PACIFIC JOURNAL OF MATHEMATICS
Vol. 79, No. 2, 1978

FINITE GROUPS WITH A STANDARD SUBGROUP ISOMORPHIC TO $PSU(4, 2)$

KENSAKU GOMI

The combined work of M. Aschbacher, G. Seitz, and I. Miyamoto classified finite groups G with a standard subgroup L isomorphic to $PSU(4, 2^n)$ such that either $n>1$ or $C_G(L)$ has noncyclic Sylow 2-subgroups. In this paper, we study the case that $n=1$ and $C_G(L)$ has cyclic Sylow 2-subgroups.

Introduction. A group L is *quasisimple* if L is its own commutator group and, modulo its center, L is simple. A quasisimple subgroup L of a finite group G is *standard* if its centralizer in G has even order, L is normal in the centralizer of every involution centralizing L, and L commutes with none of its conjugates. This definition of standard subgroups is equivalent to the original one given by M. Aschbacher in his fundamental paper [1].

I. Miyamoto has classified [23] finite groups G containing a standard subgroup L isomorphic to $PSU(4, 2^n)$ with $n > 1$ such that $C_G(L)$ has cyclic Sylow 2-subgroups. Part of his argument, however, failed to apply to $PSU(4, 2)$. This exceptional nature of $PSU(4, 2)$ may be explained by the isomorphism

$$PSU(4, 2) \cong PSp(4, 3) \cong P\Omega(5, 3) .$$

Because of this, certain groups of characteristic 3 have standard subgroups isomorphic to $PSU(4, 2)$.

In this paper, we prove the following theorem.

THEOREM. *Let G be a finite group and suppose L is a standard subgroup of G with $L \cong PSU(4, 2)$. Furthermore, assume that $C_G(L)$ has cyclic Sylow 2-subgroups, and let X denote the normal closure of L in G. Then one of the following holds.*

(1) *$X/O(X)$ is a simple group of sectional 2-rank 4.*

(2) *$X \cong PSL(4, 4)$ or $PSU(4, 2) \times PSU(4, 2)$.*

(3) *$N_G(L)/C_G(L) \cong \mathrm{Aut}(L)$, and for each central involution z of L, $C_G(z)$ has a quasisimple subgroup K that satisfies the following conditions:*

(3.1) *$z \in K$ and $W = O_2(K)$ is cyclic of order 4.*

(3.2) *$K/\langle z\rangle$ is a standard subgroup of $C_G(z)/\langle z\rangle$ and W is a Sylow 2-subgroup of $C_G(K/\langle z\rangle)$.*

(3.3) *Either $K/O(K) \cong SU(4, 3)$ or $K/Z(K)$ has a Sylow 2-subgroup isomorphic to a Sylow 2-subgroup of $PSL(6, q)$, $q \equiv 3 \bmod 4$.*

(3.4) $[K, O(C_G(z))] = 1$.

REMARK. In Case (1), the structure of $X/O(X)$ can be determined by a theorem of D. Gorenstein and K. Harada [14]; we can show that $X/O(X)$ is isomorphic to $PSp(4, 3)$, $PSp(4, 9)$, $PSU(4, 3)$, $PSL(4, 3)$, or $PSL(5, 3)$. Case (3) occurs in the automorphism group of $PSU(5, 3)$ with $K \cong SU(4, 3)$.

The proof of the theorem begins with a study of fusion of an involution t of $C_G(L)$. Let A be the unique elementary abelian subgroup of order 16 of a Sylow 2-subgroup of L. We show that the conjugacy class of t in $N_G(\langle t\rangle A)$ contains 1, 6, or 16 elements. If it contains 1 or 6 elements, then after determining the possible structure of a Sylow 2-subgroup of G, we show $t \notin G'$ by a transfer argument. It then follows that $N_X(A)/C_X(A) \cong A_5, \Sigma_5, A_6$ or Σ_6, and that $A \in \mathrm{Syl}_2(C_X(A))$. If $N_X(A)/C_X(A) \cong A_5, \Sigma_5$ or A_6, a theorem of Harada [17] shows that $r(X) = 4$. When $N_X(A)/C_X(A) \cong \Sigma_6$, we appeal to a theorem of G. Stroth [26]. Using an additional information, we show that this case does not occur. The analysis of the case where there are 16 conjugates of t follows the same line of arguments as in previous papers of Miyamoto and the author [11], [23] (we refer the reader to the introduction of [11]), although some additional argument is needed in the analysis of a subcase leading to Case (3) of the theorem.

Finally, we remark that the solvability of groups of odd order [6] is used implicitly throughout this paper.

Notation and Terminology. Our notation is standard and mainly taken from [12]. Possible exceptions are the use of the following:

$m(X)$	the 2-rank of X.
$r(X)$	the sectional 2-rank of X.
$I(X)$	the set of involutions of X.
$\mathscr{E}^*(X)$	the set of maximal elementary abelian subgroups of X.
X^∞	the final term of the derived series of X.
$J_r(X)$	the subgroup of X generated by the abelian 2-subgroups of maximal rank.
X^2	the subgroup of X generated by the squares of elements of X.
$E(X)$	the product of the quasisimple subnormal subgroups of X.
$L(X)$	the 2-layer of X.
X wreath Y	the wreath product of X by Y.

$X * Y$	a central product of X and Y.
$f(X \bmod Y)$	the preimage in X of $f(X/Y)$, where f is a function from groups to groups.
Z_{2^n}	the cyclic group of order 2^n.
E_{2^n}, $n \geqq 2$	the elementary abelian group of order 2^n.
D_n, $n \geqq 6$	the dihedral group of order n.
Q_8	the quaternion group.
A_n, Σ_n, $n \geqq 3$	the alternating and symmetric group of degree n.
$\boldsymbol{F}_q$	the field of q elements.
$V(2, F)$	the vector space of 2-dimensional row vectors with coefficients in the field F.
$M(4, F)$	the set of 4×4 matrices with entries in F.

An A_{2^n}-*subgroup* is an abelian subgroup of order 2^n, while an E_{2^n}-*subgroup* is an elementary abelian subgroup of order 2^n. Suppose $G \cong SL(2, 4) \cong A_5$. Then G has two types of "natural" modules over $\boldsymbol{F}_2$. The one is $V(2, \boldsymbol{F}_4)$ viewed as an $SL(2, 4)$-module in an obvious way. We call this the *natural module for* $G \cong SL(2, 4)$. The other is the unique nontrivial irreducible constituent of the permutation module for A_5. We call this the *natural module for* $G \cong A_5$. We use the "bar" convention for homomorphic images. Thus if G is a group, N is a normal subgroup, and $\bar{G}$ denotes the factor group G/N, then for any subset X of G, $\bar{X}$ will denote the image of X under the natural projection $G \rightarrow \bar{G}$. A similar convention will be used when a group G has a permutation representation on a set Ω, where we write X^{Ω} instead of $\bar{X}$.

1. In this section, we collect a number of preliminary lemmas to be used in later sections.

LEMMA (1A). *Let R be a nonabelian 2-group with a cyclic maximal subgroup Q, and let $t \in I(Q)$ and $u \in I(R - Q)$. Then u is conjugate to tu in R.*

Proof. This is a consequence of the classification of nonabelian 2-groups with a cyclic maximal subgroup. See Theorem 5.4.4. of [12].

LEMMA (1B). *Let G be a group which contains a direct product $H \times K$ of subgroups H and K. Assume that $|G: HK| = 2$ and that an element of $G - HK$ interchanges H and K. Then $G - HK$ contains involutions and they are all conjugate in G.*

Proof. Let $g \in G - HK$, and let $g^2 = hk$ with $h \in H$ and $k \in K$.

Then $hk = (hk)^g = k^g h^g$, so $h^g = k$ and $k^g = h$. Hence

$$\begin{aligned}(gh^{-1})^2 &= gh^{-1}gh^{-1} \\ &= g^2g^{-1}h^{-1}gh^{-1} \\ &= (hk)k^{-1}h^{-1} \\ &= 1 .\end{aligned}$$

Thus $G - HK$ contains an involution.

Now let $g \in G - HK$ and $g^2 = 1$. Let $h \in H$ and $k \in K$, and assume that ghk is an involution. Then $(hk)^g = (hk)^{-1}$, so $h^g = k^{-1}$ and $k^g = h^{-1}$. Hence $h^{-1}gh = gg^{-1}h^{-1}gh = ghk$. That is, ghk is conjugate to g. The proof is complete.

LEMMA (1C). *Let E be an elementary abelian 2-subgroup of a group G, and let t be an involution of $N_G(E)$. Then the following holds.*

(1) $|E: C_E(t)| \leqq |C_E(t)|$, *and equality holds if and only if* $I(tE) = t^E$.

(2) *If* $|E: C_E(t)| \geqq 4$, *then*

$$N_G(\langle E, t\rangle) \leqq N_G(\langle C_E(t), t\rangle) \cap N_G(E) .$$

Proof. Commutation by t induces a homomorphism from E onto $[E, t]$, and so $|[E, t]| = |E: C_E(t)|$. Also, $[E, t] \leqq C_E(t)$. Hence $|E: C_E(t)| \leqq |C_E(t)|$. Since $|I(tE)| = |C_E(t)|$ and $|t^E| = |E: C_E(t)|$, equality holds if and only if $I(tE) = t^E$.

Under the hypothesis of (2), E and $\langle C_E(t), t\rangle$ are the only maximal elementary abelian subgroups of $\langle E, t\rangle$, and they have different orders. Hence (2) follows.

LEMMA (1D). *Let G be a finite group and let $g \in G$. Then* $|C_G(g)| \geqq |G: G'|$.

Proof. For any $x \in G$, $g^{-1}g^x = [g, x] \in G'$. Hence $|G: C_G(g)| = |g^G| \leqq |G'|$.

LEMMA (1E). *Let R be an S_2-subgroup of a finite group G and S a normal subgroup of R with R/S abelian. Let x be an involution of $R - S$ and suppose that each extremal conjugate of x in R is contained in xS. Then $x \notin G'$.*

Proof. Let T be a subgroup of R with $S \leqq T \leqq R$ and $x \notin T$ subject to $|T|$ maximal. Then since R/S is abelian, R/T is cyclic. Also, each extremal conjugate of x in R is contained in xT. Therefore, Lemma (1E) follows from [27], Corollary 5.3.2.

LEMMA (1F). *Let T be an S_2-subgroup of a finite group G, and let S be a normal subgroup of T such that $T/S \cong E_4$ and $S \leqq G^\infty$. Let $a \in I(T-S)$ and $b \in I(T-\langle a, S\rangle)$, and suppose $(ab)^2 = 1$, $a^G \cap \langle b, S\rangle = \varnothing$, $b^G \cap S = \varnothing$, and $(ab)^G \cap S = \varnothing$. Then $S \in \mathrm{Syl}_2(G^\infty)$.*

Proof. By Lemma (1E), $a \notin G'$ and so $T \cap G' = S$, $\langle b, S\rangle$, or $\langle ab, S\rangle$. If $T \cap G' \neq S$, then $T \cap G'' = S$ again by Lemma (1E). Thus $S \in \mathrm{Syl}_2(G^\infty)$.

LEMMA (1G). *Let T be an S_2-subgroup of a finite group G, and let S be a normal subgroup of T such that $T/S \cong D_8$ and $S \leqq G^\infty$. Let $Z/S = Z(T/S)$, and let E/S and F/S be the fours subgroups of T/S. Let $a \in I(Z-S)$ and $b \in I(E-Z)$, and suppose $a^G \cap F \leqq aS$ and $b^G \cap F = \varnothing$. Then $S \in \mathrm{Syl}_2(G^\infty)$.*

Proof. By Lemma (1E), $b \notin G'$ and so $E \cap G' = S$ or Z, since $bS \sim abS$ in T. If $E \cap G' = S$, then $T \cap G' = S$ as $S \leqq T \cap G' \lhd T$. Suppose that $E \cap G' = Z$. Then either $T \cap G' = F$ or $T \cap G'/S$ is cyclic. Hence $a^G \cap T \cap G' \leqq aS$ and so $a \notin G''$ by Lemma (1E). Thus $T \cap G'' = S$. Therefore, $S \in \mathrm{Syl}_2(G^\infty)$.

LEMMA (1H). *Let A be a standard subgroup of a finite group G, and assume that $C_G(A)$ has a cyclic S_2-subgroup. Then the following holds.*

(1) *$AO(G) \lhd G$ if and only if an involution t of $C_G(A)$ is contained in $Z^*(G)$.*

(2) *$AO(G)/O(G)$ is a standard subgroup of $G/O(G)$ and $C_G(AO(G)/O(G))$ has a cyclic S_2-subgroup.*

(3) *If $AO(G) \ntriangleleft G$, then either $\langle A^G\rangle O(G)/O(G)$ is simple or $\langle A^G\rangle O(G)/O(G) \cong A/Z(A) \times A/Z(A)$. In either case, $C_G(\langle A^G\rangle O(G)/O(G)) = O(G)$.*

(4) *If $AO(G) \ntriangleleft G$ and if there is a t-invariant 2-subgroup P of $\langle A^G\rangle$ such that $1 \neq [P, t] \leqq C_G(C_{O(G)}(t))$, then $[\langle A^G\rangle, O(G)] = 1$.*

Proof. Let $t \in I(C(A))$ and let $\bar{G} = G/O(G)$. Then $\bar{t} \in I(\bar{G})$ and $\bar{A}$ is a quasisimple normal subgroup of $C(\bar{t})$. Let $\bar{x} \in C(\bar{A}) \cap C(\bar{t})$. We may choose $x \in C(t)$. Then $[x, A] \leqq A \cap O(G) \leqq Z(A)$, so $[x, A] = 1$. Thus $C(\bar{A}) \cap C(\bar{t}) = \overline{C(A) \cap C(t)}$. Therefore, $C(\bar{A})$ has cyclic S_2-subgroups and (2) follows.

Assume that $\bar{A} \lhd \bar{G}$. Then $C(\bar{A}) \lhd \bar{G}$ and so $C(\bar{A})$ is a cyclic 2-group and $\bar{t} \in Z(\bar{G})$. Conversely, if $\bar{t} \in Z(\bar{G})$, then $\bar{A} \lhd C(\bar{t}) = \bar{G}$. This proves (1).

Assume that $\bar{A} \not\trianglelefteq \bar{G}$. Then by a result of Aschbacher, $F^*(\bar{G}) = \langle \bar{A}^{\bar{G}} \rangle$ and either $F^*(\bar{G})$ is simple or $\bar{A}$ is simple, $F^*(\bar{G}) \cong \bar{A} \times \bar{A}$, and $\bar{t}$ interchanges two components of $F^*(\bar{G})$. Let $L = \langle A^G \rangle O(G)$ and assume that there is a t-invariant 2-subgroup P of L such that $1 \neq [P, t]$ and $[[P, t], C_{O(G)}(t)] = 1$. Then $[[P, t], O(G)] = 1$ by [11, (1J)]. Hence $C_L(O(L)) \not\leqq O(L)$. Since $\bar{L} = L/O(L)$ is simple or a direct product of simple groups interchanged by t, it follows that $L = C_L(O(L))O(L)$. Thus $\langle A^G \rangle \leqq C_L(O(L))$ and (4) follows.

LEMMA (1I). *Let $K = PSL(n, q)$, $n \geqq 2$, or $PSU(n, q)$, $n \geqq 3$, q odd, and let α be an involutory automorphism of K that is not a product of an inner automorphism and a diagonal automorphism. Then $C_K(\alpha)$ is solvable only if $K = PSL(2, 9)$, $PSL(3, 3)$, $PSL(4, 3)$, $PSU(3, 3)$, or $PSU(4, 3)$. If $C_K(\alpha)$ is not solvable, then the structure of $C_K(\alpha)^\infty$ is given on the following table.*

K	$C_K(\alpha)^\infty$
$PSL(n, q)$	$P\Omega^\pm(n, q)$, $PSp(n, q)$, n even, $PSL(n, p)$, $q = p^2$, $PSU(n, p)$, $q = p^2$,
$PSU(n, q)$	$P\Omega^\pm(n, q)$, $PSp(n, q)$, n even.

Proof. Consider the case $K = PSL(n, q)$ first. Set $G = GL(n, q)$ and $H = SL(n, q)$. Let τ be the transpose-inverse mapping of G, and if $q = p^2$, let σ be the automorphism of G induced by that of $\boldsymbol{F}_q$ of order 2. Then α is induced on $K = H/Z(H)$ by an element x of τG, σG or $\tau\sigma G$ such that $x^2 \in Z(G)$.

First, assume that $x \in \tau G$. Then $n \geqq 3$. Let $x = \tau a$, $a \in G$. Then as $x^2 \in Z(G)$, it follows that ${}^t a = a$ or $-a$, where ${}^t a$ is the transposed matrix of a. We also have that

$$C_G(x) = \{y \in G \mid {}^t y a y = a\} .$$

That is, $C_G(x)$ is the orthogonal or symplectic group defined by the symmetric or alternating matrix a. Now $\mathrm{Aut}(\langle x, Z(G) \rangle)$ is solvable, so $N_G(\langle x, Z(G) \rangle)^\infty \leqq C_G(x)$. Also, $C_G(x)^\infty \leqq H$. Thus $C_K(\alpha)^\infty = C_G(x)^\infty Z(H)/Z(H)$, and so $C_K(\alpha)$ is solvable only if $(n, q) = (3, 3)$ or $(4, 3)$, and if $C_K(\alpha)$ is nonsolvable then $C_K(\alpha)^\infty \cong P\Omega^\pm(n, q)$ or $PSp(n, q)$.

Next, consider the case $x \in \sigma G$. Let $x = \sigma a$, $a \in G$. Then as $x^2 \in Z(G)$, we see that $c = a^\sigma a$ is a scalar matrix such that $c^{p-1} = 1$.

Hence there is a scalar matrix $d \in G$ such that $d^{p+1}c = 1$, so that $(da)^{\sigma}da = 1$. Replacing x by xd, we may assume that $a^{\sigma}a = 1$. By [20, Proposition 3], there is an element $g \in G$ such that $a = g^{\sigma}g^{-1}$. Thus $x^g = \sigma$ and we may assume from the outset that $x = \sigma$. Therefore, $C_G(x) \cong GL(n, p)$, and so $C_K(\alpha)$ is solvable only if $(n, q) = (2, 9)$, and if $C_K(\alpha)$ is nonsolvable, then $C_K(\alpha)^{\infty} \cong PSL(n, p)$.

Assume, therefore, $x \in \tau\sigma G$. Let $x = \tau\sigma a$, $a \in G$. As above, we may assume that $a^{\tau\sigma}a = 1$. That is, a is a hermitian matrix. Thus $C_G(x)$ is the unitary group defined by a over F_q, and so $C_K(\alpha)$ is solvable only if $(n, q) = (2, 9)$, and if $C_K(\alpha)$ is nonsolvable, then $C_K(\alpha)^{\infty} \cong PSU(n, p)$.

Now consider the case $K = PSU(n, q)$. In this case, we set $G^* = GL(n, q^2)$, $G = U(n, q)$, and $H = SU(n, q)$. Let τ be the transpose-inverse mapping of G^* and let σ be the automorphism of G^* induced by that of F_{q^2} of order 2. Then we may regard $G = C_{G^*}(\sigma\tau)$, and assume that α is induced on $K = H/Z(H)$ by an element x of $\sigma Z(G^*)G$ such that $x^2 \in Z(G^*)$. As before, we may assume that $x = \sigma a$, $a \in Z(G^*)G$, and $a^{\sigma}a = 1$. Let $a = a_1a_2$ with $a_1 \in Z(G^*)$ and $a_2 \in G$. Then

$$a^{\sigma\tau} = a_1^{-q}a_2 = a_1^{-q-1}a = e^{-1}a \ , \tag{1}$$

where $e = a_1^{q+1}$. Now there is an element $g \in G^*$ such that $a = g^{\sigma}g^{-1}$ by [20, Proposition 3]. Hence by (1), $(g^{\sigma}g^{-1})^{\sigma\tau} = e^{-1}(g^{\sigma}g^{-1})$. That is,

$$eg^{\tau}g^{-\sigma\tau} = g^{\sigma}g^{-1} \ . \tag{2}$$

Now $(\sigma\tau)^g = \sigma\tau g^{-\sigma\tau}g$, so let $h = g^{-\sigma\tau}g$. Then $h^{\tau} = g^{-\sigma}g^{\tau} = e^{-1}g^{-1}g^{\sigma\tau} = e^{-1}h^{-1}$ by (2), so

$${}^th = eh \ .$$

Hence

$$e = \pm 1 \ .$$

Also,

$$h^{\sigma} = g^{-\tau}g^{\sigma} = e^{-1}g^{-\sigma\tau}g = eh$$

by (2). Choose an element $d \in Z(G^*)$ such that $d^{q-1} = e^{-1}$ and set $h_1 = dh$. Then ${}^th_1 = eh_1$ and $h_1^q = d^qeh = d^{q-1}eh_1 = h_1$. Thus h_1 is a symmetric or alternating matrix in $C_{G^*}(\sigma) = GL(n, q)$. Now $x^g = \sigma$ as $a^{\sigma}a = 1$, so

$$\begin{aligned} C_G(x) &= C_{G^*}(x) \cap C_{G^*}(\sigma\tau) \\ &\cong C_{G^*}(\sigma) \cap C_{G^*}((\sigma\tau)^g) \\ &= C_{G^*}(\sigma) \cap C_{G^*}(\sigma\tau h) \\ &= C_{G^*}(\sigma) \cap C_{G^*}(\tau h_1) \ . \end{aligned}$$

Thus $C_G(x) \cong O^{\pm}(n, q)$ or $Sp(n, q)$ by a previous discussion. Hence $C_K(\alpha)$ is solvable only if $(n, q) = (3, 3)$ or $(4, 3)$, and if $C_K(\alpha)$ is nonsolvable, then $C_K(\alpha)^{\infty} \cong P\Omega^{\pm}(n, q)$ or $PSp(n, q)$.

