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IMAGES OF SK_1ZG

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The computation of SK_1ZG for finite nonabelian groups G remains a difficult problem. Few examples are known in which SK_1ZG is nontrivial. One way to uncover nontrivial elements is to examine the homomorphic images of SK_1ZG under $K_1(-)$ of ring maps $ZG \to \Lambda$. Such images are investigated here in the cases where Λ is a commutative ring, a noncommutative order or a semisimple artinian image of ZG. Even trivial images illuminate the structure of SK_1ZG through K-theory exact sequences.

2. Terminology. The word "ring" refers to an associative ring with an identity. The group of units of a ring Λ is denoted Λ^*. "Map" meas homomorphism. Unless otherwise specified, G denotes a finite group, and R, the ring of integers in an algebraic number field F.

3. The origin of K_1. In 1950 J. H. C. Whitehead introduced the notion of simple homotopy equivalence. A natural question in this theory is the following: Which CW-complexes are equivalent, relative to a common subcomplex L, under deformations which add and delete cells in a "simple" way along the cell structure? (See [3] for details.)

The answer lies in the computation of the Whitehead group $Wh(L)$, which depends only on the fundamental group $\pi_1(L)$. In fact it is obtained by the following algebraic construction: Let ZG denote the integral group ring of a (possibly infinite) group G. Then $GL(ZG)$ is the group of all invertible matrices over ZG, with matrices A and B identified if $A = \begin{bmatrix} B & 0 \\ 0 & I_n \end{bmatrix}$ for some size identity matrix I_n. The commutator subgroup $E(ZG)$ of $GL(ZG)$ is generated by all elementary matrices, obtained from the identity by adding a ZG multiple of one row to another. The quotient $GL(ZG)/E(ZG)$ is written K_1ZG. The trivial units $\pm G$ of ZG are 1×1 matrices in $GL(ZG)$. The Whitehead group $Wh_1(G)$ of G is $K_1ZG/\text{Image}(\pm G)$. If L is a CW-complex, $Wh(L) = Wh_1(\pi_1(L))$.

Group maps $G \to H$ extend to ring maps $ZG \to ZH$, and entry-wise on representative matrices to groups $K_1ZG \to K_1ZH$. This makes $K_1Z(-)$ a functor from groups to abelian groups. Replacing ZG, the same construction provides the functor $K_1(-)$ from rings to abelian groups. However, the group G plays a special role in the computation of K_1ZG, which apparently has no natural analog in $K_1\Lambda$ for an arbitrary ring Λ.

Assume henceforth that G denotes a finite group. H. Bass proved [2] that K_1ZG, hence also Wh_1G, is a finitely generated abelian group of rank $r - q$, where r and q are the numbers of inequivalent irreducible real and rational representations of G, respectively. C. T. C. Wall showed [11] that the torsion parts are $\operatorname{tor} K_1ZG = SK_1ZG \times \pm G^{ab}$ and $\operatorname{tor} Wh_1(G) = SK_1ZG$, where SK_1ZG is the kernel of a determinant map, defined on K_1ZG in the next section.

4. Three functors called SK_1. Let Λ be an R order in a finite dimensional semisimple F algebra Γ. (The motivating case is $R = Z$, $F = Q$, $\Lambda = ZG$, and $\Gamma = QG$.) There is a direct product decomposition $\Gamma \overset{\approx}{\to} \prod_{i=1}^{s} \sum_i$, where each $\sum_i$ is a full matrix ring over a division ring whose center C_i contains F. There exists [4, p. 96] a number field E containing each C_i, for which there are isomorphisms: $E \otimes_{C_i} \sum_i \overset{\approx}{\to} M_{n_i}(E)$. Application of the embedding $\Lambda \to \prod_i M_{n_i}(E)$ in each entry provides a map $GL_n(\Lambda) \to \prod_i GL_{nn_i}(E)$. Following this by the determinant in each component defines a group map, det: $K_1\Lambda \to \prod_i E^*$, with kernel denoted $SK_1\Lambda$.

