact set is an H_σ-set and every H_σ-set is an F_σ-set.

COROLLARY 42. *Every H_σ-subspace of an h-normal space is normal.*

Proof. The H_σ-sets of X are F_σ-sets of $X \cup K$, and F_σ-sets of a normal space are normal.

COROLLARY 43. *Every hard subset of an h-normal space is C-embedded.*

Proof. A hard set H is closed in normal $X \cup K$, and every closed subset of a normal space is C-embedded ([2], 3D1).

EXAMPLE 44. The Sorgenfrey plane S is a realcompact space which is not h-normal. Let W be the space of ordinals with countable predecessors and $W^* = W \cup \{\omega_1\}$ be its compactification. Put $X = [W^* \times \beta S] - [\{\omega_1\} \times (\beta S - S)]$. Then X is pseudocompact ([2], 9K) and $\{\omega_1\} \times S$ is a closed, C^*-embedded subset of h-normal X which fails to be h-normal.

EXAMPLE 45. Let X be a normal, realcompact but not paracompact space. (By Moran's result [7], barring measurable cardinals, normal and metacompact imply realcompact. Hence Michael's example in [6] is such a space.) Then by Tamano's theorem ([10], Th. 2) $X \times \beta X$ is realcompact and not normal, hence not h-normal. Thus the product of a normal space and a compact space can fail to be h-normal.

THEOREM 46. *Let X and Y have nonmeasurable cardinals. If υX is paracompact and Y is a locally compact, paracompact space, then $X \times Y$ is h-normal.*

Proof. For Tychonov spaces with nonmeasurable cardinals, paracompact implies normal and realcompact. From [1], if Y is locally compact and real-compact, then for any X, $\upsilon(X \times Y) = (\upsilon X) \times Y$.

In [11] the following remarks are made about Zenor's maps φ_A (see Proposition 18 above):

1. X is normal if and only if φ_A is a quotient map for each closed set A in X.
2. Each closed set is completely separated from every disjoint zero set in X if and only if φ_A is a quotient map for each zero set A in X.

In like vein, we observe:

LEMMA 47. *X is h-normal if and only if φ_A is a quotient map for each hard set A in X.*

From [11], we also have

PROPOSITION 48 (Zenor). (a) *X is normal if and only if every Z-map is closed.* (b) *Each closed set is completely separated from every disjoint zero set in X if and only if every WZ-map is a Z-map.*

THEOREM 49. *For any space X, the following are equivalent.*
(a) *X is h-normal.*
(b) *Every WZ-map on X is an H-map.*
(c) *Every δ-perfect WZ-map on X is closed.*

Proof. (a) implies (b). Let $f: X \to Y$ be a WZ-map and let H be a hard set in X. Suppose $y \in Y - f(H)$. Then $f^{\leftarrow}(y)$ is closed in X and disjoint from H, whence $f^{\leftarrow}(y)$ and H are completely separated. So $cl_{\beta X} f^{\leftarrow}(y) \cap cl_{\beta X}(H) = \emptyset$ and y is not in $f_\beta[cl_{\beta X} H]$. But $f_\beta[cl_{\beta X} H] \cap Y$ is closed in Y and contains $f(H)$. Thus $f(H)$ is closed.

(b) implies (a). Let H be a hard set of X and F a closed set disjoint from H. Consider the Zenor map φ_F. It is a WZ-map, so $\varphi_F[H]$ is closed and $\varphi_F(F) \notin \varphi_F(H)$. Since Y is completely regular, $\varphi_F(F)$ and $\varphi_F(H)$ are completely separated, whence F and H are completely separated.

(a) implies (c). Let $f: X \to Y$ be a δ-perfect WZ-map and let B be a closed subset of X. Let $p \in Y - f(B)$. Then $f^{\leftarrow}(p)$ is hard in X and disjoint from B, hence B and $f^{\leftarrow}(p)$ are completely separated. Therefore $cl_{\beta X} B$ and $cl_{\beta X} f^{\leftarrow}(p) = f_\beta^{\leftarrow}(p)$ are disjoint, so p is not in $f_\beta[cl_{\beta X} B]$. Since f_β is a closed map, $f_\beta[cl_{\beta X} B]$ is a closed set containing $f(B)$. Therefore $p \notin cl_Y f(B)$ and $f(B)$ is closed.