LEMMA (1J). *Let E be an elementary abelian group of order 16 on which $M \cong SL(2, 4) \cong A_5$ acts. Let $R \in \mathrm{Syl}_2(M)$.*

(1) *If $|C_E(R)| = 4$, then E is a natural module for $M \cong SL(2, 4)$.*

(2) *If $|C_E(R)| = 2$, then E is a natural module for $M \cong A_5$.*

Proof. (1) follows from [11, (1K)]. Assume that $|C_E(R)| = 2$. Let $a_1, a_2, \cdots, a_5$ be the nontrivial fixed points on E of S_2-subgroups of M, so that $\{a_1, a_2, \cdots, a_5\}$ is M-invariant. Since M acts irreducibly on E, we have $a_1 a_2 \cdots a_5 = 1$ and $E = \langle a_1, a_2, \cdots, a_5 \rangle$. Now let V be the direct product of E and a group $\langle a \rangle$ of order 2, and let M act on V in an obvious fashion. Then, by the above remark, $\{aa_1, aa_2, \cdots, aa_5\}$ is an M-invariant set which generates V. Thus V is a permutation module for $M \cong A_5$ and E is a nontrivial irreducible constituent of V. This proves (2).

LEMMA (1K). *Let E be an elementary abelian group of order 2^8, and let K and L be subgroups of* $\mathrm{Aut}(E)$ *such that $SL(2, 4) \cong K \leqq L \cong SL(2, 16)$. Let $R \in \mathrm{Syl}_2(K)$, and let $R \leqq S \in \mathrm{Syl}_2(L)$. Assume that $|C_E(S)| = 4$. Then there is no nontrivial K-invariant subgroup A of E such that $C_A(R) < C_E(S)$.*

Proof. Let $W = C_E(S)$ and assume, by way of contradiction, that A is a K-invariant subgroup of E such that $1 \neq C_A(R) < W$. Clearly, $N_L(S)$ normalizes W. As $N_K(R) \leqq N_L(S)$ and $N_K(R)$ centralizes $C_A(R)$ which is a subgroup of W of order 2, we have that $[N_K(R), W] = 1$. As $|N_L(S)/S| = 15$ and $N_K(R)S/S$ is an S_3-subgroup of $N_L(S)/S$, it follows that $[N_L(S), W] = 1$.

Let $s \in I(L - N_L(S))$ and set $H = N_L(S) \cap N_L(S^s)$. Notice that H is a complement for S in $N_L(S)$. Furthermore, $W \cap W^s = C_E(L) = 1$, as $L = \langle S, S^s \rangle$ and L acts irreducibly on E by [8, (4B)].

Now $[H, WW^s] = 1$, as $[H, W] = 1$ by the first paragraph and $H^s = H$. For any $w \in W^{\#}$, let $\hat{w} = ww^s$. Then as $\langle H, s \rangle \leqq C_L(\hat{w})$ and $\langle H, s \rangle$ is a maximal subgroup of L, we have that $C_L(\hat{w}) = \langle H, s \rangle$. Consequently, $|\hat{w}^L| = |L: \langle H, s \rangle| = 136$. As $136 \times 2 = 272 > 255 = |E^{\#}|$, it follows that $\hat{w}_1 \sim \hat{w}_2$ for any $w_1, w_2 \in W^{\#}$. Choose $x \in L$ so that $\hat{w}_1^x = \hat{w}_2$. Then $\langle H, s \rangle^x = C_L(\hat{w}_1)^x = C_L(\hat{w}_2) = \langle H, s \rangle$, and so $x \in N_L(\langle H, s \rangle) = \langle H, s \rangle$. This is a contradiction as we may choose $\hat{w}_1 \neq \hat{w}_2$.

Now we define some subgroups of $SL(4, 4)$. Let M^*, R^*, D^*, and E^* be the groups consisting of the following matrices, respectively.

$$\begin{pmatrix} A & \\ & I \end{pmatrix}, \ A \in SL(2, 4), \text{ and } I \text{ is the } 2 \times 2 \text{ unit matrix,}$$

$$\begin{pmatrix} 1 & & & \\ a & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \ a \in \boldsymbol{F}_4 ,$$

$$\begin{pmatrix} a^{-1} & & & \\ & a^{-1} & & \\ & & a & \\ & & & a \end{pmatrix}, \ a \in \boldsymbol{F}_4 - \{0\} ,$$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ a & b & 1 & \\ c & d & & 1 \end{pmatrix}, \ a, b, c, d \in \boldsymbol{F}_4 .$$

Thus $R^* \in \mathrm{Syl}_2(M^*)$, and M^* and D^* normalize E^*. Let f^* be the field automorphism of $SL(4, 4)$ and let t^* be the graph-field automorphism of $SL(4, 4)$. That is, f^* is induced by the involution of $\mathrm{Aut}(\boldsymbol{F}_4)$ and t^* is the transpose-inverse mapping followed by f^* and conjugation by $\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$. Let $L^* = M^* M^{*t^*}$.

We shall consider the following situation.

Hypothesis (1.1). E is an elementary abelian group of order 2^8, and N is a subgroup of $\mathrm{Aut}(E)$ which has a normal subgroup L satisfying the following conditions.

(1) $L = M \times M^t$, $t \in I(N)$, $M \cong SL(2, 4)$.

(2) $C_N(L) = O(N)$.

(3) For $R \in \mathrm{Syl}_2(M)$, $W = C_E(RR^t)$ is a fours group.

LEMMA (1L). *Assume Hypothesis* (1.1). *Furthemore, assume the following.*

(4) $C_E(M) = 1$.

(5) *For a complement H for R in $N_M(R)$,* $[W, htht] = 1$ *for all* $h \in H$.

Then there is a monomorphism σ from the semidirect product *of N and E into $\langle M^*, E^*, D^*, t^*, f^*\rangle$ such that $M^\sigma = M^*$, $R^\sigma = R^*$, $O(N)^\sigma \leqq D^*$, $E^\sigma = E^*$, $t^\sigma = t^*$, and $f^\sigma = f^*$ if f is an element of $C(t) \cap N_N(M)$ acting as a field automorphism on $C_L(t) \cong SL(2, 4)$.*

Proof. Let $r \in I(N_M(H))$ and set $s = rtrt$. We use the additive notation for E. As $M = \langle R, R^r\rangle$, the condition (4) implies that $C_E(R) \cap C_E(R^r) = \{0\}$. In particular, $W \cap W^r = \{0\}$, and as $W + W^r \leqq C_E(R^t)$, $|C_E(R^t)| = |C_E(R^{rt})| \geqq 2^4$. As $C_E(R^t) \cap C_E(R^{rt}) = \{0\}$, we conclude that

$$E = C_E(R^t) \oplus C_E(R^{rt}) .$$

Also,

$$C_E(R^t) = W \oplus W^r \text{ and } C_E(R^{rt}) = C_E(R^t)^{trt} = W^{trt} \oplus W^s .$$

Furthermore, as $C_E(R) \cap C_E(R^t) = W$ has order 4, Lemma (1J) shows that $C_E(R^t)$ and $C_E(R^{rt})$ are natural modules for $M \cong SL(2, 4)$. This proves that we can identify M with M^* so that $E \cong E^*$ as modules for M. More precisely, if $w \in C_W(t)^\#$, $H = \langle h\rangle$, and $\boldsymbol{F}_4 = \{0, 1, x, x^2\}$, then E and E^* can be identified by the mapping which associates with $w^{h^a} + w^{h^b r} + w^{h^c trt} + w^{h^d s}$, where $a, b, c, d \in \{0, 1, 2\}$, the matrix

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ x^c & x^d & 1 & \\ x^a & x^b & & 1 \end{pmatrix}$$

and the action of an element of M on E identified with E^* is the conjugation by the corresponding element of M^*. In this identification, R^* corresponds to R.

Using the condition (5), we have that for each $i \in \{0, 1, 2\}$,

$$\begin{aligned} (w^{h^i})^t &= w^{h^{-i}} , \\ (w^{h^i r})^t &= w^{h^{-i} trt} , \\ (w^{h^i trt})^t &= w^{h^{-i} r} , \\ (w^{h^i s})^t &= w^{h^{-i} s} . \end{aligned}$$

This shows that we can identify t with t^*. Thus $\langle M, t\rangle E \cong \langle M^*, t^*\rangle E^*$.

Suppose $O(N) \neq 1$. The $A \times B$-lemma [12, Theorem 5.3.4] shows that $O(N)$ acts regularly on $W^\#$. Hence $|O(N)| = 3$ and there is an element $z \in O(N)$ such that $w^z = w^h$. Then a computation similar to the above shows that $O(N)$ can be identified with D^*.

If $LO(N) = N(M)$, then $N = \langle M, O(N), t\rangle$ so the above paragraphs prove the lemma. Suppose, therefore, that $LO(N) < N(M)$.

Now let f be an element of $N(M)$ satisfying the following conditions:

$$(*) \qquad f \text{ inverts } H, f \in C(s), \text{ and } f \in C(w) .$$

The second condition implies that f centralizes r and trt. Therefore, by a computation similar to that in previous paragraphs, we can show that f can be identified with f^*.

Suppose that $C(M^t) \neq MO(N)$. Then there is an involution $f \in C(M^t) \cap N(R)$ that satisfies the first two conditions in (*). The f normalizes RR^t and so acts on W. Hence if $O(N) \neq 1$, there is an element $z \in O(N)$ such that $w^f = w^z$, and so fz^{-1} satisfies (*). Assume that $O(N) = 1$. Then $[f, tft] \in C(M) \cap C(M^t) \leqq C(L) = O(N) = 1$, so that $(ft)^4 = 1$. Thus $\langle f, t \rangle$ is a 2-group acting on W, and so it centralizes some nontrivial element of W. As $C_W(t) = \langle w \rangle$, it follows that $w^f = w$. Therefore, we can always choose an element $f \in C(M^t)$ that satisfies (*). By the above paragraph, f acts as the field automorphism on E. It follows that $[f, t]$ centralizes E, and therefore $[f, t] = 1$. But then $f = tft \in C(M^t) \cap C(M) = O(N)$, which is a contradiction. Therefore, $C(M^t) = MO(N)$. This implies that $LO(N)$ has index 2 in $N(M)$.

Let K/L be an S_2-subgroup of N/L with $t \in K$. Notice that $K/L \cong E_4$. As $I(Lt) = t^L$ by Lemma (1B), $K = LC_K(t)$ and so $|C_K(t) \cap N(M): C_L(t)| = 2$. As $C_L(t) = \{xtxt \mid x \in M\} \cong M \cong SL(2, 4)$, $N(M) \cap C(C_L(t)) = C(L) = O(N)$ and it follows that $C_K(t) \cap N(M) \cap C(C_L(t)) = 1$. Thus we may choose an involution $f \in C(t) \cap N(M)$ which acts on $C_L(t)$ as the field automorphism. Then f acts as the field automorphism both on M and on M^t. In particular, f inverts H and centralizes s. Moreover, $f \in N(RR^t)$, so f centralizes $\langle w \rangle = C_W(t)$. Thus f satisfies (*) and therefore f can be identified with f^*. As $N = \langle M, O(N), t, f \rangle$, we have proved the lemma.

LEMMA (1M). *Assume Hypothesis* (1.1). *Furthermore, assume the following conditions.*

(4) $C_E(M) \neq 1$.

(5) $W \cap W^{rtrt} = 1$ *for* $r \in I(M - R)$.

Then $E = C_E(M) \times C_E(M^t)$, *and* $C_E(M^t)$ *is a natural module for* $M \cong A_5$.

Proof. Set $s = rtrt$. Then

$$\begin{aligned} W^s &= (C_E(R) \cap C_E(R^t))^{rtrt} \\ &= C_E(R)^r \cap C_E(R^t)^{trt} \\ &= C_E(R)^r \cap C_E(R)^{rt} . \end{aligned}$$

As $M = \langle R, R^r \rangle$, we may deduce as follows:

$$\begin{aligned} & C_E(M) \cap C_E(M^t) \\ & \quad = C_E(R) \cap C_E(R^r) \cap C_E(R^t) \cap C_E(R^{rt}) \\ & \quad = (C_E(R) \cap C_E(R^t)) \cap (C_E(R^r) \cap C_E(R^{rt})) \\ & \quad = W \cap W^s \\ & \quad = 1 . \end{aligned}$$

In particular, M acts on $C_E(M^t)$ nontrivially, and so $|C_E(M^t)| \geqq 2^4$. As $|E| = 2^8$, we must have that $E = C_E(M) \times C_E(M^t)$. Moreover, as R normalizes $C_E(M)$ and $C_E(M^t)$, it follows that

$$\begin{aligned} C_E(R) &= (C_E(M) \cap C_E(R)) \times (C_E(M^t) \cap C_E(R)) \\ &= C_E(M) \times (C_E(M^t) \cap C_E(R)) . \end{aligned}$$

Therefore,

$$\begin{aligned} W &= C_E(R) \cap C_E(R^t) \\ &= (C_E(M) \cap C_E(R^t)) \times (C_E(M^t) \cap C_E(R)) . \end{aligned}$$

Since $|W| = 4$, we conclude that $|C_E(M^t) \cap C_E(R)| = 2$. Thus, $C_E(M^t)$ is a natural module for $M \cong A_5$ by Lemma (1J).

LEMMA (1N). *Let t be an involution of a finite group G, and assume that $C(t)$ has a normal subgroup L isomorphic to $SL(2, 4)$ such that $\langle t \rangle \in \mathrm{Syl}_2(C(L) \cap C(t))$. Furthermore, assume that an S_2-subgroup R of L is contained in an $N(R) \cap C(t)$-invariant E_{16}-subgroup S of G. Then $X = \langle L^G \rangle$ is isomorphic to $SL(2, 16)$ or $SL(2, 4) \times SL(2, 4)$, $C(X) = O(G)$, and $S \in \mathrm{Syl}_2(X)$.*

Proof. Let bars denote images in $G/O(G)$. Then by Lemma (1H), $\bar{L}$ is a standard subgroup of $\bar{G}$ and $C(\bar{L})$ has a cyclic S_2-subgroup. Let H be an S_3-subgroup of $N_L(R)$. Then commutation by t induces an H-isomorphism $S/R \to R$, and since $R = [R, H]$, it follows that $S = [S, H]$. Thus $S \leqq X$, and in particular, $m(X) \geqq 4$. Appealing to [16], we now get that $\bar{X} \cong SL(2, 16)$, $SL(2, 4) \times SL(2, 4)$ or $PSL(3, 4)$. If $\bar{X} \cong PSL(3, 4)$, then we must have that $\bar{t}$ acts on $\bar{X}$ as a graph automorphism. But then $\bar{t}$ does not normalize any E_{16}-subgroup of $\bar{X}$. Therefore, $\bar{X} \cong SL(2, 16)$ or $SL(2, 4) \times SL(2, 4)$ and so $S \in \mathrm{Syl}_2(X)$. Since $R = [S, t] \leqq L$, (3) and (4) of Lemma (1H) show that $C(X) = O(G)$ and $X \cong SL(2, 16)$ or $SL(2, 4) \times SL(2, 4)$.

LEMMA (1P). *Let G be a finite group and t an involution of G. Assume that $C(t) = K \times \langle t \rangle \times O(C(t))$ with $K \cong Sp(4, 2)$. Assume* *furthermore that G has a t-invariant subgroup M isomorphic to the commutator subgroup of a maximal parabolic subgroup of $Sp(4, 4)$ and that conjugation by t induces the same automorphism of M as the involutory field automorphism of $Sp(4, 4)$. Then $E(G) \cong Sp(4, 4)$ and $C(E(G)) = O(G)$.*

Proof. Let S be a t-invariant S_2-subgroup of M and let $T = \langle S, t\rangle$. We show that $I(St) = t^T = t^G \cap T$. Our assumption on the action of t on M in particular implies that $I(St) = t^T$, so $I(St) \leqq t^G \cap T$. By assumption, $m(C_S(x)) = 6$ for any $x \in I(S)$ and so, as $m(C(t)) = 4$, $t^G \cap S = \varnothing$. Thus $t^G \cap T = I(St)$.

Let $T \leqq U \in \mathrm{Syl}_2(N(T))$. Then as $t^G \cap T = t^T$, $U = TC_U(t)$. By hypothesis, $C_S(t)$ is isomorphic to an S_2-subgroup of $Sp(4, 2)$, so $C_T(t) \in \mathrm{Syl}_2(C(t))$. Therefore, $C_U(t) = C_T(t)$ and $U = T$. This shows that $T \in \mathrm{Syl}_2(G)$.

Since $t^G \cap S = \varnothing$, Lemma (1E) shows that $t \notin G'$, and since $M = M' \leqq G'$, it follows that $S \in \mathrm{Syl}_2(G')$. Thus, $X = \langle K'^G\rangle$ has S_2-subgroups of class at most 2. Now, $K' \cong A_6$ is standard in G and $C(K')$ has cyclic S_2-subgroups. Moreover, $K'O(G) \ntriangleleft G$ by Lemma (1H) as $t \notin Z^*(G)$. Hence if bars denote images in $G/O(G)$, the same lemma shows that $C(\bar{X}) = 1$ and either $\bar{X}$ is simple or $\bar{X} \cong A_6 \times A_6$. In the first case, $\bar{X}$ is of known type by [9], and in either case $\bar{G}^\infty = \bar{X}$. Thus $\bar{M} = \bar{M}^\infty \leqq \bar{X}$ and $\bar{S} \in \mathrm{Syl}_2(\bar{X})$. Therefore, $\bar{X} \cong Sp(4, 4)$. Let E be an E_{64}-subgroup of S. By hypothesis, $[E, t] = C_E(t) \cong E_8$, and hence $[[E, t], O(C(t))] = 1$ by the structure of $C(t)$. Therefore, $E(G) \cong Sp(4, 4)$ and $C(E(G)) = O(G)$ by (3) and (4) of Lemma (1H).

LEMMA (1Q). *Let G be a finite simple group containing an E_{16}-subgroup A such that $N(A)/C(A) \cong A_6$ and $A \in \mathrm{Syl}_2(C(A))$. Then $G \cong M_{22}$, $PSL(4, q)$ ($q \equiv 5 \bmod 8$), or $PSU(4, q)$ ($q \equiv 3 \bmod 8$).*

Proof. The proof of Lemma 12 of [17] shows that G has S_2-subgroups of type $\hat{A}_8$ or $\hat{A}_{10}$. Then by [13] and [21], G is isomorphic to one of the following groups: Mc, M_{22}, M_{23}, $PSL(4, q)$ ($q \equiv 5 \bmod 8$), $PSU(4, q)$ ($q \equiv 3 \bmod 8$), and Ly. The groups Mc, M_{23}, and Ly have no E_{16}-subgroup whose automizer is isomorphic to A_6 (see a table on p. 543 of [7] and Proposition 9.1 of [13]). Thus we have the result.

LEMMA (1R). *Let $\hat{G}$ be a finite group and $\hat{Z}$ a subgroup of $Z(\hat{G})$ isomorphic to Z_4. Set $G = \hat{G}/\hat{Z}$ and let A be an E_{16}-subgroup of G satisfying the following conditions.*

(1) $N_G(A)/C_G(A) \cong \Sigma_6$.

(2) $A \in \mathrm{Syl}_2(C_G(A))$.

(3) $|G: N_G(A)|$ *is even.*

(4) *The preimage of* A *in* $\hat{G}$ *is not abelian.*

Furthermore, let t *be an involution acting on* $\hat{G}$ *and* G *in the following fashion.*

(5) $A \leqq C_G(t) \leqq N_G(A)$.

(6) $C_G(t)C_G(A)/C_G(A) \cong \Sigma_3$ *wreath* Z_2.

(7) $N_G(A)/A = C_{N_G(A)/A}(t) \cdot C_G(A)/A$.

(8) $[\hat{Z}, t] \neq 1$.

Then there is a quasisimple characteristic subgroup $\hat{H}$ *of* $\hat{G}$ *containing* $\hat{Z}$ *such that* $C_{\hat{G}}(\hat{H}) = \hat{Z}O(\hat{G})$. *Either* $\hat{H}/O(\hat{H}) \cong SU(4, 3)$ *or* $\hat{H}/Z(\hat{H})$ *has* S_2*-subgroups isomorphic to those of* $PSL(6, q)$, $q \equiv 3$ mod 4.

Proof. Let bars denote images in $G/O(G)$. Assume that $\bar{Q} = O_2(\bar{G}) \neq 1$. Then $\bar{Q} \cap C(\bar{A}) \neq 1$ and so, as $C(\bar{A}) = \bar{A}O(C(\bar{A}))$ by (2), it follows that $1 \neq \bar{Q} \cap \bar{A} \lhd N(\bar{A})$. The condition (1) implies that $N(\bar{A})$ acts irreducibly on $\bar{A}$. Therefore, $\bar{A} \leqq \bar{Q}$, but $\bar{A} \neq \bar{Q}$ as $|\bar{G}: N(\bar{A})|$ is even. But now $\bar{A} < N_{\bar{Q}}(\bar{A}) \lhd N(\bar{A})$, which is a contradiction because $O_2(N(\bar{A})) = \bar{A}$ by (1). Thus, $O_2(\bar{G}) = 1$.

By the above, $F^*(\bar{G})$ is a product of nonabelian simple groups. Let $\bar{K} = F^*(\bar{G})$, $\bar{A} \leqq \bar{T} \in \mathrm{Syl}_2(\bar{G})$, and $\bar{U} = \bar{T} \cap \bar{K}$. Then $1 \neq \bar{U} \lhd \bar{T}$ by [6]. Hence we have that $\bar{U} \cap \bar{A} \neq 1$ and then, as $\bar{U} \cap \bar{A} = \bar{K} \cap \bar{A} \lhd N(\bar{A})$, we have that $\bar{A} \leqq \bar{U} \leqq \bar{K}$ just as above. However, $\bar{A} \neq \bar{U}$ by (3), so $\bar{A} < N_{\bar{U}}(\bar{A}) \leqq N_{\bar{K}}(\bar{A}) \lhd N(\bar{A})$. It now follows from (1) that $N_{\bar{K}}(\bar{A})/C_{\bar{K}}(\bar{A}) \cong A_6$ or Σ_6. Let $\bar{L}$ be a component of $\bar{K}$ and let $\bar{V} = \bar{U} \cap \bar{L}$. Then $1 \neq \bar{V} \cap \bar{A} = \bar{L} \cap \bar{A} \lhd N_{\bar{K}}(\bar{A})$ and then $\bar{A} \leqq \bar{V} \leqq \bar{L}$ as before. As $C(\bar{A})$ is solvable, we conclude that $\bar{K}$ is simple.

Now the conditions (5), (6), and (7) imply that there is an S_2-subgroup S of $N(A)$ such that $1 \neq [S, t] \leqq A$. Also, $[C_{O(G)}(t), A] \leqq [O(C_G(t)), O_2(C_G(t))] = 1$. Therefore, $[O(G), [S, t]] = 1$ by [11, (1J)]. Thus, $C_A(O(G)) \neq 1$ and, since $N(A)$ is irreducible on A, we have $[O(G), A] = 1$.