Let $\mathscr{O}$ denote the category of R orders in finite dimensional semisimple F algebras, and of R algebra maps. The map det factors through $K_1\Gamma \to \prod_i E^*$, which is an injection [12]; so $SK_1\Lambda$ is also the kernel of $K_1\Lambda \to K_1\Gamma$ (induced by inclusion $\Lambda \hookrightarrow \Gamma$). Any map $\Lambda \to \Lambda'$ in $\mathscr{O}$ extends to an F algebra map $F\Lambda \to F\Lambda'$. Application of $K_1(-)$ to the resulting commutative square shows that $K_1(\Lambda \to \Lambda')$ takes $SK_1\Lambda$ into $SK_1\Lambda'$. So $SK_1(-)$ is a functor on $\mathscr{O}$.

Any group map $G \to H$ extends to an R algebra map $RG \to RH$. So $SK_1R(-)$ is a functor from finite groups to abelian groups. The application of this functor to inclusions $G \hookrightarrow H$ has been used in connection with the Artin and Berman-Witt induction theorems to compute SK_1ZH from the groups SK_1ZG as G ranges over certain classes of subgroups of H. (See [5], [7], and [11].)

When G is abelian, FG is a direct product of fields, and det: $K_1RG \to (RG)^*$ is the ordinary determinant. If A is any commutative ring, the determinant on $GL(A)$ induces a group map, δ: $K_1A \to A^*$, split by $GL_1(A) \to GL(A) \to K_1A$. Define SK_1A as the kernel of δ; so $K_1A = SK_1A \times A^*$. Since determinants commute with ring maps, $SK_1(-)$ is a functor commutative rings.

Any ring map from RG into a commutative ring S factors as a composite: $RG \to R[G^{ab}] \to S$ of an R algebra map followed by a map between commutative rings. Since the definitions of $SK_1(-)$ agree on the middle term, SK_1RG is mapped into SK_1S. If we replace RG by another R order this may fail, as the next section shows.

5. Twisted group rings. The noncommutativity in RG is due to the group G. Sometimes a map may be found in $\mathscr{O}$ which replaces some noncentral group elements by units in an extended coefficient ring. This shifts noncommutativity between group elements to a "twist" between coefficients and group elements.

An example of such a twisted group ring is described as follows. Let E/F be a Galois extension of number fields, with Galois group H. Let $E \circ H$ be the F algebra with E basis given by the elements of H and distributive multiplication subject to the relations in H and $he = e^h h (e \in E, h \in H)$. Restrict coefficients to the integers S of E to obtain the twisted group ring $S \circ H$, an R order in $E \circ H$.

Assume H is abelian. Let I be the ideal of S generated by $\{s - s^h; s \in S, h \in H\}$. Since I is H invariant, $I \circ H$ is an $S \circ H$ ideal, and the group ring $(S/I)H$ is a commutative quotient ring of $S \circ H$. The quotient map is universal for maps from $S \circ H$ to a commutative ring. Suppose $I \neq S$.

The composite: $H \hookrightarrow GL_1(S \circ H) \to K_1(S \circ H) \to K_1(S/I)H \xrightarrow{\delta} ((S/I)H)^*$ is just inclusion. So $H \subseteqq K_1(S \circ H)$, and the elements $h \neq 1$ in H do *not* map into $SK_1(S/I)H$. In fact, $(S/I)H$ is a finite commutative ring; so $SK_1(S/I)H = 1$ [1, p. 267].

EXAMPLE 1. Let G be the metacyclic group $\langle x, y: x^{p^r} = y^p = 1, y^{-1}xy = x^\alpha, \alpha = p^{r-1} + 1\rangle$ for an odd prime p. Replacing x by a primitive p^r root of unity ζ is a ring map $ZG \twoheadrightarrow Z[\zeta] \circ H$, where $H = \langle y: y^p = 1\rangle$. This map is one projection of the decomposition, $QG \xrightarrow{\approx} QG^{ab} \times Q(\zeta) \circ H$. Computation of the determinant of a matrix representation [7, Ch. 6]: $Q(\zeta) \circ H \xrightarrow{\approx} M_p(Q(\zeta^p))$ reveals that $H \subseteqq SK_1(Z[\zeta] \circ H)$. So, in this example, $SK_1(S \circ H) \not\twoheadrightarrow SK_1(S/I)H$.