(c) implies (a). Suppose X is not h-normal. There is a closed set F and a hard set H which is disjoint to it, but not completely separated from it. Consider the Zenor map φ_H. By Lemma 19, φ_H is a δ-perfect WZ-map. If $\varphi_H(F)$ is closed in Y, then there is some $Z_1 = Z(f_1) \in Z(Y)$ such that $\varphi_H(F) \subseteqq Z_1 \subseteqq Y - \varphi_H(H)$. Thus $\varphi_H(H) \in Y - Z_1$ which is open. Hence there is a $Z_2 = Z(f_2) \in Z(Y)$ such that $\varphi_H(H) \in \text{int}_Y Z_2 \subseteqq Z_2 \subseteqq Y - Z_1$. Now $f_i \circ \varphi_H: X \to R$ is continuous, $i = 1, 2$, and $Z(f_1 \circ \varphi_H)$ and $Z(f_2 \circ \varphi_H)$ are disjoint zero sets in X completely separating F and H, contradiction. Whence φ_H is not closed.

We observe that the closed image of a normal space is normal. If $X = \delta X$, then every map on X is an H-map. Hence by Lemma 24, every δ-perfect map $f: X \to Y$ is a δ-perfect WZ-map. (Notice that since δX is an h-normal space, such an f must be closed by Theorem 49(c).) By Corollary 17, $Y = \delta Y$ is also h-normal. More generally,

THEOREM 50. *Let X be an h-normal space and $f: X \to Y$ be a δ-perfect WZ-map. Then Y is h-normal if and only if for every δ-perfect WZ-map g on Y, $g \circ f$ is a WZ-map.*

Proof. (*If*). $g \circ f$ is δ-perfect by Corollary 14. Whence by Theorem 49(c), $g \circ f$ is closed. Thus g is closed and since g is an arbitrary δ-perfect WZ-map on Y, Y is h-normal.

(Only if). f and g are closed maps, whence $g \circ f$ is a closed map. But closed maps are WZ.

It is not true in general that the composition of WZ-maps is a WZ-map; in fact an example due to M. Henriksen shows more.

EXAMPLE 51 (Henriksen). A closed map and a Z-map whose composition is not a WZ-map. Consider the subspace of the product of ordinal spaces given by

$$X = W(\omega_1 + 1) \times W(\omega_2 + 1) - \{\omega_1\} \times [W(\omega_2 + 1) - W(\omega_1)] .$$

We observe that X is pseudocompact and $\beta X = W(\omega_1 + 1) \times W(\omega_2 + 1)$. Let $Y = W(\omega_1 + 1) \times W(\omega_1 + 1) - \{(\omega_1, \omega_1)\}$ and define $t: X \to Y$ by $t(a, b) = (a, \omega_1)$ if $b \geqq \omega_1$, $t(a, b) = (a, b)$ otherwise. Since $[W(\omega_2 + 1) - W(\omega_1)]$ is compact, it follows that t is a closed map.

Let $\varphi: Y \to W(\omega_1 + 1)$ be given by $\varphi(a, b) = a$. Isiwata has shown ([4], 3.5) that φ is an open Z-map which is not closed. Consider $\varphi \circ t: X \to W(\omega_1 + 1)$. We have $cl_{\beta X}(\varphi \circ t)^{\leftarrow}(\omega_1) = cl_{\beta X}[\{\omega_1\} \times W(\omega_1)] = \{\omega_1\} \times W(\omega_1 + 1)$. But $(\varphi \circ t)_{\beta}^{\leftarrow}(\omega_1) = \{\omega_1\} \times W(\omega_2 + 1)$, so $\varphi \circ t$ is not a WZ-map.

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