Let K be the full inverse image of $F^*(\bar{G})$ in G. Then $A \leqq C_K(O(K))$. In particular, $C_K(O(K)) \nleqq O(K)$ and so, since $K/O(K)$ is simple, we have that $K = C_K(O(K))O(K)$. Thus K is a central product of K^∞ and $O(K)$. Now we set $H = K^\infty$. Then H is quasisimple and $Z(H) = O(H)$. Furthermore, $A \leqq O^{2'}(K) = H$ and consequently, $N_H(A)/C_H(A) \cong A_6$ or Σ_6.

Now define $\hat{H}$ and $\hat{A}$ to be the subgroups of $\hat{G}$ such that $\hat{H}/\hat{Z} = H$ and $\hat{A}/\hat{Z} = A$, respectively. Then, clearly $\hat{H} \lhd \hat{G}$. We show that $\hat{H}$ is perfect. Suppose false. Then there is a subgroup $\hat{J}$ of $\hat{H}$ of index 2 such that $\hat{H} = \hat{J}\hat{Z}$. Let $\hat{B} = \hat{A} \cap \hat{J}$. Then $|\hat{B}| = 32$, $\hat{B}/\hat{Z} \cap \hat{B} \cong E_{16}$, and A_6 acts on $\hat{B}/\hat{Z} \cap \hat{B}$ nontrivially. This forces $\hat{B}$ to be elementary. But then $\hat{A} = \hat{B}\hat{Z}$ is abelian, contrary to (4). Therefore, $\hat{H}$ is perfect. Furthermore, since H is quasisimple, so also is $\hat{H}$.

We check that $\hat{H}$ is the desired subgroup of $\hat{G}$. By definition, $\hat{Z} \leqq \hat{H}$ and $C_{\hat{G}}(\hat{H}) = \hat{Z}O(\hat{G})$ since $\bar{H} = F^*(\bar{G})$ is simple. To prove the second assertion, assume first that $N_H(A)/C_H(A) \cong A_6$. Then $H/Z(H) \cong M_{22}$, $PSL(4, q)$ ($q \equiv 5 \bmod 8$) or $PSU(4, q)$ ($q \equiv 3 \bmod 8$) by Lemma (1Q). The Schur multipliers of these simple groups are known [5], and so we can determine the structure of $\hat{H}$. We see that $\hat{H}/O(\hat{H}) \cong SL(4, q)$ or $SU(4, q)$. As (5) and (6) imply that $C_G(t)$ is solvable, Lemma (1I) and (8) show that $\hat{H}/O(\hat{H}) \cong SU(4, 3)$. Therefore, assume that $N_H(A)/C_H(A) \cong \Sigma_6$. In this case, a similar argument and the theorem of [26] yield that $\hat{H}/Z(\hat{H})$ has S_2-subgroups of type $PSL(6, q)$, $q \equiv 3 \bmod 4$. The proof is complete.

2. In this section, we fix notation for $L = PSU(4, 2) \cong SU(4, 2)$ and set down some facts about L and its automorphisms.

By choosing a suitable basis of the underlying hermitian space, we identify the elements of L with the 4×4 matrices x with entries in $\boldsymbol{F}_4$ satisfying

$$(2.1) \qquad {}^t x \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \bar{x} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \text{ and } \det x = 1\,,$$

where ${}^t x$ denotes the transposed matrix of x and $\bar{x}$ is the matrix obtained by squaring each entries of x.

Denote by P the group of matrices

$$(2.2) \qquad \begin{pmatrix} 1 & & & \\ a & 1 & & \\ c & b & 1 & \\ d & a^2 b + c^2 & a^2 & 1 \end{pmatrix}$$

where $b^2 = b$ and $d^2 = ac^2 + a^2 c + d$. Define A_1 to be the group of matrices (2.2) with $b = 0$, and define A_2 to be the group of matrices (2.2) with $a = 0$. Let Z be the group of matrices (2.2) with $a = b = c = 0$.

Let e be a primitive cube root of unity in $\boldsymbol{F}_4$ and set

$$a_1 = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & & & \\ e^2 & 1 & & \\ & & 1 & \\ & & e & 1 \end{pmatrix},$$

$$b_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & & & 1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \end{pmatrix},$$

$$b_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ e^2 & & 1 & \\ & e & & 1 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{pmatrix},$$

$$c_1 = b_0, \ c_2 = b_3, \ c_3 = b_0 b_1 b_3 ,$$

$$c_4 = b_0 b_2 b_3, \ c_5 = b_0 b_1 b_2 b_3 ,$$

$$s_1 = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}.$$

Denote by H the group generated by the matrix

$$j = \begin{pmatrix} e & & & \\ & e^2 & & \\ & & e^2 & \\ & & & e \end{pmatrix}.$$

Denote by K_1 the group of matrices

$$\begin{pmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, 2)$$

and denote by K_2 the group of matrices

$$\begin{pmatrix} a & b & & \\ c & d & & \\ & & a^2 & b^2 \\ & & c^2 & d^2 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, 4) .$$

Now we list some facts about L and its automorphisms. Proofs will be mostly omitted because the assertions are consequences of straightforward calculations involving matrices.

LEMMA (2A).

(1) $|P| = 64$ *and* $P \in \mathrm{Syl}_2(L)$.

(2) *P is generated by the involutions* $a_1, a_2, b_0, b_1, b_2, b_3$, *and the following commutator relations hold*:

$$[a_1, b_2] = b_0,\ [a_1, b_3] = b_0 b_1\ ,$$
$$[a_2, b_1] = b_0,\ [a_2, b_3] = b_0 b_2\ .$$

All other commutators are trivial.

(3) A_1 *is generated by* a_1, a_2, b_1, b_2.

(4) A_2 *is generated by* b_0, b_1, b_2, b_3.

(5) $Z(P) = Z = \langle b_0 \rangle$, $Z_2(P) = \langle b_0, b_1, b_2 \rangle$.

(6) $\mathscr{E}_{16}(P) = \{A_2\}$.

(7) $\mathscr{E}^*(P/Z) = \{A_1/Z, A_2/Z\}$.

(8) $P = A_1 A_2$.

In the above lemma, (1) follows from the fact that $|L| = 2^6 \cdot 3^4 \cdot 5$.

LEMMA (2B).

(1) $N_L(P) = HP$.

(2) *The following relations hold*:

$$a_1^j = a_2,\ a_2^j = a_1 a_2,\ b_1^j = b_2,\ b_2^j = b_1 b_2\ .$$

j centralizes other generators of P listed in Lemma (2A)(2).

(3) *H acts regularly on* $(P/A_2)^\#$, $(A_1/A_1 \cap A_2)^\#$, *and* $(A_1 \cap A_2/Z)^\#$.

LEMMA (2C).

(1) $N_L(A_1) = (K_1 \times H)A_1$.

(2) $A_1 \cong D_8 * D_8 \cong Q_8 * Q_8$ *and* $Z(A_1) = Z = \langle b_0 \rangle$.

(3) *Under the action of* $N_L(A_1)$, $(A_1/Z)^\#$ *decomposes into two orbits of length s* 9 *and* 6, *the former corresponding to involutions of* $A_1 - Z$ *and the latter corresponding to elements of order* 4 *of* A_1. $O_3(K_1) \times H = \langle s_1 b_3 \rangle \times \langle j \rangle$ *acts regularly on the orbit of length* 9.

(4) $C_L(A_1/Z) = A_1$.

(5) $O^{2,2'}(K_1 A_1) = A_1$.

(4) and (5) above are consequences of (1), (2), and (3).

LEMMA (2D).

(1) $N_L(A_2) = K_2 A_2$.

(2) A_2 *is a natural module for* $K_2 \cong A_5$.

(3) $C_L(A_2) = A_2$.

(4) *Under the action of* K_2, $A_2^\#$ *decomposes into two orbits of lengths* 5 *and* 10, *the former consisting of* c_1, c_2, c_3, c_4, *and* c_5.

LEMMA (2E).

(1) *L has two conjugacy classes of involutions, and we may choose* b_0 *and* b_1 *as the representatives of these classes.*

(2) $C_P(b_0) = P$ *and* $C_L(b_0) = N_L(A_1)$.

(3) $C_P(b_1) = \langle a_1, A_2\rangle$ *and* $C_L(b_1) = \langle a_1, s_2\rangle A_2$.

(4) *Involutions of* $A_1 - Z$ *are conjugate to* b_1 *in* $N_L(A_1)$.

(5) *Central involutions of L contained in* A_2 *are* c_1, c_2, c_3, c_4, c_5, *and so they are all conjugate in* $N_L(A_2)$.

Let $A = \mathrm{Aut}(L)$ and identify L with $\mathrm{Inn}(L)$. Then $A = \langle f\rangle L$, where f is the automorphism of L induced by the automorphism of $\boldsymbol{F}_4$ of order 2. Let $R = \langle f\rangle P$.

LEMMA (2F).

(1) $R \in \mathrm{Syl}_2(A)$.

(2) *The following relations hold*:

$$a_1^f = a_1,\ a_2^f = a_1a_2\ ,$$

$$b_0^f = b_0,\ b_1^f = b_1,\ b_2^f = b_1b_2,\ b_3^f = b_3\ .$$

(3) $r(R) = 4$.

(4) $Z(R) = Z(P) = Z,\ R' = \langle a_1, b_0, b_1, b_2\rangle$.

(5) *R has exactly four* E_{16}*-subgroups*: A_2, $\langle C_{A_1}(f), f\rangle = \langle a_1, b_0, b_1, f\rangle$, $\langle C_{A_2}(f), f\rangle = \langle b_0, b_1, b_3, f\rangle$, *and* $\langle C_{A_2}(f), f\rangle^{a_2} = \langle b_0, b_1, b_2b_3, a_1f\rangle$. *All these are self-centralizing in R.*

(6) $J_r(R) = \langle C_{A_1}(f), A_2, f\rangle = \langle a_1, b_0, b_1, b_2, b_3, f\rangle$, $ZJ_r(R) = \langle b_0, b_1\rangle$.

For the proof of (3) above, see [17, Lemma 2]. (6) is a direct consequence of (5).

LEMMA (2G).

(1) $N_A(A_1) = \langle f\rangle N_L(A_1)$.

(2) $N_A(A_1)/A_1 \cong K_1 \times \langle f\rangle H \cong \Sigma_3 \times \Sigma_3$.

(3) $C_A(A_1/Z) = A_1$.

(4) $O_2(N_A(A_1)) = A_1$.

(2), (3), and (4) above are consequences of (1) and Lemma (2C). See Lemma (2D) for the proof of the next lemma.

LEMMA (2H).

(1) $N_A(A_2) = \langle f\rangle N_L(A_2)$.

(2) $N_A(A_2)/A_2 \cong \langle f\rangle K_2 \cong \Sigma_5$.

(3) $C_A(A_2) = A_2$.

(4) $O_2(N_A(A_2)) = A_2$.

LEMMA (2I). $N_A(\langle C_{A_1}(f), f\rangle) = \langle f\rangle K_1A_1$.

Proof. Observe that b_0 is the only central involution of L contained in A_1. By Lemma (2E)(2), we have

$$N_A(\langle C_{A_1}(f), f\rangle) \leqq N_A(C_{A_1}(f)) \leqq C_A(b_0) = \langle f\rangle N_L(A_1) .$$

Thus, using Lemma (2C)(1), we obtain the result.

LEMMA (2J).
(1) $C_A(C_{A_2}(f)) = \langle A_2, f\rangle$.
(2) $N_A(\langle C_{A_2}(f), f\rangle) = \langle f, a_1, s_2, A_2\rangle$.

Proof. Use Lemma (2E)(3) to prove (1). Once (1) is proved, then $N_A(\langle C_{A_2}(f), f\rangle) \leqq N_A(C_{A_2}(f)) \leqq N_A(\langle A_2, f\rangle) \leqq N_A(A_2)$, hence (2) follows easily.

LEMMA (2K).
(1) $C_L(f) \cong Sp(4, 2) \cong \Sigma_6$.
(2) $C_L(fb_0) = C_L(f) \cap C_L(b_0) = \langle a_1, b_0, b_1, b_3, s_1\rangle$.
(3) *If* $x \in I(A - L)$, *then* $x \sim f$ *or* fb_0 *in* A *and* $x^A \cap C_L(x)x \neq \{x\}$.
(4) *If* $x \in I(N_A(P) - L)$ *and* $C(x) \cap N_L(A_2)$ *is an extension of* E_8 *by* $SL(2, 2)$, *then* $x \in f^A$.

Proof. For the proof of (1), (2), and (3), see [3, § 19]. For (4), suppose $(fb_0)^g = x, g \in L$. Since $C_L(fb_0)$ is also an extension of E_8 by $SL(2, 2)$ by (2), we have $C_L(fb_0)^g = C(x) \cap N_L(A_2)$, hence $\langle a_1, b_0, b_1\rangle^g = O_2(C_L(fb_0))^g = O_2(C(x) \cap N_L(A_2)) = C(x) \cap A_2$. Since $b_0 \in C(x) \cap A_2$ and since b_0 is strongly closed in A_1 with respect to L by Lemma (2E), we have $b_0^g = b_0$, hence $g \in C_L(b_0) = N_L(A_1)$. But $C_L(fb_0)^g \leqq N_L(A_1) \cap N_L(A_2) = N_L(P)$, a contradiction. Therefore, $x \in f^A$.

3. In this section, we begin the proof of the theorem stated in the introduction.

Let G be a finite group which contains a standard subgroup L isomorphic to $PSU(4, 2)$, and assume that $C(L)$ has a cyclic S_2-subgroup.

We identify L with the group of 4×4 matrices x satisfying (2.1). The symbols used in § 2 for various objects defined for $PSU(4, 2)$ will retain their meaning for the balance of the paper. Thus P is an S_2-subgroup of L consisting of matrices (2.2).

Let t be an involution of $C(L)$ and set $C = C(t)$. We first prove the following.

LEMMA (3A). *If* $t^G \cap LC_G(L) = \{t\}$, *then* $r(\langle L^G\rangle) = 4$.

Proof. Assume that $t^G \cap LC_C(L) = \{t\}$. Let $T \in \mathrm{Syl}_2(C_C(L))$, $Q = PT$, and $Q \leqq R \in \mathrm{Syl}_2(C)$. Then $t \in Z(R)$ and $Z(R) \leqq Q$ by Lemma (2F). Therefore, $t^G \cap Z(R) = \{t\}$ by our assumption, and hence $N(R) \leqq C$. This implies that $R \in \mathrm{Syl}_2(G)$.

Now if $t \in Z^*(G)$, then $LO(G) \lhd G$ by Lemma (1H). Therefore, we may assume that $t^G \cap R \neq \{t\}$ by [10].

Let $t \neq u \in t^G \cap R$. Then $u \notin Q$ by our assumption, and so $|R: Q| = 2$. Notice that $Q/P \cong T$ is cyclic by our hypothesis. Hence if R/P is nonabelian, then $uP \sim tuP$ in R by Lemma (1A), and so $t^G \cap tuP \neq \varnothing$. If R/P is abelian, then by Lemma (1E), either $t^G \cap \langle tu \rangle P \neq \varnothing$ or $t \notin G'$. In the latter case, $R \cap G' = P$ or $P\langle tu \rangle$ as $P \leqq L \leqq G'$. Hence $r(\langle L^G \rangle) = 4$ by Lemma (2F). Therefore, we may assume that $t^G \cap tuL \neq \varnothing$ for all $u \in t^G \cap C$, $u \neq t$.

Suppose $tu \in t^G$ for all $u \in t^G \cap C$ with $u \neq t$. Let $t^g \in C - \{t\}$. If $t \notin L^g C(L^g)$, then there exists an element $x \in C_{L^g}(t)^\#$ with $tx \in t^{L^g}$ by Lemma (2K). Then $x = t(tx) \in t^G$, so $x^{g^{-1}} \in t^G \cap L$, contrary to our assumption. If $t \in L^g C(L^g)$, then $t \neq t^{g^{-1}} \in t^G \cap LC_C(L)$, contrary to our assumption. Thus there is a conjugate $t^g \in C - \{t\}$ such that $tt^g \not\sim t$.

Choose $t^g \in C - \{t\}$ so that $tt^g \not\sim t$, and let $t^h \in tt^g L$. If $C_L(t^h) \cong C_L(tt^g) = C_L(t^g)$, then $t \sim t^h \sim tt^g$ by Lemma (2K), a contradiction. Hence $C_L(t^h) \not\cong C_L(t^g)$. If R/P is nonabelian, we may choose $h \in gR$ by Lemma (1A). But then $C_L(t^h) \cong C_L(t^g)$, a contradiction. Therefore, R/P is abelian.

Now $Z(R) \leqq Q$ by Lemma (2F), so $P\langle tt^g \rangle$ contains no extremal conjugates of t in R. Thus $t \notin G'$ by Lemma (1E), and $r(\langle L^G \rangle) = 4$ as before. The proof is complete.

In view of Lemma (3A), we shall make the following hypothesis.

Hypothesis (3.1). $t^G \cap LC_C(L) \neq \{t\}$.

We next prove

LEMMA (3B). *Under Hypothesis* (3.1), $\langle t \rangle \in \mathrm{Syl}_2(C_C(L))$.

Proof. Let $T \in \mathrm{Syl}_2(C_C(L))$ and let $t \neq t^g \in LC_C(L)$. We may assume $t^g \in PT$ so $T \leqq C(t^g) = C^g$. Lemma (2E) shows that $C_L(t^g) = L \cap C^g$ contains an E_{16}-subgroup A. The image of $A \times T$ in $C^g/C_C(L)^g$ has rank at least 4 and its exponent is equal to that of T as $T \cap C_C(L)^g = 1$. Thus Lemma (2F)(5) forces $|T| = 2$.

DEFINITION (3.1). Let $Q = P\langle t \rangle$, and $B_i = A_i\langle t \rangle$ for $i \in \{1, 2\}$.

LEMMA (3C). *We have* $t^G \cap L = \varnothing$.

Proof. This is obvious if $t^G \cap LC_G(L) = \{t\}$. Therefore, we may assume Hypothesis (3.1). Suppose $t^g \in L$ for some $g \in G$. By Lemma (2E), we may assume $t^g = b_0$ or b_1, so that $C_P(t^g) \in \mathrm{Syl}_2(C_L(t^g))$ and t^g has a square root in P. In particular, t has a square root in C. Hence, if $Q \leqq R \in \mathrm{Syl}_2(C)$, then $R/P \cong Z_4$ by Lemma (3B). Thus $I(C) \leqq L\langle t\rangle$. But then $C_P(t^g) = \Omega_1(C_P(t^g)) \leqq L^g\langle t^g\rangle$, and therefore, $t^g \in C_P(t^g)^2 \leqq L^g$. This is a contradiction proving the lemma.

LEMMA (3D). *If C contains an S_2-subgroup of G, then $r(\langle L^G\rangle)=4$.*

Proof. We may assume Hypothesis (3.1) by Lemma (3A). Let $Q \leqq R \in \mathrm{Syl}_2(C)$, so that $R \in \mathrm{Syl}_2(G)$. Suppose that $t \in G'$. As $|R/P|$ is at most 4 by Lemma (3B), Lemmas (1E) and (3C) show that there is an element $u \in t^G \cap (R - Q)$ and, moreover, $\langle u\rangle P$ contains an extremal conjugate v of t in R. However, since $Z(R) \leqq Q$, we have $v \in P$, which is impossible by Lemma (3C). Therefore, $t \notin G'$ and so $r(\langle L^G\rangle) = 4$ as in the third paragraph of the proof of Lemma (3A).

LEMMA (3E). $N(B_2) \leqq N(A_2)$.

Proof. If $N(B_2) \leqq C$, then $N(B_2)$ normalizes $B_2 \cap L = A_2$. If $N(B_2) \nleqq C$, then $\Omega = t^{N(B_2)} \neq \{t\}$. By Lemma (3C), $\Omega \leqq A_2t$, so $A = \langle ab \mid a, b \in \Omega\rangle$ is a nonidentity $N(B_2)$-invariant subgroup of A_2. As $K_2(\leqq N(B_2))$ acts irreducibly on A_2, $A_2 = A$. Thus $N(B_2) \leqq N(A_2)$.

LEMMA (3F). *$|C(A_2) \cap N(B_2)\colon C(B_2)|$ is a power of 2.*

Proof. As $C(A_2) \cap N(B_2)$ stabilizes the series $1 < A_2 < B_2$, the assertion follows from [12, Corollary 5.3.3].

LEMMA (3G). *Let $\Omega = t^{N(B_2)}$. Then $\Omega = \{t\}$, $\{t, c_1t, c_2t, c_3t, c_4t, c_5t\}$ or A_2t. If $\Omega \neq \{t\}$, $N(B_2)^\Omega$ is a primitive permutation group on Ω, and $C(\Omega) = C(B_2)$.*

Proof. By Lemma (3C), $\Omega \leqq A_2t$. Under the action of K_2, which is contained in $N_G(B_2)$, A_2 decomposes into two orbits of lengths 5 and 10, the former consisting of c_1, c_2, c_3, c_4, and c_5. Hence it is enough to show that $|\Omega| \neq 11$. Suppose $|\Omega| = 11$. Then by Lemmas (3E) and (3F), $C(A_2) \cap N(B_2) = C(B_2)$ and then $N(B_2)/C(B_2)$ is isomorphic to a subgroup of $\mathrm{Aut}(A_2) \cong GL(4, 2)$. This is a contradiction because $|GL(4, 2)|$ is not divisible by 11.

LEMMA (3H). *Let $f \in I(C - LC_G(L))$ and suppose that the action of f on L is induced by the involution of* $\mathrm{Aut}(F_4)$. *If*

$t^G \cap \langle b_0, b_1, b_3, t\rangle \leqq t^{N(B_2)}$, *then no element of* G *interchanges* B_2 *and* $\langle C_{A_2}(f), f, t\rangle$ *by conjugation.*

Proof. If an element g of G interchanges B_2 and $\langle C_{A_2}(f), f, t\rangle$, then g normalizes their intersection $\langle b_0, b_1, b_3, t\rangle$ and so $t^{gh} = t$ for some $h \in N(B_2)$ by hypothesis. However, $gh \in C$ and $\langle C_{A_2}(f), f, t\rangle^{gh} = B_2$ which is a contradiction as $\langle C_{A_2}(f), f, t\rangle \not\leqq \langle L, t\rangle$ while $B_2 \leqq \langle L, t\rangle$.