In short, the prevalent definitions of $SK_1(-)$ are unambiguous where applied, but are not part of a general "subfunctor" of $K_1(-)$ on rings.

6. Exact sequences. If J is an ideal of a ring A, let $GL(A, J)$ denote the kernel of $GL(A \to A/J)$, and let $E(A, J)$ be the normal subgroup generated by elementary matrices in $GL(A, J)$. Define $K_1(A, J)$ to be $GL(A, J)/E(A, J)$. Suppose A is either commutative or an R order in a finite dimensional semisimple F algebra. Since $GL(A, J) \subseteqq GL(A)$, $SK_1(A, J)$ may be defined as the kernel of the appropriate determinant, δ or det, on $K_1(A, J)$. A sequence of J. Milnor's [8, p. 54] restricts to the *relative exact sequence*:

$$K_2A \longrightarrow K_2(A/J) \longrightarrow SK_1(A, J) \longrightarrow SK_1A \longrightarrow K_1(A/J)\ .$$

Call a commutative square of surjective ring maps:

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & A_1 \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\nu} \\ A_2 & \xrightarrow[\mu]{} & A' \end{array}$$

a *surjective pullback* if the sequence of additive groups:

$$0 \longrightarrow A \xrightarrow{(\kappa,\lambda)} A_1 \oplus A_2 \xrightarrow{\mu - \nu} A' \longrightarrow 0$$

is exact. If all four rings are commutative or $A_2 \overset{\kappa}{\twoheadleftarrow} A \overset{\lambda}{\twoheadrightarrow} A_1$ is in $\mathcal{O}$ (see §4), there is an exact *Mayer-Vietoris sequence* [6, §4]:

$$K_2A_1 \times K_2A_2 \longrightarrow K_2A' \xrightarrow{\partial_m} SK_1A \xrightarrow{(\kappa.,\lambda.)} SK_1A_1 \times SK_1A_2 \xrightarrow{\nu./\mu.} K_1A'$$

where $\kappa.$, $\lambda.$, $\mu.$, and $\nu.$ denote $K_1(-)$ of the corresponding maps κ, λ, μ, and ν.

In particular a surjective group map $G \twoheadrightarrow H$ decomposes FG as a direct product of F algebras: $FG \overset{\approx}{\rightarrow} FH \times \Sigma$. Projections of RG to the two factors are the maps κ and λ in the following surjective pullback [6, §5]:

$$\begin{array}{ccc} RG & \xrightarrow{\lambda} & \Lambda \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\nu} \\ RH & \xrightarrow[\mu]{} & (R/nR)H \end{array} \quad (n = |G|/|H|) .$$

Consider its Mayer-Vietoris sequence:

$$(1) \quad K_2(R/nR)H \xrightarrow{\partial_m} SK_1RG \xrightarrow{(\kappa.,\lambda.)} SK_1RH \times SK_1\Lambda \xrightarrow{\nu./\mu.} K_1(R/nR)H .$$

Computation of the first term is complicated. In some cases, when $K_2(R/nR)H = 1$, this sequence has been used to prove $SK_1ZG = 1$ [6]. But when nontrivial generators of $K_2(R/nR)H$ are known, it is difficult to determine whether or not their images in SK_1RG are trivial.

Consider, rather, the maps to the right of SK_1RG.

7. **Right exactness of $SK_1R(-)$.** Since any ring map from RG to a commutative ring factors through $RG \rightarrow R[G^{ab}]$, the latter is the most informative about SK_1RG. Whether or not $SK_1R(-)$ is right exact is an open question, but the Mayer-Vietoris sequence provides a strong partial result:

THEOREM 1. *If R is the ring of integers in a number field and*

G is a finite group with abelian quotient H, then the quotient map induces a surjection: $SK_1RG \twoheadrightarrow SK_1RH$.

Proof. By inspection of the sequence (1), $\kappa.$ is surjective if and only if $\mu.(SK_1RH) \subseteqq \nu.(SK_1\Lambda)$. If H is abelian, $\mu.(SK_1RH) \subseteqq SK_1(R/nR)H$. Since $(R/nR)H$ is a finite commutative ring, $SK_1(R/nR)H = 1$.