LEMMA (3I). *Let* f *be as in* (3H) *and suppose that* $\langle C_{A_1}(f), f, t\rangle^g = B_2$ *for some* $g \in G$. *Then* $A_1^g \leqq O^{2,2'}(N(B_2))$.

Proof. As $\langle L, f, t\rangle = L\langle f\rangle \times \langle t\rangle$ and as $K_1A_1 = N_L(\langle C_{A_1}(f), f\rangle)$ by Lemma (2I), we have that $X = N_{\langle L,f,t\rangle}(\langle C_{A_1}(f), f, t\rangle)$ is equal to $\langle K_1A_1, f, t\rangle$. Thus $O^{2,2'}(X) = O^{2,2'}(K_1A_1) = A_1$ by Lemma (2C), and hence $A_1 \leqq O^{2,2'}(N(\langle C_{A_1}(f), f, t\rangle))$. Therefore, $A_1^g \leqq O^{2,2'}(N(B_2))$.

LEMMA (3J). *Under Hypothesis* (3.1), *the following conditions hold.*

(1) $N(Q) \leqq N(B_1) \cap N(B_2)$.

(2) $m(C) = 5$.

(3) *C does not have an* E_{32}*-subgroup* X *such that* $SL(2, 2) \times SL(2, 2) \hookrightarrow N_C(X)/C_C(X)$.

Proof. By Lemma (2A), $\mathscr{E}^*(Q/Z(Q)) = \{B_1/Z(Q), B_2/Z(Q)\}$, hence (1) follows. (2) is a direct consequence of Lemma (2F)(5). By the same lemma, if X is an E_{32}-subgroup of C, then $X \sim B_2$, $\langle C_{A_1}(f), f, t\rangle$, or $\langle C_{A_2}(f), f, t\rangle$ in C, where f is an involution acting on L as a field automorphism. Hence $N_C(X)/C_C(X) \hookrightarrow \Sigma_5$ or $Z_2 \times SL(2, 2)$ by Lemmas (2H)—(2J). Thus (3) holds.

4. In this section, we shall work under the following hypothesis.

Hypothesis (4.1). $t^{N(B_2)} = \{t\}$.

We prove the following theorem.

THEOREM (4A). *Under Hypothesis* (4.1), $r(\langle L^G\rangle) = 4$.

The proof involves a series of reductions. First, if $t^G \cap LC_G(L) = \{t\}$, then Theorem (4A) holds by Lemma (3A). Therefore, we assume that G satisfies Hypothesis (3.1). Then $\langle t\rangle \in \mathrm{Syl}_2(C_C(L))$ by Lemma (3B).

LEMMA (4B). *If $t \notin G'$, then Theorem* (4A) *holds.*

Proof. By Hypothesis (4.1), $N(B_2) \leqq C$ so that $N(B_2) \cap C(A_2) = C(B_2)$. This implies that $B_2 \in \mathrm{Syl}_2(C(A_2))$ as $C(B_2) = B_2O(C)$ by Lemmas (2H) and (3B). Hence $N(A_2) = N(B_2)C(A_2) = N_C(B_2)C(A_2)$ by a Frattini argument, and so $N(A_2)/C(A_2) \cong A_5$ or Σ_5 by Lemmas (2D) and (2H). We also have that $N_L(A_2) \leqq N_{G'}(A_2)$ since $L \leqq G'$. Therefore, $N_{G'}(A_2)/C_{G'}(A_2) \cong A_5$ or Σ_5. Also, $A_2 \leqq C_{G'}(A_2) \lhd C(A_2)$. Since $B_2 \in \mathrm{Syl}_2(C(A_2))$ and $t \notin G'$, it follows that $A_2 \in \mathrm{Syl}_2(C_{G'}(A_2))$. Thus, $r(G') = 4$ by [17, Theorem 3] and hence $r(\langle L^G\rangle) = 4$. The proof is complete.

Let $Q \leqq R \in \mathrm{Syl}_2(C)$. The following lemma follows from Lemma (3D).

LEMMA (4C). *If $R \in \mathrm{Syl}_2(G)$, then Theorem* (4A) *holds.*

In view of Lemmas (4B) and (4C), we shall form now on assume that

$$t \in G' \text{ and } R \notin \mathrm{Syl}_2(G) .$$

We shall eventually derive a contradiction from this hypothesis.

LEMMA (4D). *There is an involution $f \in C$ whose action on $L = PSU(4, 2)$ is induced by the automorphism of $\boldsymbol{F}_4$ of order* 2.

Proof. It is enough to show that $I(R - Q) \neq \varnothing$. Since $R \notin \mathrm{Syl}_2(G)$, $N(R) \nleqq C$ so that $N(R) \nleqq N(B_2)$ as $N(B_2) \leqq C$ by Hypothesis (4.1). If $I(R) \leqq I(Q)$, then B_2 would be the only E_{32}-subgroup of R by Lemma (2A), and so $N(R) \leqq N(B_2)$. Therefore, $I(R - Q) \neq \varnothing$, as required.

We assume without loss of generality that $f \in R$. Notice that $R = Q\langle f\rangle$. Let $S \in \mathrm{Syl}_2(N(R))$. Then $R < S$, so we may choose $g \in S - R$.

LEMMA (4E). *The following conditions hold.*

(1) *$S = R\langle g\rangle$ and $g^2 \in R$.*

(2) *$t^g = b_0t$ and $b_0^g = b_0$.*

(3) *g interchanges B_2 and $\langle C_{A_1}(f), f, t\rangle$ by conjugation.*

(4) $g \in N(A_1) \cap N(B_1)$.

Proof. As $C_S(t) = R < S$, $\{t\} < t^S$. Also, $t^S \leqq Z(R)$. As $Z(R) = \langle b_0, t\rangle$ by Lemma (2F) and as $t \not\sim b_0$ by Lemma (3C), it follows that $t^S = \{t, b_0 t\}$. Therefore, $|S: R| = 2$ and $S \leqq C(b_0)$. Hence (1) and (2) follow.

By Lemma (2F), B_2, $\langle C_{A_1}(f), f, t\rangle$, $\langle C_{A_2}(f), f, t\rangle$, and $\langle C_{A_2}(f), f, t\rangle^x$, where $x \in P - C_{A_1}(f)A_2$, are the only E_{32}-subgroups of R. Since $N(B_2) \leqq C$ by Hypothesis (4.1), $B_2 \neq B_2^g \lhd R$. Thus (3) holds. Then Lemma (3I) shows that $A_1^g \leqq O^{2,2'}(N(B_2))$. Since $N(B_2) \leqq C$, $O^{2,2'}(N(B_2)) = N_L(A_2)$ by Lemma (2D). Hence $A_1^g \leqq R \cap N_L(A_2) = P$. Also, $b_0 = b_0^g \in A_1^g$. Since $A_1/\langle b_0\rangle$ is the only E_{16}-subgroup of $P/\langle b_0\rangle$ by Lemma (2A), we have that $A_1^g = A_1$. Since $B_1 = \langle A_1, t\rangle$ and $t^g = b_0 t \in A_1 t$, $g \in N(B_1)$. The proof is complete.

LEMMA (4F). *We may choose f so that the following conditions hold.*

(1) *g interchanges $A_1 \cap A_2$ and $C_{A_1}(f)$ by conjugation.*
(2) *g interchanges P and $\langle A_1, f\rangle$ by conjugation.*
(3) $g \in N(\langle P, f\rangle)$.
(4) $t^G \cap \langle P, f\rangle = \varnothing$.

Proof. Using Lemma (4E), we may deduce as follows:

$$\begin{aligned}(A_1 \cap A_2)^g &= (A_1 \cap B_2)^g \\ &= A_1 \cap \langle C_{A_1}(f), f, t\rangle \\ &= C_{A_1}(f)\,.\end{aligned}$$

Since $g^2 \in R \leqq N(A_1 \cap A_2)$, $C_{A_1}(f)^g = A_1 \cap A_2$. Now A_2^g is a maximal subgroup of $\langle C_{A_1}(f), f, t\rangle$ containing $C_{A_1}(f)$. Since $t^g \cap L = \varnothing$ by Lemma (3C), $A_2^g \neq \langle C_{A_1}(f), t\rangle$. Therefore, $A_2^g = \langle C_{A_1}(f), f\rangle$ or $\langle C_{A_1}(f), ft\rangle$. Replacing f by ft in the latter case, we may choose f so that $A_2^g = \langle C_{A_1}(f), f\rangle$. Then

$$\begin{aligned}P^g &= (A_1 A_2)^g \\ &= A_1\langle C_{A_1}(f), f\rangle \\ &= \langle A_1, f\rangle\,,\end{aligned}$$

and $\langle A_1, f\rangle^g = P$ as $g^2 \in R \leqq N(P)$. Hence g normalizes $\langle P, A_1, f\rangle = \langle P, f\rangle$. Since $A_2^g = \langle C_{A_1}(f), f\rangle$ and $t^G \cap A_2 = \varnothing$, $t^G \cap \langle C_{A_1}(f), f\rangle = \varnothing$. By Lemma (2K), every involution of Pf is conjugate to an element of $C_{A_1}(f)f$. Therefore, $t^G \cap \langle P, f\rangle = \varnothing$. The proof is complete.

LEMMA (4G). *The following conditions hold.*

(1) $N(R) \leqq N(B_1)$.

(2) $S \in \mathrm{Syl}_2(N(B_1))$.

Proof. Since $Z(B_1) = \langle b_0, t\rangle$ by Lemma (2C), $t^{N(B_1)} \leqq \{t, b_0t\}$. By Lemma (4E), $g \in N(B_1) - C$. Hence $|N(B_1): N_C(B_1)| = 2$ and $N(B_1) = N_C(B_1)\langle g\rangle$. Similarly, $N(R) = N_C(R)\langle g\rangle$. Since $N_C(R) \leqq N_C(B_1)$ by Lemma (3J), (1) follows. Now $R \in \mathrm{Syl}_2(N_C(B_1))$, so $S = R\langle g\rangle \in \mathrm{Syl}_2(N(B_1))$. The proof is complete.

LEMMA (4H). $I(S) \nleqq I(R)$.

Proof. Suppose this is false. Then $\Omega_1(S) = R$, so $N(S) \leqq N(R)$, and Lemma (4G) yields that $S \in \mathrm{Syl}_2(G)$. Also, $t^G \cap S = t^G \cap R \leqq \langle P, f\rangle t$ by Lemma (4F)(4). As $\langle P, f\rangle \lhd S$ and $|S/\langle P, f\rangle| = 4$ by Lemma (4F), Lemma (1E) forces $t \notin G'$ against our hypothesis. Therefore, $I(S) \nleqq I(R)$.

Now let bars denote images in $C(b_0)/\langle b_0\rangle$. Then S acts on $\bar{A}_1$ by Lemma (4E). In the following two lemmas, we collect necessary information on this action. Notice that we may choose $\bar{a}_1, \bar{b}_2, \bar{a}_2, \bar{b}_1$ as a basis of $\bar{A}_1$.

LEMMA (4I). *The following conditions hold.*
(1) $\bar{a}_1^{b_3} = \bar{a}_1\bar{b}_1, \bar{b}_2^{b_3} = \bar{b}_2, \bar{a}_2^{b_3} = \bar{b}_2\bar{a}_2, \bar{b}_1^{b_3} = \bar{b}_1$.
(2) $\bar{a}_1^f = \bar{a}_1, \bar{b}_2^f = \bar{b}_2\bar{b}_1, \bar{a}_2^f = \bar{a}_1\bar{a}_2, \bar{b}_1^f = \bar{b}_1$.
(3) $\bar{a}_1^{b_3f} = \bar{a}_1\bar{b}_1, \bar{b}_2^{b_3f} = \bar{b}_2\bar{b}_1, \bar{a}_2^{b_3f} = \bar{a}_1\bar{b}_2\bar{a}_2\bar{b}_1, \bar{b}_1^{b_3f} = \bar{b}_1$.
(4) $C_{\bar{A}_1}(b_3) = \langle\bar{b}_2, \bar{b}_1\rangle$.
(5) $C_{\bar{A}_1}(f) = \langle\bar{a}_1, \bar{b}_1\rangle$.
(6) $C_{\bar{A}_1}(b_3f) = \langle\bar{a}_1\bar{b}_2, \bar{b}_1\rangle$.

Proof. (1), (2), and (3) follow from relations listed in Lemmas (2A) and (2F). (4), (5), and (6) are consequences of (1), (2), and (3), respectively.

Now choose f as in Lemma (4F). So far g was an arbitrary element of $S - R$. We now prove

LEMMA (4J). *We may choose g so that $g^2 \in A_1$ and the following relations hold:*

$$\bar{a}_1^g = \bar{b}_2, \bar{b}_2^g = \bar{a}_1, \bar{a}_2^g = \bar{a}_2, \bar{b}_1^g = \bar{b}_1 .$$

For g satisfying these relations, we have that

$$C_{\bar{A}_1}(g) = \langle\bar{a}_1\bar{b}_2, \bar{a}_2, \bar{b}_1\rangle .$$

Proof. Lemma (4I) shows that b_3, f, and b_3f have the following matrix forms with respect to the basis $\bar{a}_1$, $\bar{b}_2$, $\bar{a}_2$, $\bar{b}_1$ of $\bar{A}_1$, respectively.

$$\begin{pmatrix} 1 & & & 1 \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & \\ & 1 & & 1 \\ 1 & & 1 & \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & 1 \\ & 1 & & 1 \\ 1 & 1 & 1 & 1 \\ & & & 1 \end{pmatrix}$$

Choosing a suitable element $g \in S - R$, we determine the matrix form of g. By Lemma (4F), g interchanges $\bar{A}_1 \cap \bar{A}_2 = \langle \bar{b}_1, \bar{b}_2 \rangle$ and $\overline{C_{A_1}(f)} = \langle \bar{a}_1, \bar{b}_1 \rangle$, and so g normalizes $\langle \bar{b}_1 \rangle$. Therefore, g has the following matrix form.

$$\begin{pmatrix} & 1 & & a \\ 1 & & & b \\ c & d & 1 & e \\ & & & 1 \end{pmatrix}$$

By Lemma (4H), we may assume from the outset that $g^2 \in A_1$. Then g induces an involutory automorphism on $\bar{A}_1$, and so the square of the matrix of g is equal to the unit matrix. Hence we have that $a = b$ and $c = d$. Thus g has the following matrix form.

$$\begin{pmatrix} & 1 & & a \\ 1 & & & a \\ c & c & 1 & e \\ & & & 1 \end{pmatrix}$$

Now $P^g = \langle A_1, f \rangle$ by Lemma (4F), so $gb_3g \equiv f \bmod A_1$. This implies that

$$\begin{pmatrix} & 1 & & a \\ 1 & & & a \\ c & c & 1 & e \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & 1 \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} & 1 & & a \\ 1 & & & a \\ c & c & 1 & e \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & 1 \\ 1 & & 1 & \\ & & & 1 \end{pmatrix}.$$

Hence we have that $a = c$, and so g has the following matrix form.

$$\begin{pmatrix} & 1 & & a \\ 1 & & & a \\ a & a & 1 & e \\ & & & 1 \end{pmatrix}$$

We compute that b_3fg has the following matrix form.

$$\begin{pmatrix} & 1 & & a+1 \\ 1 & & & a+1 \\ a+1 & a+1 & 1 & e+1 \\ & & & 1 \end{pmatrix}$$

Hence replacing g by b_3fg if $a = 1$, we may assume that g has the following matrix form.

$$\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & 1 & e \\ & & & 1 \end{pmatrix}$$

This implies that $a_2^g = a_2b_1^e$ or $a_2b_1^eb_0$. Since a_2^g is an involution, it follows that $e = 0$. This implies that the relations listed in Lemma (4J) hold. The latter half of the lemma follows from this easily.

Now choose g as in Lemma (4J). We next prove

LEMMA (4K). *The following conditions hold.*
(1) $\langle P, f, g\rangle/A_1 \cong D_8$ *and* $Z(\langle P, f, g\rangle/A_1) = \langle A_1, b_3f\rangle/A_1$.
(2) $\bar{S} = \langle \bar{P}, \bar{f}, \bar{g}\rangle \times \langle \bar{t}\rangle$.
(3) $Z(S) = \langle b_0\rangle$.
(4) $Z_2(S) = \langle b_0, b_1, t\rangle$.

Proof. By the choice of g, $g^2 \in A_1$ and g interchanges $P = \langle A_1, b_3\rangle$ and $\langle A_1, f\rangle$. Hence (1) follows. By Lemma (4E)(2), $\bar{t} \in Z(\bar{S})$. Since $\langle P, f, g\rangle \cap R = \langle P, f\rangle$, $t \notin \langle P, f, g\rangle$. Thus (2) holds. Now $Z(S) \leqq C_S(t) = R$, so $Z(S) \leqq Z(R) = \langle b_0, t\rangle$. Since $t^g = b_0t$ by Lemma (4E), (3) follows. By (2), $Z(\bar{S}) = Z(\langle \bar{P}, \bar{f}, \bar{g}\rangle) \times \langle \bar{t}\rangle$. Since $[b_3f, \bar{A}_1] \neq 1$ and since $\langle A_1, b_3f\rangle/A_1 = Z(\langle P, f, g\rangle/A_1)$, we have that $C_{\langle \bar{P}, \bar{f}, \bar{g}\rangle}(\bar{A}_1) = \bar{A}_1$. Hence $Z(\langle \bar{P}, \bar{f}, \bar{g}\rangle) = C_{\bar{A}_1}(\langle \bar{b}_3, \bar{f}, \bar{g}\rangle) = \langle \bar{b}_1\rangle$ by Lemmas (4I) and (4J). Thus $Z(\bar{S}) = \langle \bar{b}_1, \bar{t}\rangle$. This proves (4).

LEMMA (4L). $S \notin \mathrm{Syl}_2(G)$.

Proof. Assume that $S \in \mathrm{Syl}_2(G)$. Then $\langle P, f, g\rangle$ contains an extremal conjugate u of t in S by Lemma (1E), since $t \in G'$. Since $t^G \cap \langle P, f\rangle = \emptyset$ by Lemma (4F), $u \equiv g$ or b_3fg mod A_1, and we may assume that $u \equiv g$ mod A_1. Then $C_{\bar{A}_1}(u) = \langle \bar{a}_1\bar{b}_2, \bar{a}_2, \bar{b}_1\rangle$ by Lemma (4J) and $C_{\langle P,f,g\rangle/A_1}(u) = \langle A_1, g, b_3fg\rangle/A_1$, so $|C_{\langle P,f,g\rangle}(u)| \leqq 2^6$ and $|C_S(u)| \leqq 2^7$. However, $|C|_2 = |R| = 2^8$. This is a contradiction. Therefore, $S \notin \mathrm{Syl}_2(G)$.

Now let $T \in \mathrm{Syl}_2(N(S))$.

LEMMA (4M). *The following conditions hold.*
(1) $|T: S| = 2$.
(2) $t^T = \langle b_0, b_1\rangle t$.
(3) $T \in \mathrm{Syl}_2(G)$.

Proof. By Lemma (4L), $S < T$ and so $t^T = |T: C_T(t)| = |T: R| \geqq 4$. On the other hand, $t^T \leqq Z_2(S) = \langle b_0, b_1, t\rangle$ by Lemma (4K), so $t^T \leqq \langle b_0, b_1\rangle t$ since $t^G \cap L = \varnothing$. Hence (1) and (2) follow.

Now $Z(T) = \langle b_0\rangle$ since $Z(T) \leqq C_T(t) \leqq S$ and $Z(S) = \langle b_0\rangle$. Hence $Z_2(T) \leqq N_T(B_1) = S$ by Lemma (4G)(2), and so $Z_2(T) \leqq Z_2(S) = \langle b_0, b_1, t\rangle$. Now (2) shows that $\langle b_0, b_1\rangle \lhd T$, so $\langle b_0, b_1\rangle \leqq Z_2(T)$. It also follows from (2) and Lemma (4E)(2) that $t^h = b_1 t$ or $b_0 b_1 t$ for $h \in T - S$. This implies that $t \notin Z_2(T)$. Therefore, $Z_2(T) = \langle b_0, b_1\rangle$.

Let $X = Z_3(T)$. Then $X \leqq N_T(B_1) = S$, and $[X, S] \leqq \langle b_0, b_1\rangle$. Hence $[\bar{X}, \bar{S}] \leqq \langle \bar{b}_1\rangle = Z(\bar{T})$. Now $\langle \bar{b}_1, \bar{t}\rangle = Z(\bar{S}) \lhd \bar{T}$, so $\langle \bar{b}_1, \bar{t}\rangle \leqq \bar{X}$. In particular, $\bar{t} \in \bar{X}$ and so, if $\bar{Y} = \bar{X} \cap \langle \bar{P}, \bar{f}, \bar{g}\rangle$, then $\bar{X} = \bar{Y}\langle \bar{t}\rangle$ by Lemma (4K)(2). We have that

$$[\bar{Y}, \langle \bar{P}, \bar{f}, \bar{g}\rangle] \leqq \langle \bar{b}_1\rangle \leqq \bar{A}_1 .$$

Hence $\bar{Y} \leqq Z(\langle \bar{P}, \bar{f}, \bar{g}\rangle \bmod \bar{A}_1) = \langle \bar{A}_1, \bar{b}_3\bar{f}\rangle$ by Lemma (4K)(1). From Lemma (4I)(3), we get that $[\bar{b}_3\bar{f}, \bar{a}_2] = \bar{a}_1\bar{b}_2\bar{b}_1 \notin \langle \bar{b}_1\rangle$. Hence, $\bar{Y} \leqq \bar{A}_1$ and using Lemmas (4I), (4J), we get that $\bar{Y} \leqq \langle \bar{a}_1\bar{b}_2, \bar{b}_1\rangle$. Therefore, $\langle \bar{b}_1, \bar{t}\rangle \leqq \bar{X} \leqq \langle \bar{a}_1\bar{b}_2, \bar{b}_1, \bar{t}\rangle$. That is, $\langle b_0, b_1, t\rangle \leqq Z_3(T) \leqq \langle a_1 b_2, b_0, b_1, t\rangle$. Hence $\Omega_1(Z_3(T)) = \langle b_0, b_1, t\rangle$.

Now let $U \in \mathrm{Syl}_2(N(T))$. Then $t^U \leqq \langle b_0, b_1, t\rangle$ by the above, and so $t^U = \langle b_0, b_1\rangle t$. This shows that $|U: R| = 4$. Hence $U = T$ and $T \in \mathrm{Syl}_2(G)$. The proof is complete.

LEMMA (4N). $t \notin G'$.

Proof. Let $h \in T - S$. Then $R \cap R^h \lhd T$ as $h^2 \in S \leqq N(R)$ by Lemma (4M). Since $R = C_T(t)$ and $t^h \in \langle b_0\rangle b_1 t$ by Lemmas (4E) and (4M),

$$R \cap R^h = C_R(t^h) = C_R(b_1) = \langle a_1, b_0, b_1, b_2, b_3, f, t\rangle .$$

Now $t \sim b_0 t \sim b_1 t$ by Lemma (4M), and since every involution of L is conjugate in L to b_0 or b_1, it follows that $t \sim xt$ for all $x \in I(L)$. Since $P^g = \langle A_1, f\rangle$ and $t^g = b_0 t$, we also have that $b_0 t \sim (f b_0) b_0 t = ft$. Hence $t \sim ft$. Also, $t^G \cap \langle a_1, b_0, b_1, b_2, b_3, f\rangle = \varnothing$ by Lemma (4F)(4). Therefore, we conclude that the subgroup generated by the products of two elements of $t^G \cap \langle a_1, b_0, b_1, b_2, b_3, f, t\rangle$ is equal to $\langle a_1, b_0, b_1, b_2, b_3, f\rangle$. This shows that $\langle a_1, b_0, b_1, b_2, b_3, f\rangle \lhd T$. Hence $\langle P, f\rangle \cap \langle P, f\rangle^h = \langle a_1, b_0, b_1, b_2, b_3, f\rangle$. Thus $N = \langle P, f\rangle\langle P, f\rangle^h$ is a normal subgroup of T of index 4, and moreover, $t \notin N$ as $S = \langle N, t\rangle$.

Let u be an extremal conjugate of t in T. Assume that $u \in S$. Notice that $\langle b_0, t\rangle \lhd S$ and $S/\langle b_0, t\rangle \cong \langle P, f, g\rangle/\langle b_0\rangle$ by Lemma (4K). Hence if $u \notin R$, then $u \equiv g$ or b_3fg mod B_1, and so $|C_{S/B_1}(u)| = 4$ and $|C_{B_1/\langle b_0, t\rangle}(u)| = 8$ by Lemma (4J). Since $|C_T(u)| = 2^8$ by assumption, we get that $C_{\langle b_0, t\rangle}(u) = \langle b_0, t\rangle$. But then $u \in C_T(t) = R$, a contradiction. Hence $u \in R$ and so $u \in \langle P, f\rangle t \leqq Nt$ by Lemma (4F)(4).

Assume that $u \notin S$. Then we may choose $h = u$. Now $\bar{B}_1^h$ is an E_{32}-subgroup of $\bar{S}$, and $\bar{B}_1^h \neq \bar{B}_1$ since $S \in \mathrm{Syl}_2(N(B_1))$ by Lemma (4G). Also, $\bar{t} \in Z(\bar{S}) \leqq \bar{B}_1^h$ by Lemma (4K)(4). Therefore, $\bar{B}_1^h = \bar{X}\langle \bar{t}\rangle$ for some E_{16}-subgroup $\bar{X}$ of $\langle \bar{P}, \bar{f}, \bar{g}\rangle$ different from $\bar{A}_1$ by Lemma (4K)(2). Thus $\bar{X}\bar{A}_1/\bar{A}_1$ is a nonidentity elementary abelian subgroup of $\langle \bar{P}, \bar{f}, \bar{g}\rangle/\bar{A}_1$ which centralizes the subgroup $\bar{X} \cap \bar{A}_1$ of $\bar{A}_1$. We argue that $\bar{X}\bar{A}_1 = \langle \bar{A}_1, \bar{b}_3\bar{f}, \bar{g}\rangle$. If not, then using Lemma (4I)(4), (5), (6), and Lemma (4J), we get that $\bar{X}\bar{A}_1 = \langle \bar{A}_1, \bar{g}\rangle$ or $\langle \bar{A}_1, \bar{g}_3\bar{f}\bar{g}\rangle$. Conjugating, we may assume the former. Then $\bar{X} \cap \bar{A}_1 = Z(\langle \bar{A}_1, \bar{g}\rangle) = \langle \bar{a}_1\bar{b}_2, \bar{a}_2, \bar{b}_1\rangle$ by Lemma (4J). But then $a_2 \in B_1^h \leqq R^h$, so $a_2 \in R \cap R^h = \langle a_1, b_0, b_1, b_2, b_3, f, t\rangle$, which is a contradiction. Therefore, $\bar{X}\bar{A}_1 = \langle \bar{A}_1, \bar{b}_3\bar{f}, \bar{g}\rangle$ and so $\bar{B}_1\bar{B}_1^h = \langle \bar{B}_1, \bar{b}_3\bar{f}, \bar{g}\rangle$. This implies that $B_1 \cap B_1^h$ has index 4 in B_1, so that $|B_1 \cap B_1^h| = 2^4$. We also have that $B_1 \cap R^h = B_1 \cap (R \cap R^h) = \langle a_1, b_0, b_1, b_2, t\rangle$. Hence $|B_1 \cap R^h| = 2^5$. Now consider the following normal series of T.

$$B_1 \cap B_1^h \leqq (B_1 \cap R^h)(B_1^h \cap R) \leqq R \cap R^h \leqq RR^h = S \leqq T .$$

The factors of this series have order 2 except for $(B_1 \cap R^h)(B_1^h \cap R)/B_1 \cap B_1^h$ and $RR^h/R \cap R^h$, which are fours groups. Therefore, the centralizer of h in each factor has order 2. There are 4 factors and $|C_T(h)| = 2^8$ by the choice of h. Hence h must centralize $B_1 \cap B_1^h$. But then, as $t \in Z_2(S) \leqq B_1 \cap B_1^h$, $h \in C_T(t) \leqq S$, which is a contradiction. Therefore, $u \in S$ and so $u \in Nt$ as shown before.

We have shown that each extremal conjugate of t in T is contained in Nt. Thus Lemma (1E) shows that $t \notin G'$.

Lemma (4N) conflicts with our assumption. Therefore, we have proved Theorem (4A).

5. In this section, we shall make the following hypothesis.

Hypothesis (5.1). $t^{N(B_2)} = \{t, c_1t, c_2t, c_3t, c_4t, c_5t\}$.

The purpose of this section is to prove the following.

THEOREM (5A). *Under Hypothesis* (5.1), $r(\langle L^G\rangle) = 4$.

The proof of this theorem is similar to that of Theorem (4A), although the arguments involved in this section are much more complicated than in § 4. We begin the proof by studying the permutation representation of $N(B_2)$ on $\Omega = t^{N(B_2)}$. Let

$$n_1 = t \text{ and } n_i = c_{i-1}t$$

for $i \in \{2, 3, 4, 5, 6\}$, so that

$$\Omega = \{n_1, n_2, n_3, n_4, n_5, n_6\} .$$

LEMMA (5B). *$N(B_2)^\Omega \cong N(B_2)/C(B_2) \cong \Sigma_6$ or A_6.*

Proof. First, observe that $\langle\Omega\rangle = B_2$. Hence $C(\Omega) = C(B_2)$ and $N(B_2)^\Omega \cong N(B_2)/C(B_2)$. By Hypothesis (5.1), $|N(B_2): N_C(B_2)| = 6$. Since $N_C(B_2)/C(B_2) \cong \Sigma_5$ or A_5 by Lemmas (2D) and (2H), it follows that $|N(B_2)/C(B_2)| = 720$ or 360. Thus $N(B_2)^\Omega$ is a subgroup of the symmetric group on Ω of index 1 or 2. Hence $N(B_2)^\Omega \cong \Sigma_6$ or A_6.

Notice that Hypothesis (5.1) implies Hypothesis (3.1). Therefore, $\langle t\rangle \in \mathrm{Syl}_2(C_C(L))$ by Lemma (3B).

LEMMA (5C). *The following conditions hold.*
(1) $N(A_2)/C(A_2) \cong N(B_2)/C(B_2)$.
(2) $N(B_2) \cap C(A_2) = C(B_2) = B_2O(C)$.
(3) $B_2 \in \mathrm{Syl}_2(C(A_2))$.

Proof. Since $\langle t\rangle \in \mathrm{Syl}_2(C_C(L))$, Lemma (2H) shows that $C(B_2) = B_2O(C)$. By Lemma (5B), $N(B_2)/C(B_2)$ has no nonidentity normal 2-subgroups. Since $N(B_2) \cap C(A_2)/C(B_2)$ is a normal 2-subgroup of $N(B_2)/C(B_2)$ by Lemmas (3E) and (3F), it follows that $N(B_2) \cap C(A_2) = C(B_2)$. This proves (3), since $B_2 \in \mathrm{Syl}_2(C(B_2))$. Finally, (1) holds by a Frattini argument.

Now $O(C(B_2)) = O(C)$ by Lemma (5C)(2), so let bars denote images in $N(B_2)/O(C)$. Then since $C(B_2) = B_2O(C)$, $\overline{N(B_2)}/\bar{B}_2 \cong \Sigma_6$ or A_6 by Lemma (5B). Choose the subgroup $\bar{M}$ of $\overline{N(B_2)}$ such that $\bar{B}_2 \leqq \bar{M}$ and $\bar{M}/\bar{B}_2 \cong A_6$. Then since $\bar{K}_2\bar{B}_2/\bar{B}_2 \cong A_5$, $\bar{K}_2\bar{B}_2 \leqq \bar{M}$ and in particular, $\bar{Q} \leqq \bar{M}$. Now $\bar{A}_2 \lhd N(\bar{B}_2)$ by Lemma (3E). Hence $\bar{M}/\bar{A}_2$ is an extension of Z_2 by A_6, and it contains $\bar{Q}/\bar{A}_2 \cong E_8$. Therefore, the extension splits, and there is a subgroup $\bar{N}$ of $\bar{M}$ such that $\bar{A}_2 \leqq \bar{N}$ and $\bar{M}/\bar{A}_2 = \bar{N}/\bar{A}_2 \times \bar{B}_2/\bar{A}_2$. As before, $\bar{K}_2\bar{A}_2 \leqq \bar{N}$, and so $\bar{P} \leqq \bar{N}$.

DEFINITION (5.1). Let M and N be the preimages of $\bar{M}$ and $\bar{N}$, respectively. Furthermore, let $Q \leqq R \in \mathrm{Syl}_2(C)$, $R \leqq T \in \mathrm{Syl}_2(N(B_2))$, $S = T \cap M$, and $U = S \cap N$.

Thus $U \lhd T$, $T = RU$, $R \cap U = P$, and $R \cap S = Q$ by the above remark. In particular, $T/U \cong R/P$. Notice also that $N(B_2)/C(B_2) \cong \Sigma_6$ if and only if $Q < R$, as $R \in \mathrm{Syl}_2(N_C(B_2))$.

LEMMA (5D). *If T/U is cyclic, then Theorem* (5A) *holds.*

Proof. Suppose that T/U is cyclic. Then $t^G \cap T \leqq S$. Hence $t^G \cap R \leqq S \cap R = Q$, so $B_2 = \langle t^G \cap B_2 \rangle$ is weakly closed in R with respect to G by Lemma (2A). Let $t^g \in B_2$. Then $B_2^{g^{-1}} \leqq C$, so there is an element $c \in C$ such that $B_2^{g^{-1}} \leqq R^c$. By the weak closure of B_2, $B_2^{g^{-1}} = B_2^c$ and $t^g = t^{cg} \in t^{N(B_2)}$. Therefore, $t^G \cap B_2 = t^{N(B_2)} = \Omega$.

Let $x \in t^G \cap (Q - B_2)$. Then $x \in B_1$ by Lemma (2A) and x is conjugate to an element of $B_1 \cap B_2$ in $N_C(B_1)$ by Lemma (2E). Since $t^G \cap B_1 \cap B_2 = \Omega \cap B_1 = \{t, c_1t\}$ and since t and $c_1t \in Z(N_C(B_1))$, $x = t$ or c_1t and so $x \in B_2$, which is a contradiction. Therefore, $t^G \cap Q = t^G \cap B_2$. This in turn implies that $t^G \cap S = t^G \cap B_2$, as M/B_2 has one conjugacy class of involutions by the definition of M. Thus $t^G \cap T = t^G \cap B_2 = \Omega$. Hence $N(T) \leqq N(B_2)$ and so $T \in \mathrm{Syl}_2(G)$. Also, $t^G \cap T \leqq Ut$. Therefore, $t \notin G'$ by Lemma (1E). Since $U \leqq N' \leqq G'$, we conclude that $U \in \mathrm{Syl}_2(G')$.

Now $N(A_2)/C(A_2) \cong \Sigma_6$ or A_6 by Lemmas (5C) and (5B). As $N_{N'}(A_2)/C_{N'}(A_2) \cong A_6$ and $U \in \mathrm{Syl}_2(N_{G'}(A_2))$, it follows that $N_{G'}(A_2)/C_{G'}(A_2) \cong A_6$. Also, since $B_2 \in \mathrm{Syl}_2(C(A_2))$ and since $t \notin G'$, $A_2 \in \mathrm{Syl}_2(C_{G'}(A_2))$. Thus by [17, Theorem 3], $r(G') = 4$ and hence $r(\langle L^G \rangle) = 4$. The proof is complete.

In view of Lemma (5D), we shall assume from now on that T/U is not cyclic. This implies that $T/U \cong E_4$. Let bars denote images in $N(B_2)/O(C)$. Then since $\overline{N(B_2)}/\bar{N} \cong \bar{T}/\bar{U}$, there is a subgroup $\bar{K}$ of $\overline{N(B_2)}$ such that $\bar{N} < \bar{K}$ and $\overline{N(B_2)}/\bar{A}_2 = \bar{K}/\bar{A}_2 \times \bar{B}_2/\bar{A}_2$.

DEFINITION (5.2). Let K be the preimage of $\bar{K}$ in $N(B_2)$ and set $V = T \cap K$.

Since $R/P \cong E_4$, we may assume without loss of generality that there is an involution $f \in R - Q$ whose action on L is induced by the automorphism of F_4 of order 2.

Now $A_2 \lhd R$, so R acts on A_2 by conjugation. In the following lemma, we collect information on this action. For the proof, see Lemmas (2A) and (2F).

LEMMA (5E). *The following conditions hold.*

(1) $b_0^{a_1} = b_0$, $b_1^{a_1} = b_1$, $b_2^{a_1} = b_0b_2$, $b_3^{a_1} = b_0b_1b_3$.

(2) $b_0^{a_2} = b_0$, $b_1^{a_2} = b_0b_1$, $b_2^{a_2} = b_2$, $b_3^{a_2} = b_0b_2b_3$.

(3) $b_0^f = b_0$, $b_1^f = b_1$, $b_2^f = b_1b_2$, $b_3^f = b_3$.

(4) $C_{A_2}(a_1) = \langle b_0, b_1\rangle$.

(5) $C_{A_2}(a_2) = \langle b_0, b_2\rangle$.

(6) $C_{A_2}(f) = \langle b_0, b_1, b_3\rangle$.

Permutation representations of a_1, a_2, and f on Ω can be computed by using Lemma (5E) and the expressions of c_i's in terms of b_i's given in § 2. We have that

$$a_1^\Omega = (n_3, n_4)(n_5, n_6),\ a_2^\Omega = (n_3, n_5)(n_4, n_6),\ f^\Omega = (n_5, n_6)\ .$$

Therefore, we may assume without loss of generality that

$$T^\Omega = \langle a^\Omega, f^\Omega, a_1^\Omega, a_2^\Omega\rangle\ ,$$

where

$$a^\Omega = (n_1, n_2)\ .$$

That is, $t^a = c_1t$, $(c_1t)^a = t$, and $(c_it)^a = c_it$ for $i \in \{2, 3, 4, 5\}$. Noticing that $c_i = (c_it)t$, we get that $c_1^a = c_1$ and $c_i^a = c_1c_i$ for $i \in \{2, 3, 4, 5\}$. Thus we can determine the action of a on B_2, using the relations $b_0 = c_1$, $b_1 = c_4c_5$, $b_2 = c_1c_2c_4$, and $b_3 = c_2$. Furthermore, we can compute $[B_2, a]$ and $C_{B_2}(a)$. Also, $C_T(\Omega) = B_2$ and a^Ω is an involution which centralizes a_1^Ω, a_2^Ω, and f^Ω. Thus we have the following result.

LEMMA (5F). *There is an element $a \in T - R$ which satisfies the following conditions.*

(1) a^2, $[a_1, a]$, $[a_2, a]$, *and* $[f, a] \in B_2$.

(2) $b_0^a = b_0$, $b_1^a = b_1$, $b_2^a = b_2$, $b_3^a = b_0b_3$, $t^a = b_0t$.

(3) $[B_2, a] = \langle b_0\rangle$.

(4) $C_{B_2}(a) = \langle b_0, b_1, b_2, b_3t\rangle$.

Our next result shows that T has the unique structure.

LEMMA (5G).

(1) *We may choose a in Lemma* (5F) *and f so that* $a^2 = [a_1, a] = [a_2, a] = [f, a] = 1$.

(2) *If P^*/A_2 is an E_4-subgroup of U/A_2 different from P/A_2, then $\mathscr{E}^*(P^*)$ consists of two E_{16}-subgroups.*

Proof. Observe first that $V \cap R = \langle P, f\rangle$ or $\langle P, ft\rangle$. Replacing f by ft in the latter case, we may assume that $f \in V$.

Choose an element $a \in T - R$ as in Lemma (5F), and let bars denote images in $N(B_2)/C(B_2)$. Then $\bar{T} = \langle \bar{a} \rangle \times \langle \bar{a}_1, \bar{a}_2, \bar{f} \rangle \cong Z_2 \times D_8$ and $Z(\bar{T}) = \langle \bar{a}, \bar{a}_1 \rangle$.

Now $\bar{a}_1 \in Z(\bar{T})$, so $\langle a_1 \rangle A_2 \lhd V$. Also, $C_{A_2}(a_1) = \langle b_0, b_1 \rangle$ and so $I(a_1 A_2) = a_1^{A_2}$ by Lemma (1C). Thus $V = C_V(a_1)A_2$, and consequently $|C_V(a_1)| = 64$.

Now $\langle a_1, a_2, f, b_0 \rangle \leqq N(\langle a_1, a_2 \rangle) \cap C_V(a_1)$. Suppose that equality holds here. Then $C_V(a_1) \cap C_V(a_2) = C(a_2) \cap \langle a_1, a_2, f, b_0 \rangle = \langle a_1, a_2, b_0 \rangle$ and so $|C_V(a_1): C_V(a_1) \cap C_V(a_2)| = 8$. This shows that $|a_2^{C_V(a_1)}| = 8$. However, since $\langle \bar{a}_1, \bar{a}_2 \rangle \lhd \bar{T}$, $\langle a_1, a_2, C_{A_2}(a_1) \rangle \lhd C_V(a_1)$. Similarly, $\langle a_1, C_{A_2}(a_1) \rangle \lhd C_V(a_1)$. Hence $a_2^{C_V(a_1)} \leqq a_2 \langle a_1, C_{A_2}(a_1) \rangle$, whereas $|I(a_2 \langle a_1, C_{A_2}(a_1) \rangle)| = 4$ as $C(a_2) \cap \langle a_1, C_{A_2}(a_1) \rangle = \langle a_1, b_0 \rangle$ has order 4. This contradiction shows that $\langle a_1, a_2, f, b_0 \rangle \neq N(\langle a_1, a_2 \rangle) \cap C_V(a_1)$, so $N(\langle a_1, a_2 \rangle) \cap C_V(a_1)$ has index 2 in $C_V(a_1)$.

Now $C_{A_2}(a_1) \nleqq N(\langle a_1, a_2 \rangle)$, so that by the above paragraph,

$$C_V(a_1) = (N(\langle a_1, a_2 \rangle) \cap C_V(a_1)) C_{A_2}(a_1) .$$

Thus $V = N_V(\langle a_1, a_2 \rangle) A_2$ and so we may assume $a \in N_V(\langle a_1, a_2 \rangle)$. Then, since $[\bar{a}, \langle \bar{a}_1, \bar{a}_2 \rangle] = 1$, $[a, \langle a_1, a_2 \rangle] = 1$. Also, since $\bar{a}^2 = (\overline{af})^2 = 1$, a^2 and $(af)^2 \in N_{A_2}(\langle a_1, a_2 \rangle) = \langle b_0 \rangle$. Using the relation $t^a = b_0 t$, we may deduce as follows:

$$\begin{aligned}(atf)^2 = (aft)^2 &= (af)^2 (af)^{-1} t (af) t \\ &= (af)^2 t^{af} t \\ &= (af)^2 t b_0 t \\ &= (af)^2 b_0 .\end{aligned}$$

Also,

$$(at)^2 = a^2 t^a t = a^2 (b_0 t) t = a^2 b_0 .$$

If $a^2 = b_0$, let $a_0 = at$. Then $a_0^2 = 1$ and $(a_0 f)^2 = (af)^2 b_0 \in \langle b_0 \rangle$ by the above. If $(a_0 f)^2 = b_0$, let $f_0 = ft$. Then $(a_0 f_0)^2 = (af)^2 = (a_0 f)^2 b_0 = 1$. If $a^2 = 1$ and $(af)^2 = b_0$, then $(af_0)^2 = (af)^2 b_0 = 1$. Therefore, replacing a and f by at and ft, if necessary, we may assume that $a^2 = (af)^2 = 1$. This proves (1).

Now $(af)^\Omega = (n_1, n_2)(n_5, n_6)$ by definition, so $af \in S$ and $S = \langle a_1, a_2, af \rangle B_2$. Since $P^* B_2 / B_2$ is an E_4-subgroup of S/B_2 different from PB_2/B_2 and since $PB_2 = \langle a_1, a_2 \rangle B_2$, it follows that $P^* B_2 = \langle a_1, af \rangle B_2$. Hence if $x \in P^* - A_2$, then $C_{A_2}(x) = C_{A_2}(a_1)$, $C_{A_2}(af)$ or $C_{A_2}(a_1 af)$, and so using Lemmas (5E) and (5F), we have that $C_{A_2}(x) = \langle b_0, b_1 \rangle$. Now (1) shows that $\langle a, a_1, a_2, f \rangle$ is a complement for B_2 in T, so that B_2 has a complement Y in $N(B_2)$ by Gaschütz's theorem [19, Hauptsatz 17.4]. Then Y' is a complement for A_2 in N', and so there is a fours group X such that $XA_2 = P^*$ and $X \cap A_2 = 1$. Since $C_{A_2}(x) =$

$\langle b_0, b_1\rangle$ for $x\in X^\#$, [11, (1C)] shows that $\mathscr{E}^*(P^*) = \{A_2, X\langle b_0, b_1\rangle\}$. This proves (2).