Note 1. There are algorithms for the computation of SK_1ZH when H is a finite abelian group [10].

Note 2. Even if H is not abelian, a *split* surjection of finite groups $G \twoheadrightarrow H$ induces a split surjection of abelian groups $SK_1RG \twoheadrightarrow SK_1RH$.

8. **The image $\lambda.(SK_1RG)$.** Example 1 shows that $SK_1ZG \rightarrow SK_1\Lambda$ need not be surjective, if it is induced by projection of ZG to its image Λ in a noncommutative factor of QG. Indeed, in that example, the composite $K_1ZG \rightarrow K_1(S \circ H) \rightarrow K_1(S/I)H$ kills SK_1ZG, but the second map does not kill $H \subseteqq SK_1(S \circ H)$.

Suppose in the sequence (1) that $\mu.(SK_1RH) = 1$, as happens when H is abelian. The following exact sequence may be extracted:

$$SK_1RG \xrightarrow{\lambda.} SK_1\Lambda \xrightarrow{\nu.} K_1(R/nR)H \ .$$

This extends to the left as the relative sequence for λ. From the relative sequence of ν it is clear that $\lambda.(SK_1RG)$ is the image of the natural map $SK_1(\Lambda, J) \rightarrow SK_1\Lambda$, where J is the kernel of ν. If H is G^{ab}, J is generated by the set of all $ab - ba (a, b \in \Lambda)$.

9. **Maps to semisimple artinian rings.** Since SK_1RG is the kernel of $K_1(RG \hookrightarrow FG)$, it may be expected to appear in the kernel of $K_1(RG \rightarrow \Sigma)$ for other semisimple artinian rings Σ. Specifically, the Wedderburn theorems describe Σ as a direct product of matrix rings over division rings, providing a determinant map on $K_1\Sigma$. A connection might be expected between the map det and the composite $K_1RG \rightarrow K_1\Sigma \rightarrow$ determinant $(K_1\Sigma)$.

THEOREM 2. *If $\mathfrak{P}$ is a maximal ideal of R not dividing the order of G, then $K_1(-)$ of the quotient map $RG \rightarrow (R/\mathfrak{P})G$ kills SK_1RG.*

(My thanks to Frank Demeyer for suggesting the following use of localization).

Proof. Let E be a number field which splits every simple component of FG. Then there is an isomorphism $\rho: EG \xrightarrow{\approx} \prod_{i=1}^{s} M_{n_i}(E)$.

Let $\mathfrak{S}$ denote a prime lying over $\mathfrak{P}$ in the integers S of E. The order of G is a unit in the localization $S_{\mathfrak{S}}$; so $S_{\mathfrak{S}}G$ is a maximal $S_{\mathfrak{S}}$ order in EG [9, Theorem 41.1]. Since $S_{\mathfrak{S}}$ is a discrete valuation ring, $S_{\mathfrak{S}}G$ is conjugate in EG to the maximal $S_{\mathfrak{S}}$ order $\rho^{-1}(\prod_{i=1}^{s} M_{n_i}(S_{\mathfrak{S}}))$ [9, Theorem 18.7].

The following diagram of ring maps commutes:

$$\begin{array}{ccccccc}
RG & \hookrightarrow & R_{\mathfrak{P}}G & \twoheadrightarrow & (R/\mathfrak{P})G & \xrightarrow{\approx} & \prod_{j=1}^{t} M_{m_j}(L_j) \\
\cap\downarrow & & \cap\downarrow & b & \cap\downarrow & d & \cap\downarrow \\
EG & \hookleftarrow & S_{\mathfrak{S}}G & \twoheadrightarrow & (S/\mathfrak{S})G & \xrightarrow{\approx} & \prod_{j=1}^{t} M_{m_j}((S/\mathfrak{S}) \otimes_{R/\mathfrak{P}} L_j) \\
\downarrow\wr & a & \downarrow\wr & c & \downarrow\wr & & \\
\prod_{i=1}^{s} M_{n_i}(E) & \hookleftarrow & \prod_{i=1}^{s} M_{n_i}(S_{\mathfrak{S}}) & \twoheadrightarrow & \prod_{i=1}^{s} M_{n_i}(S/\mathfrak{S}) & . &
\end{array}$$