Now choose an element $a\in T-R$ as in Lemma (5G). As remarked in the proof of Lemma (5G)(2), $T = \langle a, a_1, a_2, f\rangle B_2$ and $\langle a, a_1, a_2, f\rangle \cap B_2 = 1$.

LEMMA (5H). *The following conditions hold.*

(1) $Z(T) = \langle b_0\rangle$.

(2) $Z_2(T) = \langle a, b_0, b_1, t\rangle$.

Proof. As $Z(T) \leqq C_T(t) = R$, $Z(T) \leqq Z(R) = \langle b_0, t\rangle$. As $t^a = b_0t$ by Lemma (5F)(2), $Z(T) = \langle b_0\rangle$.

Now $Z_2(T)\leqq C_T(B_2/\langle b_0\rangle)\leqq Z(T \bmod B_2) = \langle a, a_1\rangle B_2$. Since $[a, B_2] = \langle b_0\rangle$ by Lemma (5F)(3) and since $[a_1, B_2] = \langle b_0, b_1\rangle$ by Lemma (5E)(1), we have that $\langle a\rangle \leqq Z_2(T) \leqq \langle a\rangle B_2$. Hence if $X = B_2 \cap Z_2(T)$, then $Z_2(T) = \langle a\rangle X$.

By definition $X \leqq Z_2(Q) = \langle b_0, b_1, b_2, t\rangle$. Clearly, $b_0\in X$. We have that $[\langle a, a_1, a_2, f\rangle, b_1] = \langle b_0\rangle$ by Lemmas (5E) and (5F). Also, $[\langle a, a_1, a_2, f\rangle, t] = \langle b_0\rangle$. Hence b_1 and $t\in X$. However, $b_2\notin X$ since $[f, b_2] = b_1$ by Lemma (5E)(3). Therefore, $X = \langle b_0, b_1, t\rangle$ and so $Z_2(T) = \langle a, b_0, b_1, t\rangle$.

LEMMA (5I). *The following conditions hold.*

(1) $C_T(b_1t) = \langle aa_2, a_1, f, B_2\rangle$.

(2) *B_2 and $D = \langle a_1, f, b_0, b_1, t\rangle$ are E_{32}-subgroups of $C_T(b_1t)$ and both are normal in T.*

(3) $C_T(a) = \langle a, a_1, a_2, f, b_0, b_1, b_2, b_3t\rangle$.

(4) $C_T(ab_1) = \langle a, a_1, f, b_0, b_1, b_2, b_3t, a_2t\rangle$.

(5) *$E = \langle a, b_0, b_1, b_2, b_3t\rangle$ and $F = \langle a, a_1, f, b_0, b_1\rangle$ are E_{32}-subgroups of $C_T(a)$ and $C_T(ab_1)$, and both E and F are normal in T.*

Proof. Since B_2 is abelian, $C_T(b_1t) = C_{\langle a,a_1,a_2,f\rangle}(b_1t)B_2$. By Lemma (5E), a_1 and f centralize b_1t. Also, $(b_1t)^{aa_2} = (b_1b_0t)^{a_2} = b_0b_1b_0t = b_1t$ by Lemmas (5E) and (5F). However, $a\notin C(b_1t)$ by Lemma (5F)(2). Thus $C_{\langle a,a_1,a_2,f\rangle}(b_1t) = \langle aa_2, a_1, f\rangle$ and hence (1) follows.

To prove (2), it is enough to show that $a\in N(D)$ as $D=\langle C_{A_1}(f), f, t\rangle \lhd R$ by Lemma (2F). By Lemmas (5F) and (5G), a centralizes a_1, f, b_0, b_1. Also, $t^a = b_0t$. Thus $a\in N(D)$. (3) is a direct consequence of Lemmas (5G)(1) and (5F)(4).

As a consequence of (3), we have that E is elementary of order 32. Also, F is elementary of order 32 as $\langle a, a_1, f\rangle$ centralizes $\langle b_0, b_1\rangle$ by Lemmas (5E) and (5F). Thus E and $F\leqq C_T(ab_1)$. Now $(ab_1)^{a_2t} = (ab_0b_1)^t = (ab_0)b_0b_1 = ab_1$ by Lemmas (5E) and (5F)(2). Hence

$\langle E, F, a_2t\rangle \leqq C_T(ab_1)$ and as $\langle E, F, a_2t\rangle$ is maximal in T and $ab_1 \notin Z(T)$ by Lemma (5H), we conclude that $C_T(ab_1) = \langle E, F, a_2t\rangle = \langle a, a_1, f, b_0, b_1, b_2, b_3t, a_2t\rangle$.

Now $\langle a_1, a_2, f\rangle$ centralizes a and normalizes $\langle b_0, b_1, b_2, b_3t\rangle$ by Lemmas (5E) and (5F). Also, $[B_2, a] = \langle b_0\rangle$ and B_2 centralizes $\langle b_0, b_1, b_2, b_3t\rangle$. Thus $T = \langle a_1, a_2, f, E, B_2\rangle$ normalizes E.

Similarly, we see that a_2 normalizes $\langle a, a_1, f\rangle$ and $\langle b_0, b_1\rangle$. Furthermore, $[\langle a, a_1, f\rangle, B_2] \leqq \langle b_0, b_1\rangle$ and B_2 centralizes $\langle b_0, b_1\rangle$. Hence $T = \langle a_2, F, B_2\rangle$ normalizes F.

LEMMA (5J). *$t^G \cap \langle A_1, t\rangle = t^T = \{t, b_0t\}$ and $t^G \cap B_2 = t^{N(B_2)}$.*

Proof. Suppose that $t \sim b_1t$. Since $R \in \mathrm{Syl}_2(C(t))$, t is extremal in an S_2-subgroup of G containing T. Therefore, there is an element $g \in G$ such that $(b_1t)^g = t$ and $C_T(b_1t)^g = R$. By Lemma (2F), B_2 and D are the only normal E_{32}-subgroup of R, so Lemma (5I)(2) shows that $\{B_2, D\}^g = \{B_2, D\}$. Since $b_1t \notin t^{N(B_2)}$ by Hypothesis (5.1), $g \in N(B_2)$ and therefore, $D^g = B_2$.

Now $T \leqq N(C_T(b_1t)) \cap N(D)$ by Lemma (5I), so $T^g \leqq N(B_2) \cap N(R)$. Also, $T \leqq N(B_2) \cap N(R)$. Hence there is an element $h \in g(N(B_2) \cap N(R))$ such that $T^h = T$. Thus $b_0^h = b_0$ since $Z(T) = \langle b_0\rangle$, $D^h = B_2$, and $(b_1t)^h = t$ or b_0t since $Z(R) = \langle b_0, t\rangle$.

It follows from Lemma (3I) that $A_1^h \leqq T \cap O^{2,2'}(N(B_2)) = U$ as $O^2(N(B_2)) = N$. Suppose that $A_1^h = A_1$. Then $B_1^h = \langle A_1, b_1t\rangle^h = \langle A_1, t\rangle$ or $\langle A_1, b_0t\rangle$, so $h \in N(B_1) \leqq N(Z(B_1))$. However, $Z(B_1) = \langle b_0, t\rangle$ and $t^{h^{-1}} = b_1t$ or $b_0b_1t \notin Z(B_1)$. This is a contradiction. Therefore, $A_1^h \neq A_1$ and so $A_1^h \nleqq P$ since $A_1/\langle b_0\rangle$ is the unique E_{16}-subgroup of $P/\langle b_0\rangle$. Hence $A_1^hA_2/A_2$ is contained in the E_4-subgroup P^*/A_2 of U/A_2 different from P/A_2, and so $A_1^h \leqq P^*$. However, $|\mathscr{E}^*(P^*)| = 2$ by Lemma (5G), whereas $|\mathscr{E}^*(A_1)| > 2$. This is a contradiction. Therefore, $t \nsim b_1t$ and then $t^G \cap B_2 = t^{N(B_2)}$ by Lemma (2D).

Now $t^G \cap A_1 = \varnothing$ by Lemma (3C). Also, (2E) shows that involutions in $A_1t - \{t, b_0t\}$ are conjugate to b_1t. Thus $t^G \cap \langle A_1, t\rangle \leqq \{t, b_0t\}$. Since $b_0t = t^a$ and $R = C_T(t)$ has index 2 in T, we conclude that $t^G \cap \langle A_1, t\rangle = \{t, b_0t\} = t^T$.

LEMMA (5K). *Let $T_1 \in \mathrm{Syl}_2(N(T))$. Then the following holds.*

(1) $|T_1 : T| \leqq 2$.

(2) *If $g \in T_1 - T$, then $\langle b_0, b_1, t\rangle^g = \langle a, b_0, b_1\rangle$, $B_2^g = F$, $F^g = B_2$, $D^g = E$, and $E^g = D$.*

(3) *If $T < T_1$, then there is an element $g \in T_1 - T$ such that $g^2 \in \langle b_0, b_1\rangle$.*

(4) *If $T < T_1$, then there is an element $g \in T_1 - T$ such that $t^g = a$ or ab_1.*

Proof. First of all, $Z_2(T) = \langle a, b_0, b_1, t\rangle$ and $C_{\langle b_0, b_1, t\rangle}(a) = \langle b_0, b_1\rangle$ by Lemmas (5F) and (5H). Hence

$$\mathscr{E}^*(Z_2(T)) = \{\langle a, b_0, b_1\rangle, \langle b_0, b_1, t\rangle\}$$

and

$$\langle b_0, b_1\rangle = Z(Z_2(T)) \lhd T_1 .$$

Assume that $T < T_1$ and let $g \in T_1 - T$. By Lemma (5J),

$$t^G \cap \langle b_0, b_1, t\rangle = \{t, b_0t\} .$$

On the other hand, $|t^{T\langle g\rangle}| = |T\langle g\rangle : R| \geqq 4$. Hence we must have that $\langle b_0, b_1, t\rangle \ntriangleleft T\langle g\rangle$. However, $\langle b_0, b_1, t\rangle \lhd T$ by Lemma (5H). Therefore, $g \notin N(\langle b_0, b_1, t\rangle)$. Since g acts on $\mathscr{E}^*(Z_2(T))$, we conclude that

$$\langle b_0, b_1, t\rangle^g = \langle a, b_0, b_1\rangle .$$

As a consequence of this, we have that $|t^G \cap \langle a, b_0, b_1\rangle| = 2$ and moreover $t^G \cap \langle a, b_0, b_1\rangle \leqq a\langle b_0, b_1\rangle$ since $\langle b_0, b_1\rangle \lhd T_1$. Now $a^{b_3} = ab_0$ and $(ab_1)^{a_2} = ab_0b_1$ by Lemmas (5E) and (5F). Hence

$$t^G \cap \langle a, b_0, b_1\rangle = \{a, ab_0\} \text{ or } \{ab_1, ab_0b_1\} .$$

This proves (4), and we may assume that $t^g = a$ or ab_1 in proving the remaining part of (2) since B_2, D, E, and $F \lhd T$.

Now we have shown that $t^G \cap Z_2(T) = \{t, b_0t, a, ab_0\}$ or $\{t, b_0t, ab_1, ab_0b_1\}$. Therefore, $|T_1 : R| = |t^{T_1}| \leqq 4$ and $|T_1 : T| \leqq 2$.

Let $g \in T_1 - T$ and suppose $t^g = a$ or ab_1. By Lemma (2F), B_2 and D are the only normal E_{32}-subgroups of $C_T(t) = R$. Also, E and F are normal E_{32}-subgroups of $C_T(a)$ and $C_T(ab_1)$ by Lemma (5I). Hence $\{B_2, D\}^g = \{E, F\}$. Now $\langle a, B_2\rangle$ is conjugate to $\langle f, B_2\rangle$ in $N(B_2)$ since $a^\Omega = (n_1, n_2)$ and $f^\Omega = (n_5, n_6)$. Since $\mathscr{E}^*(\langle a, B_2\rangle) = \{E, B_2\}$ by Lemma (5F)(4) and since $\mathscr{E}^*(\langle f, B_2\rangle) = \{\langle C_{A_2}(f), f, t\rangle, B_2\}$, it follows that E is conjugate to $\langle C_{A_2}(f), f, t\rangle$ in $N(B_2)$. Thus $B_2^g \neq E$ by Lemma (3H) and so $B_2^g = F$ and $D^g = E$. This proves (2) as $g^2 \in T \leqq N(B_2) \cap N(D)$.

Now $\langle b_0, b_1\rangle \lhd T_1$ and $\langle b_0, b_1\rangle \nleqq Z(T)$, so $C_T(\langle b_0, b_1\rangle)$ is a subgroup of $C_{T_1}(\langle b_0, b_1\rangle)$ of index 2. Furthermore, $C_T(\langle b_0, b_1\rangle) = B_2F$ and $B_2 \cap F = \langle b_0, b_1\rangle$. The assertion (3) now follows from Lemma (1B) applied to $C_{T_1}(\langle b_0, b_1\rangle)/\langle b_0, b_1\rangle$.

LEMMA (5L). *If $T < T_1 \in \mathrm{Syl}_2(N(T))$, then the following conditions hold.*

(1) $Z(T_1) = \langle b_0\rangle$.

(2) $Z_2(T_1) = \langle b_0, b_1, at\rangle$.

(3) $Z_3(T_1) = \langle a, a_1b_2, b_0, b_1, t\rangle$.

Proof. Since $Z(T_1) \leqq C(t) \cap T_1 = R \leqq T$, $Z(T_1) \leqq Z(T) = \langle b_0\rangle$ by Lemma (5H). Hence $Z(T_1) = \langle b_0\rangle$, and consequently, $Z_2(T_1) \leqq N_{T_1}(B_2) = T$. Since $Z(T_1) = Z(T)$, $Z_2(T_1) \leqq Z_2(T) = \langle a, b_0, b_1, t\rangle$ by Lemma (5H). Now Lemma (5K)(2) shows that T_1 normalizes $\langle b_0, b_1\rangle$, so $\langle b_0, b_1\rangle \leqq Z_2(T_1)$. Furthermore, if $g \in T_1 - T$, then g interchanges $\langle a, b_0, b_1\rangle$ and $\langle b_0, b_1, t\rangle$. Hence $\langle b_0, b_1\rangle \leqq Z_2(T_1) \leqq \langle b_0, b_1, at\rangle$. We show that $at \in Z_2(T_1)$. We may assume that $t^g = a$ or ab_1 by Lemma (5K)(4). If $t^g = a$, then $a^g = t$ or b_0t since $g^2 \in T$ and $t^T = \{t, b_0t\}$. Hence $(at)^g = atb_0$ or at by Lemma (5F)(2). If $t^g = ab_1$, then $(ab_1)^g = t$ or b_0t, so $(ab_1t)^g = (ab_1t)b_0$ or ab_1t. In either case, $at \in Z_2(T_1)$. Therefore, $Z_2(T_1) = \langle b_0, b_1, at\rangle$.

It remains to prove (3). Suppose first that $Z_3(T_1) \nleqq T$. Then we may choose $g \in Z_3(T_1) - T$. However, since g normalizes $Z_2(T_1)B_2 = \langle a, B_2\rangle$ and since $\mathscr{E}^*(\langle a, B_2\rangle) = \{E, B_2\}$ by Lemma (5F), we must have that $B_2^g = E$, contrary to Lemma (5K)(2). Thus $Z_3(T_1) \leqq T$.

Let bars denote images in $T_1/\langle b_0, b_1\rangle$. Then $\overline{FB_2}$ is a normal E_{64}-subgroup of $\bar{T}_1$ by Lemma (5K)(2) d $\bar{T}_1$an $= \overline{FB_2}\langle \bar{a}_2, \bar{g}\rangle$. We choose $\bar{a}_1, \bar{f}, \bar{a}, \bar{b}_2, \bar{b}_3, \bar{t}$ as a basis of $\overline{FB_2}$ and represent $\bar{a}_2$ and $\bar{g}$ by 6×6 matrices with respect to this basis. Using Lemmas (5E) and (5F), we see that $\bar{a}_2$ has the following matrix form.

$$\begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & 1 & 1 & \\ & & & & & 1 \end{pmatrix}$$

Therefore, $Z(\bar{T}) = C_{\overline{FB_2}}(\bar{a}_2) = \langle \bar{a}, \bar{a}_1, \bar{b}_2, \bar{t}\rangle$. Then by Lemma (5K)(2), $\bar{g}$ interchanges $\langle \bar{a}, \bar{a}_1\rangle$ and $\langle \bar{b}_2, \bar{t}\rangle$ as $\langle \bar{a}, \bar{a}_1\rangle = Z(\bar{T}) \cap \bar{F}$ and $\langle \bar{b}_2, \bar{t}\rangle = Z(\bar{T}) \cap \bar{B}_2$. Also, $\bar{g}$ interchanges $\langle \bar{a}_1, \bar{f}\rangle$ and $\langle \bar{b}_2, \bar{b}_3\bar{t}\rangle$ as $\langle \bar{a}_1, \bar{f}\rangle = \bar{F} \cap \bar{D}$ and $\langle \bar{b}_2, \bar{b}_3\bar{t}\rangle = \bar{E} \cap \bar{B}_2$. Thus $\bar{g}$ interchanges $\langle \bar{a}_1\rangle$ and $\langle \bar{b}_2\rangle$, and also interchanges $\langle \bar{a}_1, \overline{af}\rangle$ and $\langle \bar{b}_2, \bar{b}_3\rangle$. Since $\bar{g}$ also interchanges $\langle \bar{t}\rangle$ and $\langle \bar{a}\rangle$ by Lemma (5K)(2), we get that the matrix of $\bar{g}$ has the following shape.

$$\begin{pmatrix} & & & 1 & & \\ & & & \alpha & 1 & 1 \\ & & & & & 1 \\ 1 & & & & & \\ \beta & 1 & 1 & & & \\ & & 1 & & & \end{pmatrix}$$

By Lemma (5K)(3), we may assume from the outset that $\bar{g}^2 = 1$. This implies that the square of the above matrix is the unit matrix. Hence $\alpha = \beta$ and $\bar{g}$ has the following matrix form.

$$\begin{pmatrix} & & & 1 & & \\ & & & \alpha & 1 & 1 \\ & & & & & 1 \\ 1 & & & & & \\ \alpha & 1 & 1 & & & \\ & & 1 & & & \end{pmatrix}$$

Now an element $\bar{x}$ of $\overline{FB}_2$ is represented by a sextuplet $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$. Using matrix forms of $\bar{a}_2$ and $\bar{g}$, we see that $[\bar{x}, \bar{a}_2]$ and $[\bar{x}, \bar{g}]$ are represented by the sextuplets $(\beta_2, 0, 0, \beta_5, 0, 0)$ and $(\beta_1 + \beta_4 + \alpha\beta_5, \beta_2 + \beta_5, \beta_3 + \beta_5 + \beta_6, \beta_1 + \alpha\beta_2 + \beta_4, \beta_2 + \beta_5, \beta_2 + \beta_3 + \beta_6)$, respectively. This shows first that $[\overline{FB}_2, \bar{a}_2] = \langle \bar{a}_1, \bar{b}_2 \rangle \nleqq \langle \overline{at} \rangle$. Therefore, $Z_3(T_1) \leqq FB_2$. Next, both $[\bar{x}, \bar{a}_2]$ and $[\bar{x}, \bar{g}]$ are contained in $\langle \overline{at} \rangle$ if and only if the following equations hold.

$$\beta_2 = \beta_5 = 0, \quad \beta_1 + \beta_4 + \alpha\beta_5 = 0, \quad \beta_2 + \beta_5 = 0\,,$$
$$\beta_3 + \beta_5 + \beta_6 = \beta_2 + \beta_3 + \beta_6, \ \beta_1 + \alpha\beta_2 + \beta_4 = 0\,.$$

These are satisfied if and only if $\beta_1 = \beta_4$ and $\beta_2 = \beta_5 = 0$. This implies that $\overline{Z_3(T_1)} = \langle \bar{a}_1\bar{b}_2, \bar{a}, \bar{t} \rangle$. Hence (3) follows.

In the course of the proof of Lemma (5L), we have proved the following.

LEMMA (5M). *Let $T_1 \in \mathrm{Syl}_2(N(T))$ and let g be an element of $T_1 - T$ such that $g^2 \in \langle b_0, b_1 \rangle$. Then g acts on $\overline{FB}_2 = FB_2/\langle b_0, b_1 \rangle$ in the following fashion.*

$$\bar{a}_1^g = \bar{b}_2, \ \bar{f}^g = \bar{b}_2^\alpha \bar{b}_3 \bar{t}, \ \bar{a}^g = \bar{t}\,,$$
$$\bar{b}_2^g = \bar{a}_1, \ \bar{b}_3^g = \bar{a}_1^\alpha \overline{fa}, \ \bar{t}^g = \bar{a}\,.$$

Here, $\alpha = 0$ or 1.

LEMMA (5N). *$N(T)$ contains an S_2-subgroup of G.*

Proof. Let $T_1 \in \mathrm{Syl}_2(N(T))$. If $T = T_1$, then $T \in \mathrm{Syl}_2(G)$. Therefore, assume that $T < T_1$. Then by Lemmas (5L), (5E), and (5F),

$$\begin{aligned} Z_3(T_1) &= \langle a, a_1b_2, b_0, b_1, t \rangle \\ &= \langle b_1 \rangle \times \langle a, t \rangle * \langle a_1b_2 \rangle \\ &\cong Z_2 \times D_8 * Z_4\,. \end{aligned}$$

Therefore, $Z_3(T_1)$ has exactly 3 abelian maximal subgroups

$$Y_1 = \langle b_1, t, a_1b_2\rangle ,$$
$$Y_2 = \langle b_1, a, a_1b_2\rangle ,$$
$$Y_3 = \langle b_1, at, a_1b_2\rangle .$$

Let $X \in \mathrm{Syl}_2(N(T_1))$. Since Y_3 contains $Z_2(T_1) = \langle b_0, b_1, at\rangle$ while Y_1 and Y_2 do not, X acts on $\{Y_1, Y_2\}$. Since $t^G \cap Y_1 = \{t, b_0t\} = t^T$ by Lemma (5J), $N_X(Y_1) \leqq N_X(\{t, b_0t\}) = T$. Thus $|X: T| \leqq 2$ and so $X = T_1$. This shows $T_1 \in \mathrm{Syl}_2(G)$.

Now let T_1 be an S_2-subgroup of G containing T.

LEMMA (5O). *The following conditions hold.*
(1) $W = \langle a, a_1, a_2, b_0, b_1, b_2, t\rangle = \langle A_1, a, t\rangle$ *is a normal subgroup of* T_1.
(2) W *is an extra-special group of order* 2^7, *and* $Z(W) = \langle b_0\rangle$.
(3) $T_1/W = \begin{cases} \langle f, b_3, W\rangle/W \cong E_4 \text{ if } T = T_1, \\ \langle f, g, W\rangle/W \cong D_8 \text{ if } g \in T_1 - T. \end{cases}$

Proof. First of all, $|T_1: T| \leqq 2$ by Lemmas (5K) and (5N). Next, using Lemmas (5E) and (5F), we have that $\mathscr{E}^*(T/B_2) = \{FB_2/B_2, \langle a, a_1, a_2\rangle B_2/B_2\}$ and that $\mathscr{E}^*(T/F) = \{B_2F/F, \langle a_2, b_2, t\rangle F/F\}$. Since T_1 permutes B_2 and F and since $B_2F \lhd T_1$ by Lemmas (5I) and (5K), it follows that T_1 permutes $\langle a, a_1, a_2\rangle B_2$ and $\langle a_2, b_2, t\rangle F$. Hence T_1 normalizes their intersection. Since $\langle a_2, b_2, t\rangle F = \langle a, a_1, a_2, f\rangle\langle b_0, b_1, b_2, t\rangle$, the intersection is equal to $\langle a, a_1, a_2\rangle\langle b_0, b_1, b_2, t\rangle = W$. Hence (1) holds.

Now $W = \langle a_1, b_2\rangle * \langle a_2, b_1\rangle * \langle a, t\rangle \cong D_8 * D_8 * D_8$ and $Z(W) = \langle b_0\rangle$. We have that $T = \langle f, b_3, W\rangle$, so $T/W \cong E_4$. Assume that $T < T_1$. Then by Lemma (5K), there is an element $g \in T_1$ such that $T_1 = \langle g\rangle T$ and $g^2 \in \langle b_0, b_1\rangle \leqq W$. Lemma (5M) shows that $f^g \in b_3W$. Thus $T_1 = \langle f, g, W\rangle$ and $T_1/W \cong D_8$. The proof is complete.

Now let bars denote images in $C(b_0)/\langle b_0\rangle$. Then T_1 acts on $\bar{W}$ by Lemma (5O). In the following two lemmas, we collect information on this action. Notice that we may choose $\bar{a}_1, \bar{b}_2, \bar{a}_2, \bar{b}_1, \bar{a}, \bar{t}$ as a basis of $\bar{W}$.

LEMMA (5P). *The following conditions hold.*
(1) $\bar{a}_1^{b_3} = \bar{a}_1\bar{b}_1,\ \bar{b}_2^{b_3} = \bar{b}_2,\ \bar{a}_2^{b_3} = \bar{b}_2\bar{a}_2,\ \bar{b}_1^{b_3} = \bar{b}_1,\ \bar{a}^{b_3} = \bar{a},\ \bar{t}^{b_3} = \bar{t}$.
(2) $\bar{a}_1^f = \bar{a}_1,\ \bar{b}_2^f = \bar{b}_2\bar{b}_1,\ \bar{a}_2^f = \bar{a}_1\bar{a}_2,\ \bar{b}_1^f = \bar{b}_1,\ \bar{a}^f = \bar{a},\ \bar{t}^f = \bar{t}$.
(3) $\bar{a}_1^{fb_3} = \bar{a}_1\bar{b}_1,\ \bar{b}_2^{fb_3} = \bar{b}_2\bar{b}_1,\ \bar{a}_2^{fb_3} = \bar{a}_1\bar{b}_2\bar{a}_2\bar{b}_1,\ \bar{b}_1^{fb_3} = \bar{b}_1,\ \bar{a}^{fb_3} = \bar{a},\ \bar{t}^{fb_3} = \bar{t}$.
(4) $C_{\bar{W}}(b_3) = \langle \bar{b}_2, \bar{b}_1, \bar{a}, \bar{t}\rangle$.

(5) $C_{\bar W}(f) = \langle \bar a_1, \bar b_1, \bar a, \bar t\rangle$.
(6) $C_{\bar W}(fb_3) = \langle \bar a_1\bar b_2, \bar b_1, \bar a, \bar t\rangle$.

Proof. (1), (2), and (3) follow from relations listed in Lemmas (2A) and (2F) together with Lemmas (5F) and (5G)(1). (4), (5), and (6) are consequences of (1), (2), and (3), respectively.

LEMMA (5Q). *If $T < T_1$, then there is an element $g \in T_1 - T$ which satisfies the following conditions.*
(1) $g^2 \in \langle A_1, at\rangle$.
(2) $\bar a_1^g = \bar b_2$, $\bar b_2^g = \bar a_1$, $\bar a_2^g = \bar a_2(\bar b_1\bar a\bar t)^\alpha$, $\bar b_1^g = \bar b_1$, $\bar a^g = \bar b_1^\alpha \bar t$, $\bar t^g = \bar b_1^\alpha \bar a$, *where* $\alpha = 0$ *or* 1.
(3) $$C_{\bar W}(g) = \begin{cases} \langle \bar a_1\bar b_2, \bar a_2, \bar b_1, \overline{at}\rangle & \text{if } \alpha = 0,\\ \langle \bar a_1\bar b_2, \bar b_1, \bar a_2^\beta \bar a^\gamma \bar t^\delta \mid \beta, \gamma, \delta \in \{0, 1\},\ \beta+\gamma+\delta=0\rangle & \text{if } \alpha=1.\end{cases}$$

Proof. Choose $\bar a_1, \bar b_2, \bar a_2, \bar b_1, \bar a, \bar t$ as a basis of $\bar W$. Lemma (5P) shows that b_3, f, and fb_3 have the following matrix forms with respect to this basis, respectively.

$$\begin{pmatrix} 1 & & & 1 & & \\ & 1 & & & & \\ & 1 & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & & & \\ & 1 & & 1 & & \\ 1 & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & 1 & & \\ & 1 & & 1 & & \\ 1 & 1 & 1 & 1 & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

Choosing a suitable element $g \in T_1 - T$, we determine the matrix of g. We choose g so that $g^2 \in \langle b_0, b_1\rangle$ by Lemma (5K)(3). From Lemmas (5L) and (5M), we get that $\langle a_1, b_0, b_1\rangle^g = \langle b_0, b_1, b_2\rangle$, $\langle b_0, b_1\rangle^g = \langle b_0, b_1\rangle$, and $\langle a, b_0, b_1\rangle^g = \langle b_0, b_1, t\rangle$. Hence g has the following matrix form.

$$\begin{pmatrix} & 1 & & \alpha & & \\ 1 & & & \beta & & \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ & & & 1 & & \\ & & & \delta & & 1 \\ & & & \varepsilon & 1 & \end{pmatrix}$$

Clearly, $\gamma_3 = 1$. Since $g^2 \in W$, the square of this matrix should be the unit matrix. Hence we have that $\alpha = \beta$, $\delta = \varepsilon$, $\gamma_1 = \gamma_2$, and $\gamma_5 = \gamma_6$, and so, changing notation, we see that g has the following matrix form.

$$\begin{pmatrix} & 1 & & \alpha & & \\ 1 & & & \alpha & & \\ \beta & \beta & 1 & \gamma & \delta & \delta \\ & & & 1 & & \\ & & & \varepsilon & & 1 \\ & & & \varepsilon & 1 & \end{pmatrix}$$

By Lemma (5M), $gb_3g \in fW$. This implies that

$$\begin{pmatrix} & 1 & & \alpha & & \\ 1 & & & \alpha & & \\ \beta & \beta & 1 & \gamma & \delta & \delta \\ & & & 1 & & \\ & & & \varepsilon & & 1 \\ & & & \varepsilon & 1 & \end{pmatrix} \begin{pmatrix} 1 & & & 1 & & \\ & 1 & & & & \\ & 1 & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} & 1 & & \alpha & & \\ 1 & & & \alpha & & \\ \beta & \beta & 1 & \gamma & \delta & \delta \\ & & & 1 & & \\ & & & \varepsilon & & 1 \\ & & & \varepsilon & 1 & \end{pmatrix}$$

is equal to the matrix of f. Hence we have that $\alpha = \beta$. Now gfb_3 has the following matrix form.

$$\begin{pmatrix} & 1 & & \alpha+1 & & \\ 1 & & & \alpha+1 & & \\ \alpha+1 & \alpha+1 & 1 & \gamma+1 & \delta & \delta \\ & & & 1 & & \\ & & & \varepsilon & & 1 \\ & & & \varepsilon & 1 & \end{pmatrix}$$

Hence, replacing g by gfb_3 if $\alpha = 1$, we may assume that $\alpha = 0$. Thus the matrix of g has the following shape.

$$\begin{pmatrix} & 1 & & & & \\ 1 & & & & & \\ & & 1 & \gamma & \delta & \delta \\ & & & 1 & & \\ & & & \varepsilon & & 1 \\ & & & \varepsilon & 1 & \end{pmatrix}$$

This in turn implies that $a_2^g \in a_2 b_1^\gamma a^\delta t^\delta \langle b_0 \rangle$ and so $1 = (a_2^g)^2 = (a_2 b_1^\gamma)^2 (a^\delta t^\delta)^2$. Hence we have that $\gamma = \delta$. Finally, $\bar{W}$ becomes a nonsingular symplectic space over $\boldsymbol{F}_2$ with respect to the bilinear form $(\bar{x}, \bar{y}) = \lambda$, where $[x, y] = b_0^\lambda$, $\lambda \in \{0, 1\}$, and the basis we have chosen is a symplectic basis. Furthermore, g induces a symplectic transformation on $\bar{W}$. This implies that the matrix of g is invariant under the transpose-inverse mapping followed by conjugation by the matrix

$$\begin{pmatrix} & 1 & & & & \\ 1 & & & & & \\ & & & 1 & & \\ & & 1 & & & \\ & & & & & 1 \\ & & & & 1 & \end{pmatrix}.$$

Hence we have that $\gamma = \varepsilon$. Thus, changing notation, we conclude that g has the following matrix form.

$$\begin{pmatrix} & 1 & & & & \\ 1 & & & & & \\ & & 1 & \alpha & \alpha & \alpha \\ & & & 1 & & \\ & & & \alpha & & 1 \\ & & & \alpha & 1 & \end{pmatrix}$$

This implies that g satisfies (2).

Now let $W_0 = \langle A_1, at\rangle$. We have chosen g so that $g^2 \in \langle b_0, b_1\rangle \leqq W_0$, and we may have replaced g by gfb_3. However, Lemma (5M) shows that $(fb_3)^g \in \langle a_1b_2, b_0, b_1, at\rangle fb_3 \leqq W_0 fb_3$ and so $(gfb_3)^2 = g^2(fb_3)^g fb_3 \in W_0$. Therefore, the property that $g^2 \in W_0$ is preserved. Thus g satisfies (1). Since (3) is a consequence of (2), we have proved the lemma.

LEMMA (5R). *W is weakly closed in T_1 with respect to G.*

Proof. Assume that T_1 contains a conjugate X of W different from W. Since $|XW: W| \leqq |T_1: W| \leqq 2^3$, $|X \cap W| \geqq 2^4$. If $Z(X) \not\leqq W$, then $(X \cap W)^2 \leqq W \cap Z(X) = 1$ and $(X \cap W)Z(X)$ is elementary abelian of order at least 2^5. However, this is impossible as X is extra-special of order 2^7. Therefore, $Z(X) \leqq W$. Then $X^2 = Z(X) \leqq W$, so XW/W is elementary abelian. Hence $|XW: W| \leqq 2^2$ by Lemma (5O), and $|X \cap W| \geqq 2^5$. Thus, $W' = (X \cap W)' = X'$ and so X centralizes $X \cap W/W'$. Since $|X \cap W/W'| \geqq 2^4$ and since no element of $T_1 - W$ centralizes a hyperplane of W/W' by Lemmas (5P) and (5Q), we have that $|X \cap W/W'| = 2^4$ and $|XW/W| = 2^2$. However, $XW = \langle f, b_3, W\rangle$ or $\langle fb_3, g, W\rangle$ by Lemma (5O) and so $|C_{W/W'}(X)| < 2^4$ by Lemmas (5P) and (5Q). Here we choose g so that $g^2 \in W$. This is a contradiction proving the lemma.

LEMMA (5S). $t \notin G'$.

Proof. Define

$$W_0 = \langle A_1, at \rangle ,$$

and

$$T_0 = \begin{cases} \langle af, b_3, W_0 \rangle \text{ if } T = T_1 , \\ \langle af, b_3, g, W_0 \rangle \text{ if } g \in T_1 - T . \end{cases}$$

We choose g as in Lemma (5Q). Lemmas (5P) and (5Q) show that f and b_3 normalize A_1 and $\langle at \rangle$, and that g normalizes W_0. Hence $W_0 \lhd T_1$. Using Lemmas (5E) and (5F), we get that $(afb_3)^2 = b_0$. Therefore, $\langle af, b_3 \rangle \cong D_8$ and $\langle af, b_3, W_0 \rangle = \langle af, b_3 \rangle W_0$ has order 2^8. By the choice of g and Lemma (5M), $(af)^g \in b_3\langle b_0, b_1, b_2 \rangle \leqq b_3 W_0$ and $b_3^g \in af\langle a_1, b_0, b_1 \rangle \leqq afW_0$. Hence g normalizes $\langle af, b_3, W_0 \rangle$ and $\langle af, b_3, g, W_0 \rangle / W_0 \cong D_8$. In particular, $|\langle af, b_3, g, W_0 \rangle| = 2^9$. Hence T_0 is a maximal subgpoup of T_1 in either case.

Assume that $t \in G'$. Then T_0 contains an extremal conjugate u of t in T_1 by Lemma (1E). We may assume that $u^x = t$ and $C_{T_1}(u)^x = C_{T_1}(t) = R$ for some $x \in G$.

Suppose $u \in W_0$. Since $u \notin Z(W) = \langle b_0 \rangle$, $|C_W(u)| = 2^6$ by Lemma (1D), and so $|C_{T_1}(u) \colon C_W(u)| = 2^2$. Hence $C_{T_1}(u)'' \leqq \langle b_0 \rangle$. Since $C_{T_1}(u)^x = R$ and since $R'' = \langle b_0 \rangle$, it follows that $x \in C(b_0)$. Now $W/\langle b_0 \rangle$ is weakly closed in $C(b_0)/\langle b_0 \rangle = \overline{C(b_0)}$ by Lemma (5R), so there exists an element $y \in N(W)$ such that $\bar{t}^y = \bar{u}$. Then $t^y = u$ or ub_0, and so $C_W(t)^y = C_W(u)$. Now $|C_{T_1}(u) \colon C_W(u)| = 2^2$, so $\overline{fb_3} \in C_{\overline{T}_1}(\bar{u})$. Hence $\bar{u} \in C_{\overline{W}_0}(\overline{fb_3}) = \langle \bar{a}_1\bar{b}_2, \bar{b}_1, \overline{at} \rangle$ by Lemma (5P). Thus $u \in \langle a_1b_2 \rangle \langle b_0, b_1 \rangle \langle at \rangle$. Also, $u \in A_1 at$ as $t^G \cap A_1 = \varnothing$. Since $u^2 = 1$, we conclude that $u \in a_1b_2at\langle b_0, b_1 \rangle$. Now $a_1b_2atb_0 = (a_1b_2at)^t$, $a_1b_2atb_1 = (a_1b_2at)^f$, and $a_1b_2atb_0b_1 = (a_1b_2at)^{ft}$. Therefore, $a_1b_2at\langle b_0, b_1 \rangle \leqq u^G \cap C_W(u)$. But now $t^G \cap C_W(t) = t^G \cap \langle A_1, t \rangle = \{t, b_0t\}$ by Lemma (5J), so $(t^G \cap C_W(t))^y = u^G \cap C_W(u)$ contains only two elements. This contradiction shows that $u \notin W_0$.

Suppose $u \in T_1 - \langle fb_3, W \rangle$. Then $\overline{C_{T_1}(u)} \leqq \overline{T}$ or $\langle \overline{fb}_3, \bar{g}, \overline{W} \rangle$, so $|C_{T_1}(u) \colon C_W(u)| \leqq 2^2$. Also, uW is conjugate to fW, b_3W, or gW in T_1, so $|C_{\overline{W}}(u)| \leqq 2^4$ by Lemmas (5P) and (5Q). But then $|C_W(u)| \leqq 2^5$ and $|C_{T_1}(u)| \leqq 2^7$, which is a contradiction. Therefore, $u \in \langle fb_3, W \rangle \cap T_0 = \langle afb_3, W_0 \rangle$ and then $u \in afb_3W_0$.

Now $(afb_3)^2 = b_0$, so $\overline{afb_3}$ is an involution which normalizes $\bar{A}_1$ and $\langle \overline{at} \rangle$. Moreover, $C_{\bar{A}_1}(\overline{afb_3}) = \langle \bar{a}_1\bar{b}_2, \bar{b}_1 \rangle$ by Lemma (5O), hence Lemma (1C) shows that $\bar{u}$ is conjugate to $\overline{afb_3}$ or $\overline{afb_3at}$ under $\bar{A}_1$. Since $u^2 = 1$, we have that u is conjugate in T_1 to an element of $afb_3at\langle b_0 \rangle$. Notice that $afb_3at\langle b_0 \rangle = fb_3t\langle b_0 \rangle$ by (5F) and (5G). So we assume that $u \in fb_3t\langle b_0 \rangle$. Then $C_{T_1}(u) = C_{T_1}(fb_3t)$. Now $C_{\overline{W}}(\overline{fb_3t}) =$

$C_{\bar{W}}(\overline{fb_3}) = \langle \bar{a}_1\bar{b}_2, \bar{b}_1, \bar{a}, \bar{t}\rangle$ by Lemma (5P), and so $C_W(fb_3t) \leqq \langle a_1b_2, b_1, a, t\rangle$. Equality does not hold here, since $(fb_3t)^{a_1b_2} = (fb_0b_1b_3t)^{b_2} = fb_1b_0b_1b_3t = fb_0b_3t$. Therefore, $|C_W(fb_3t)| \leqq 2^4$ and since $|C_{T_1}(fb_3t): C_W(fb_3t)| \leqq 2^3$, it follows that $|C_{T_1}(fb_3t)| \leqq 2^7$. This is a contradiction because $C_{T_1}(fb_3t) = C_{T_1}(u)$ has order 2^8. Therefore, $t \notin G'$.

Now we conclude the proof of Theorem (5A). Let $X = \langle L^G\rangle$ and let bars denote images in $G/O(G)$. Since $|G|_2 \leqq 2^{10}$ and $t \notin G'$, we have that $|\bar{X}|_2 \leqq 2^9$. Hence by Lemma (1H), $\bar{X}$ is a simple group and $C_{\bar{G}}(\bar{X}) = 1$. Now $N(A_2)/C(A_2) \cong \Sigma_6$ or A_6 by Lemmas (5B) and (5C). Since $O^{2'}(N) = \langle P^N\rangle \leqq N_X(A_2)$, it follows that $N_{\bar{X}}(\bar{A}_2)/C_{\bar{X}}(\bar{A}_2) \cong \Sigma_6$ or A_6. Also, since $B_2 \in \mathrm{Syl}_2(C(A_2))$ and since $t \notin X$, we get that $\bar{A}_2 \in \mathrm{Syl}_2(C_{\bar{X}}(\bar{A}_2))$. Assume that $N_{\bar{X}}(\bar{A}_2)/C_{\bar{X}}(\bar{A}_2) \cong \Sigma_6$. Then since $|\bar{X}|_2 \leqq 2^9$, [26] shows that $\bar{X}$ is isomorphic to the Higman-Sims simple group. However, the centralizer of an involution in the automorphism group of the Higman-Sims group does not have a component isomorphic to $PSU(4, 2)$ (see [2]). Hence $N_{\bar{X}}(\bar{A}_2)/C_{\bar{X}}(\bar{A}_2) \cong A_6$, and so $r(X) = 4$ by [17, Theorem 3].

6. In this section, we consider the following situation.

Hypothesis (6.1). $t^{N(B_2)} = A_2t$.

Notice that this implies Hypothesis (3.1). Hence $\langle t\rangle \in \mathrm{Syl}_2(C_G(L))$ by Lemma (3B). We prove the following theorem.

THEOREM (6A). *Under Hypothesis* (6.1), $\langle L^G\rangle \cong PSL(4, 4)$ *or* $PSU(4, 2) \times PSU(4, 2)$, *or else Case* (3) *of the main theorem occurs.*

We begin the proof by studying the structure of $N(B_2)$.

DEFINITION (6.1). Let $D_2 = O_2(N(B_2))$.

LEMMA (6B). *The following conditions hold.*

(1) $N(B_2) = N_G(B_2)D_2$ *and* $N_G(B_2) \cap D_2 = B_2$.

(2) D_2/B_2 *is elementary abelian and commutation by t induces an* $N_G(B_2)$*-isomorphism* $D_2/B_2 \to A_2$.

(3) $Z(D_2) = D_2^2 = A_2$.

Proof. By Hypothesis (6.1), $|N(B_2): N_G(B_2)| = 16$. As $N_G(B_2)/C(B_2) \cong A_5$ or Σ_5, we have that $|N(B_2)/C(B_2)| = 2^6\cdot 3\cdot 5$ or $2^7\cdot 3\cdot 5$. Then a theorem of [4] shows that $N(B_2)/C(B_2)$ is not simple; so let $C(B_2) < X \lhd N(B_2)$, $X \neq N(B_2)$. Recall from Lemma (3G) that $N(B_2)/C(B_2)$ is a primitive permutation group on $\Omega = A_2t$. Hence we have

$N(B_2) = N_C(B_2)X$. Furthermore, either $N_C(B_2) \cap X/C(B_2) \cong A_5$ or 1. Assume the former. Then $N_C(B_2)/C(B_2) \cong \Sigma_5$ as $X \neq N(B_2)$, and so $|N(B_2)/C(B_2)|_2 = 2^7$. Hence $N(B_2)/C(B_2)$ can not be embedded in $GL(4, 2)$. Thus Lemma (3E) forces $C(B_2) < C(A_2) \cap N(B_2) \lhd N(B_2)$, and so $C(A_2) \cap N(B_2)/C(B_2)$ is a nontrivial normal 2-subgroup of $N(B_2)/C(B_2)$ by Lemma (3F). Therefore, we can always choose X so that $N_C(B_2) \cap X = C(B_2)$. Let us fix such X, and let bars denote images in $N(B_2)/C(B_2)$. Then $\bar{X}^{\Omega}$ is the regular normal subgroup of $\overline{N(B_2)}^{\Omega}$ and so $\bar{X}$ is a self-centralizing elementary abelian subgroup of order 16. Let $Y = C(O(C)) \cap N(B_2)$. Then as $C(B_2) = B_2 \times O(C)$, $O(C) \lhd N(B_2)$ and $\bar{Y} \lhd \overline{N(B_2)}$. Moreover, $\bar{Y} \neq 1$ as $\bar{K}_2 \leqq \bar{Y}$. Hence we have $\bar{X} \cap \bar{Y} \neq 1$, and so $\bar{X} \leqq \bar{Y}$. This implies that $X = C_X(O(C))O(C)$. Thus X is 2-closed and, as $O_2(N_C(B_2)) = B_2$, the statement (1) follows.