The vertical maps in square a are conjugation followed by ρ. The left side of b induces the right side on the coefficient level. The left side of c sends $\mathfrak{S}S_{\mathfrak{S}}$ to itself, inducing the right side. Since $(R/\mathfrak{P})G$ is semisimple and finite ($\mathfrak{P} \nmid |G|$), there is a top isomorphism in d, where the L_j are finite fields. The bottom of d is $(S/\mathfrak{S}) \otimes_{R/\mathfrak{P}} (-)$ of the top.

Apply $K_1(-)$ to the diagram above, and extend by the appropriate determinant maps δ to obtain the following commutative diagram of abelian groups:

(2)

$$\begin{array}{cccccccc}
 & K_1RG & \to & K_1R_{\mathfrak{P}}G & \to & K_1(R/\mathfrak{P})G & \xrightarrow{\approx} & \prod K_1L_j & \xrightarrow{\delta} & \prod L_j^* \\
 & \downarrow & & \downarrow & & \downarrow\beta & & \downarrow\alpha & & \cap\downarrow \\
 & K_1EG & \leftarrow & K_1S_{\mathfrak{S}}G & \to & K_1(S/\mathfrak{S})G & \xrightarrow{\approx} & \prod K_1((S/\mathfrak{S})\otimes L_j) & \xrightarrow{\delta} & \prod ((S/\mathfrak{S})\otimes L_j)^* \\
\Delta & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & & & \\
 & \prod K_1E & \leftarrow & \prod K_1S & \to & \prod K_1(S/\mathfrak{S}) & & & & \\
 & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & & & \\
\to & \prod E^* & \hookleftarrow & \prod S_{\mathfrak{S}}^* & \xrightarrow{\gamma} & \prod (S/\mathfrak{S})^* & . & & &
\end{array}$$

Since $SK_1(-)$ of any product of fields or local commutative ring is trivial, the determinants δ are all isomorphisms. So α and hence β, is injective. The distinction between $EG = E \otimes_F FG$ and $\prod E \otimes_{C_i} \Sigma_i$ (from §4) is just a duplication of components. So the kernel of the composite Δ is SK_1RG. This is killed by the composite $K_1RG \to K_1R_{\mathfrak{P}}G \to K_1(R/\mathfrak{P})G$, which is $K_1(RG \to (R/\mathfrak{P})G)$.

COROLLARY 3. *Let $f\colon RG \twoheadrightarrow \Sigma$ be a surjective ring map, where*

Σ is a semisimple artinian ring of characteristic coprime to the order of G. Then $K_1 f(SK_1RG) = 1$.

Proof. The ring Σ is a direct product $\prod \Sigma_i$ of simple artinian rings. The center of each is a field which is a finitely generated abelian group (Σ_i being an image of RG). So $f(R)$ is a direct product of fiinite residue fields $\prod R/\mathfrak{P}_i$. Since the characteristic of Σ is the least common multiple of the characteristics of the $R/\mathfrak{P}_i$, no $\mathfrak{P}_i$ divides the order of G. Because f factors through $RG \to (R/\prod_{\mathfrak{P}_i})G = \prod (R/\mathfrak{P}_i)G$ (where the product is taken over distinct $\mathfrak{P}_i$), and because $K_1(-)$ respects direct products, the corollary follows from the case $f: RG \to (R/\mathfrak{P}_i)G$.

Let M be the set of maximal ideals of R not dividing the order of G. If $\mathfrak{P} \in M$, the (unrestricted) relative exact sequence, $K_2(R/\mathfrak{P})G \to K_1(RG, \mathfrak{P}G) \to K_1RG \to K_1(R/\mathfrak{P})G$, provides a natural identification of $K_1(RG, \mathfrak{P}G)$ with the kernel of $K_1(RG \to (R/\mathfrak{P})G)$, since K_2 of a direct product of full matrix rings over finite fields is trivial. Using the restricted relative sequence (§6), Theorem 2 says $SK_1(RG, \mathfrak{P}G) = SK_1RG$.

THEOREM 4. *If M_0 is any infinite subset of M, $SK_1RG = \bigcap_{\mathfrak{P} \in M_0} K_1(RG, \mathfrak{P}G)$.*

Proof. The preceding paragraph shows $SK_1RG \subseteqq \bigcap_{\mathfrak{P} \in M_0} K_1(RG, \mathfrak{P}G)$. Let x be in this intersection. For each $\mathfrak{P}$ in M_0 there is a prime $\mathfrak{S}$ of S over $\mathfrak{P}$, and a diagram (2). Then $\Delta(x)$ is in the kernel of γ. Since $\Delta(K_1RG) \subseteqq \prod S^*$ [1, p. 153], the components of $\Delta(x)$ are in $S^* \cap (1 + \mathfrak{S}S_{\mathfrak{S}}) \subseteqq 1 + \mathfrak{S}$. If M_0 is infinite, $\Delta(x)$ must be 1, and $x \in SK_1RG$.

Note 3. The same arguments prove Theorems 2 and 4 and Corollary 3 when RG is replaced by its image Λ under a projection to a direct factor of FG, and $\mathfrak{P}G$ is replaced by $\mathfrak{P}\Lambda$.

Note 4. It is unclear when SK_1RG is a *finite* intersection of relative groups $K_1(RG, \mathfrak{P}G)$. But for $\mathfrak{P}$ in M, SK_1RG is the torsion part of $K_1(RG, \mathfrak{P}G)$ exactly when reduction modulo $\mathfrak{P}$ restricts to an injection tor $R^* \to (R\mathfrak{P})^*$. (This follows from C. T. C. Wall's result [11, Proposition 6.5]: tor $K_1RG = SK_1RG \times$ tor $R^* \times G^{ab}$.)

10. Groups with a cyclic direct factor.

There is an isomorphism $Z[G \times H] \xrightarrow{\approx} ZG \otimes_Z ZH$, but $SK_1(-)$ does not respect tensor products. Although SK_1ZG and SK_1ZH are direct factors of $SK_1Z[G \times H]$, there is generally more.

THEOREM 5. *Let C_r denote a cyclic group of order r. If n is a square free rational integer coprime to the order of G, then there is an isomorphism: $SK_1Z[C_n \times G] \overset{\approx}{\to} SK_1AG$, where A is the integral closure of ZC_n in QC_n.*

Proof. Let ζ_r denote a primitive r root of unity. The decomposition of QC_p induces a surjective pullback of rings:

$$\begin{array}{ccc} R[C_p \times G] & \xrightarrow{\lambda} & R[\zeta_p]G \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\nu} \\ RG & \xrightarrow[\mu]{} & (R/pR)G \end{array} \qquad (p \text{ a prime})$$

whenever R is the ring of integers of a number field F not containing ζ_p. The maps λ and κ take a generator of C_p to ζ_p and 1, respectively; μ and ν are reduction of coefficients modulo p and $1 - \zeta_p$, respectively.

Suppose R is $Z[\zeta_t]$ and that p divides neither t nor the order of G. Then p is unramified in R; so $(R/pR)G$ is finite and semisimple of characteristic p. Therefore $K_2(R/pR)G = 1$.

Since $Z[\zeta_t]$ and $Z[\zeta_t][\zeta_p](=Z[\zeta_{pt}])$ are the rings of integers in number fields, Corollary 3 says $\mu.(SK_1Z[\zeta_t]G) = \nu.(SK_1Z[\zeta_{pt}]G) = 1$. The Mayer-Vietoris sequence becomes an isomorphism:

$$SK_1Z[\zeta_t][C_p \times G] \xrightarrow{\approx} SK_1Z[\zeta_t]G \times SK_1Z[\zeta_{pt}]G\ .$$

Reasoning by induction on t yields the composite:

$$SK_1Z[C_n \times G] \xrightarrow{\approx} \prod_{r|n} SK_1Z[\zeta_r]G \xrightarrow{\approx} SK_1(\prod_{r|n} Z[\zeta_r])G$$

where $\prod_{r|n} Z[\zeta_r] = A$.

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