Now $A_2 \lhd D_2$ by Lemma (3E), so $A_2 \cap Z(D_2) \neq 1$. As K_2 acts irreducibly on A_2, it follows that $A_2 \leqq Z(D_2)$. Also, $Z(D_2) \leqq C_{D_2}(t) = B_2$. Therefore, $Z(D_2) = A_2$. Consequently, (2) holds. Moreover, $A_2 \cap D_2^2 \neq 1$ and so $A_2 \leqq D_2^2 \leqq B_2$. Suppose that $D_2^2 = B_2$. Then D_2/A_2 has a cyclic subgroup X/A_2 of order 4. As $A_2 = Z(D_2)$, X is abelian. But this contradicts $C_{D_2}(t) = B_2$. Therefore, $D_2^2 = A_2$.

DEFINITION (6.2). Let $Q_2 = QD_2$, $Q_1 = N_{Q_2}(Q)$, and $F = N_{Q_2}(Q_1)$. Let $V = \langle Z, t\rangle$, $D_1 = O_2(N(B_1))$, and $D_0 = C_{D_1}(A_1)$.

REMARK. We have $Q_1/B_2 = Q/B_2 \times N_{D_2/B_2}(Q/B_2)$ and the $N_C(B_2)$-isomorphism $D_2/B_2 \to A_2$ maps $N_{D_2/B_2}(Q/B_2)$ onto $C_{A_2}(Q) = Z(P)$. Hence $|N_{D_2/B_2}(Q/B_2)| = 2$ and $|Q_1/Q| = 2$. Also, F is the product of Q and the group of elements x of D_2 such that $[Q, x] \leqq N_{D_2}(Q)$. Commutation by t maps the latter group onto the group of elements $y \in A_2$ such that $[Q, y] \leqq Z(P)$, which is equal to $A_1 \cap A_2$. Thus we have $|F/B_2| = 32$.

LEMMA (6C). *The following conditions hold.*

(1) $N(B_1) \leqq N(A_1)$.

(2) $N(B_1) = N(V)$.

(3) $N(B_1)/B_1 = N_C(B_1)/B_1 \times D_1/B_1$.

(4) $QD_1 = Q_1$.

(5) $D_1 = B_1D_0$ *and* $B_1 \cap D_0 = V$.

(6) $D_0 \cong D_8$.

(7) $D_0 \leqq D_2$.

(8) $[N_L(A_1), D_0] = 1$.

Proof. Every involution of A_1t is conjugate to an element of

A_2t under L, and so it is conjugate to t by Hypothesis (6.1). As $t^G \cap A_1 = \varnothing$ by Lemma (3C) and as $A_1 = \Omega_1(A_1)$, it follows that $A_1 = \langle ab \mid a, b \in t^G \cap B_1 \rangle$. Hence (1) follows.

Now $|Q_1 \cap D_2 : B_2| = 2$ by Lemma (6B) and so $Q_1 \cap D_2 = B_2(Q_1 \cap D_2 \cap C(HO(C)))$. Let $x \in Q_1 \cap D_2 \cap C(HO(C)) - B_2$. Then $x \in N(B_1)$ by Lemma (3J). In particular, $N_C(B_1) < N(B_1)$. Now, $N(B_1) \leqq N(V)$ as $Z(B_1) = V$, and $N_C(B_1) = N_C(V)$ as $O_2(N_L(V)) = A_1$. Moreover, $|N(V): N_C(V)| \leqq 2$ as $t^{N(V)} \leqq \{t, b_0t\}$. Hence $N(B_1) = N(V) = \langle N_C(B_1), x \rangle$. In particular, (2) holds.

Now $B_1C(B_1) = B_1 \times O(C)$ by Lemma (2G). Hence $O(C) \lhd N(B_1)$ and $X = C_{N(B_1)}(O(C))O(C)$ is a normal subgroup of $N(B_1)$ containing $B_1O(C)$. Let bars denote images in $N(B_1)/B_1O(C)$. Then $\bar{H} \lhd \overline{N_C(B_1)}$ by the structure of $N_C(B_1)$, and as $\overline{N(B_1)} = \langle \overline{N_C(B_1)}, \bar{x} \rangle$, it follows that $\bar{H} \lhd \overline{N(B_1)}$. Hence $\bar{Y} = C_{\bar{X}}(\bar{H})$ is a normal subgroup of $\overline{N(B_1)}$. Now, $\bar{x} \in \bar{Y}$ by the choice of x, and so $\bar{Y} = \langle \bar{Y} \cap \overline{N_C(B_1)}, \bar{x} \rangle$. As $\overline{N_L(A_1)} = \bar{K}_1 \times \bar{H} \leqq \bar{Y} \cap \overline{N_C(B_1)} \leqq C(\bar{H}) \cap \overline{N_C(B_1)} = \overline{N_L(A_1)}$, it follows that $\bar{Y} = (\bar{K}_1 \times \bar{H})\langle \bar{x} \rangle$. Now $\bar{K}_1 \cong \Sigma_3$. Hence $\bar{K}_1 = O^3(\bar{K}_1 \times \bar{H}) \lhd \bar{Y}$, and so, as $\mathrm{Aut}(\Sigma_3) \cong \Sigma_3$, it follows that $\bar{Y} = \bar{K}_1 \times \bar{H} \times \bar{K}$ for some subgroup $\bar{K}$ of order 2. Clearly, $\bar{K} = O_2(\bar{Y}) \lhd \overline{N(B_1)}$. Now let K denote the preimage of $\bar{K}$ in $N(B_1)$. Then as $O(C) \leqq K \leqq X$, $K = C_K(O(C))O(C)$ and thus K is 2-closed. As $O_2(N_C(B_1)) = B_1$ by Lemma (2G), (3) holds.

As a consequence of (3) we have $D_1 \leqq N(Q)$, so $D_1 \leqq N(B_2)$ by Lemma (3J). Hence D_1 normalizes $Q_2 = QD_2$. Also, $B_1 \cap B_2 < B_1 < D_1$ is a series of H-invariant normal subgroups of D_1. As H acts irreducibly on $B_1/B_1 \cap B_2$ by Lemma (2B), it follows that D_1 centralizes $B_1/B_1 \cap B_2$. Noticing that $B_1/B_1 \cap B_2 \cong Q_2/D_2$, we conclude that D_1 centralizes Q_2/D_2. However, $N(B_2)/D_2O(C) \cong A_5$ or Σ_5 by Lemma (6B) and, in particular, an S_2-subgroup of $N(B_2)/D_2$ is either E_4 or D_8. Thus we have $D_1 \leqq Q_2$, and as $D_1 \leqq N(Q)$ and $|Q_1 : Q| = 2$, (4) follows.

To prove the remaining assertions, set $D = C_{D_1}(H)$. Then as H centralizes D_1/B_1 and as $C_{B_1}(H) = V$, we have $D_1 = B_1D$ and $B_1 \cap D = V$. Consequently, $|D| = 8$ and as $C_D(t) = C_{B_1}(H) = V$, we see that $D \cong D_8$. Now $D \leqq Q_2$ by (4) and H acts regularly on Q_2/D_2 as $Q_2/D_2 \cong Q/B_2$ as H-modules. Therefore, $D \leqq D_2$, and then $D \leqq D_2^{s_1}$ as $s_1 \in N(D)$ by the definition of D. Thus by Lemma (6B), D centralizes $\langle A_2, A_2^{s_1}, H \rangle = N_L(A_1)$. In particular, $[A_1, D] = 1$ and hence it follows that $D = D_0$. Thus all parts of the lemma hold.

LEMMA (6D). *D_2 has a maximal subgroup E_2 which is either elementary abelian or homocyclic of exponent* 4 *and is inverted by t.*

Proof. Let $\Gamma = \{c_1, c_2, c_3, c_4, c_5\}$. We may choose elements $d_i \in$

D_2, $i \in \{1, 2, 3, 4, 5\}$, such that $[d_i, t] = c_i$ by Lemma (6B)(2). Let $\bar{D}_2 = D_2/B_2$ and $\Delta = \{\bar{d}_1, \bar{d}_2, \bar{d}_3, \bar{d}_4, \bar{d}_5\}$. Now Γ is the set of central involutions of L contained in A_2, so $N_C(B_2)$ acts transitively on Γ. Hence $N(B_2)$ acts transitively on Δ by Lemma (6B). We may choose each d_i to be an involution. Indeed, we can choose $d_1 \in I(D_0)$ by Lemma (6C), and then choose conjugates d_2, d_3, d_4, d_5 of d_1 under $N_C(B_2)$. Then $\langle d_i, A_2\rangle$ is elementary abelian since $A_2 = Z(D_2)$, and moreover, $C_{\langle d_i, A_2\rangle}(t) = A_2$. Hence $\mathscr{E}^*(\langle d_i, B_2\rangle) = \{\langle d_i, A_2\rangle, B_2\}$, and so if $\tilde{D}_2 = D_2/A_2$, then $\{\tilde{d}_1, \tilde{d}_2, \cdots, \tilde{d}_5\}$ is $N(B_2)$-invariant. Now $c_1c_2\cdots c_5 = 1$, so $\bar{d}_1\bar{d}_2\cdots\bar{d}_5 = 1$. Thus there are two cases: $\tilde{d}_1\tilde{d}_2\cdots\tilde{d}_5 = 1$ or $\tilde{t}$. As $A_2 = \langle c_1, c_2, \cdots, c_5\rangle$, $\bar{D}_2 = \langle \bar{d}_1, \bar{d}_2, \cdots, \bar{d}_5\rangle$ and so $\tilde{D}_2 = \langle \tilde{d}_1, \tilde{d}_2, \cdots, \tilde{d}_5, \tilde{t}\rangle$. Hence if $\tilde{d}_1\tilde{d}_2\cdots\tilde{d}_5 = 1$, then we may choose $\tilde{d}_1\tilde{t}, \tilde{d}_2\tilde{t}, \cdots, \tilde{d}_5\tilde{t}$ as a basis of $\tilde{D}_2$. If $\tilde{d}_1\tilde{d}_2\cdots\tilde{d}_5 = \tilde{t}$, then we may choose $\tilde{d}_1, \tilde{d}_2, \cdots, \tilde{d}_5$ as a basis of $\tilde{D}_2$. In either case, the basis of $\tilde{D}_2$ we have chosen is $N(B_2)$-invariant. Hence if we define $\tilde{E}_2$ to be the subgroup of $\tilde{D}_2$ generated by the elements that are the products of even number of the basis elements, then $\tilde{E}_2$ is an $N(B_2)$-invariant maximal subgroup of $\tilde{D}_2$ and $\tilde{B}_2 \cap \tilde{E}_2 = 1$.

Let E_2 be the preimage of $\tilde{E}_2$ in D_2. Then $E_2/A_2 \cong A_2$ as K_2-modules by Lemma (6B)(2), so E_2 is abelian by Theorem 1 of [24].

If $\tilde{d}_1\tilde{d}_2\cdots\tilde{d}_5 = 1$, then $\tilde{d}_1 = (\tilde{d}_2\tilde{t})(\tilde{d}_3\tilde{t})(\tilde{d}_4\tilde{t})(\tilde{d}_5\tilde{t}) \in \tilde{E}_2$ by the definition of $\tilde{E}_2$, and so E_2 is generated by involutions. If $\tilde{d}_1\tilde{d}_2\cdots\tilde{d}_5 = \tilde{t}$, then $\tilde{d}_1\tilde{t} = \tilde{d}_2\tilde{d}_3\tilde{d}_4\tilde{d}_5 \in \tilde{E}_2$. As $(d_1t)^2 = [d_1, t] = c_1$, E_2 has a basis consisting of elements of order 4 inverted by t. The proof is complete.

DEFINITION (6.3). Let $W = D_0 \cap E_2$.

Since $D_2 = E_2\langle t\rangle$ and $t \in D_0 \leqq D_2$, we have $D_0 = W\langle t\rangle$ and $W \cong Z_4$ or E_4. Also, $WA_2 = Q_1 \cap E_2$. Indeed, $A_2W \leqq Q_1 \cap E_2$ by definition, $|Q_1 \cap E_2 : A_2| = 2$ by a remark following Definition (6.2), and $W \not\leqq A_2$ as $W\langle t\rangle = D_0 \not\leqq B_2 = A_2\langle t\rangle$ by Lemma (6C).

LEMMA (6E). *The following conditions hold.*

(1) $N(B_1) \leqq N(D_0) \leqq N(D_1) \leqq N(A_1W) \leqq N(W)$.

(2) $Q_2 \cap N(D_1) = F$.

(3) *If* $N(B_1) = N(D_0)$, *let* $D = O_2(N(D_1))$. *Then* $N(D_1) = N(B_1)D$, $N(B_1) \cap D = D_1$, D/D_1 *is elementary abelian, and* $D/D_1 \cong A_1/Z$ *as* $N(B_1)$*-modules.*

(4) *If* $N(B_1) < N(D_0)$, *then the following hold.*

(4.1) $C(D_1/W) = D_1O(C)$.

(4.2) $N(D_1)/D_1O(C) \cong \Sigma_6$.

(4.3) $N(D_0)/D_1O(C) \cong \Sigma_3$ *wreath* Z_2.

(4.4) $W \cong Z_4$.

(4 L.5) $C \neq C_C(L)$.

Proof. By definition, $D_0 = C_{D_1}(A_1) \lhd N(A_1) \cap N(D_1)$. As $N(B_1) \leqq N(A_1) \cap N(D_1)$ by Lemma (6C), $N(B_1) \leqq N(D_0)$. Recall also from Lemma (6C) that $N(B_1) = N(V)$ and that $D_0 \cong D_8$. These show

(a) $$|N(D_0): N(B_1)| \leqq 2 ,$$

as V is one of the two E_4-subgroups of D_0. In particular, $N(B_1) \lhd N(D_0)$ and so, as $D_1 = O_2(N(B_1))$, we have $N(D_0) \leqq N(D_1)$. As $A_1 = C_{D_1}(D_0)$, we also have that

(b) $$N(D_0) \leqq N(A_1) .$$

We argue that $N(D_0) \leqq N(W)$ and $V \not\sim W$. If $W \cong Z_4$, this is obvious. If $W \cong E_4$, then $E_2 \cong E_{256}$ by Lemma (6D) and so $t^G \cap W = \varnothing$ as $m(C) = 5$. Thus $V \not\sim W$ and consequently $N(D_0) \leqq N(W)$. Furthermore, if $N(B_1) < N(D_0)$, then $W \cong Z_4$ as otherwise $V \sim W$ in $N(D_0)$, a contradiction. As $C_{D_1}(W) = A_1W$, it follows that $N(D_1) \cap N(W) \leqq N(A_1W)$. Finally, $N(A_1W) \leqq N(W)$ as $Z(A_1W) = W$. Thus we have proved the following.

(c) $$N(B_1) \leqq N(D_0) \leqq N(D_1) \cap N(W) \leqq N(A_1W) \leqq N(W) .$$

Let $X = N(D_1) \cap N(W)$ and $a = |X: N(D_0)|$. We shall determine the value of a and prove that $X = N(D_1)$. The statement (1) will, then, follow from (c). First, we shall obtain two expressions for $|X: N(Q)|$. It follows from the structure of $N_C(B_1)$, Lemma (3J), and Lemma (6C) that $|N(B_1): N(Q)| = 3$. Hence

(d) $$|X: N(Q)| = 3|N(D_0): N(B_1)|a .$$

Now $Q_1 = QD_1 = QD_0 = P*D_0$ by Lemma (6C), so $Z = Z(Q_1)$ and $\mathscr{E}^*(Q_1/Z) = \{A_1D_0/Z, A_2D_0/Z\}$. Thus $N(Q_1)$ normalizes $A_1D_0 = D_1$ and, in particular, $F \leqq N(D_1)$. Also, $F \leqq N(W)$ as $Q_2 = B_1E_2$ normalizes W. Therefore, $F \leqq X$. More precisely, we have that $F = Q_2 \cap X$ as $Q_2 \cap N(D_1)$ normalizes $Q_1 = D_1B_2$. The statement (2) will follow from this once we prove $X = N(D_1)$. By Lemma (3J) and the definition of D_i, $i \in \{1, 2\}$, $N(Q) \leqq N(B_i) \leqq N(D_i)$. Hence $N(Q) \leqq N(Q_i)$ and then $N(Q) \leqq N(F)$. Furthermore,

(e) $$N(D_0) \cap F = Q_1$$

as $N(D_0) \cap F$ normalizes $Q = A_1B_2$ by (b). In particular, $N(Q) \cap F = Q_1$. Thus setting $b = |X: N(Q)F|$, we have another expression:

(f) $$|X: N(Q)| = 4b .$$

Now let bars denote images in X/W. Then, as $\langle \bar{t} \rangle = \bar{D}_0$, $C(\bar{t}) = \overline{N(D_0)}$ and

$$|\bar{t}^{\bar{X}}| = |X: N(D_0)| = a .$$

Also, as $\bar{D}_1 = \langle \bar{t} \rangle \times \bar{A}_1$ and $\bar{A}_1 \lhd \bar{X}$,

$$|\bar{t}^{\bar{X}}| = 1 + |\bar{t}^{\bar{X}} \cap \bar{t}\bar{A}_1^\#| .$$

To determine the second term, consider the action of $C(\bar{t}) = \overline{N(D_0)}$ on $\bar{A}_1^\# = (A_1W/W)^\#$. By (b), $A_1W/W \cong A_1/Z$ as $N(D_0)$-modules. We know that under the action of $N_L(A_1)$, which is contained in $N(D_0)$, $(A_1/Z)^\#$ decomposes into two orbits of lengths 9 and 6, one corresponding to the involutions of $A_1 - Z$ and the other corresponding to the elements of order 4 of A_1 (see Lemma (2C)). Therefore, under the action of $C(\bar{t})$, $\bar{A}_1^\#$ decomposes into two orbits of lengths 9 and 6. Thus

$$|\bar{t}^{\bar{X}} \cap \bar{t}\bar{A}_1^\#| = 0, 6, 9 \text{ or } 15 ,$$

and hence

(g) $$a = 1, 7, 10 \text{ or } 16 .$$

Now recall that $t^G \cap A_1 = \varnothing$. This yields that $t^{N(D_1)} \subseteq I(D_1 - A_1)$, so

$$|t^{N(D_1)}| \leqq 52$$

as $D_1 \cong D_8 * D_8 * D_8$ and $A_1 \cong D_8 * D_8$. On the other hand,

$$|t^{N(D_1)}| = |N(D_1): X|\,|X: N_C(B_1)|$$

as $N(D_1) \cap C = N_C(B_1)$, so

$$|t^{N(D_1)}| = \begin{cases} 2\,|N(D_1): X|\,a \text{ if } N(B_1) = N(D_0) , \\ 4\,|N(D_1): X|\,a \text{ if } N(B_1) < N(D_0) . \end{cases}$$

Therefore,

(h) $$|N(D_1): X|\,a \leqq \begin{cases} 26 \text{ if } N(B_1) = N(D_0) , \\ 13 \text{ if } N(B_1) < N(D_0) . \end{cases}$$

Now assume that $N(B_1) = N(D_0)$. Then $3a = 4b$ by (d) and (f). Thus $a = 16$ by (g), and then $N(D_1) = X$ by (h). Assume next that $N(B_1) < N(D_0)$. Then $3a = 2b$ by (a), (d), and (f). Also, $a \leqq 13$ by (h). Therefore, $a = 10$ by (g) and then $N(D_1) = X$ by (h). Thus $a = 10$ or 16 and $N(D_1) = X$ in either case. Statements (1) and (2) follow from this as remarked before.

Now $\langle \bar{t}^{\bar{X}} \rangle = \bar{D}_1$ in either case and so $\tilde{X} = \bar{X}/C(\bar{D}_1)$ is a permutation group on $\Omega = \bar{t}^{\bar{X}}$. Furthermore, $\tilde{X}^\Omega$ is primitive in either case. We shall determine the structure of $\tilde{X}^\Omega$. By Lemma (6C), $D_1 \leqq$

$C(D_1/W)$. Also, $N(B_1) = D_1N_C(B_1)$, and $C_C(B_1/Z) = B_1O(C)$ by Lemma (2G). Hence

$$\begin{aligned} C(D_1/W) \cap N(B_1) &= D_1(C(D_1/W) \cap N_C(B_1)) \\ &= D_1(C(B_1/Z) \cap N_C(B_1)) \\ &= D_1(B_1O(C)) \\ &= D_1O(C) . \end{aligned}$$

Notice that $[D_1, O(C)] = 1$ as $O(C)$ stabilizes the series $1 \leqq B_1 \leqq D_1$.

Assume that $N(B_1) = N(D_0)$. Then $|\Omega| = 16$ and $C_{\bar{X}}(\bar{t}) = \overline{N(D_0)} = \overline{N(B_1)}$, and consequently, $C(\bar{D}_1) = \overline{D_1O(C)}$ by the above. Thus $|\tilde{X}: C_{\tilde{X}}(\tilde{t})| = 16$ and $C_{\tilde{X}}(\tilde{t}) \cong N_C(B_1)/B_1O(C) \cong \Sigma_3 \times Z_3$ or $\Sigma_3 \times \Sigma_3$ by Lemma (2C) and Lemma (2G). This shows that $\tilde{X}$ is a $\{2, 3\}$-group that has no nonidentity normal 3-subgroup. Then by Burnside's theorem [12, Theorem 4.3.3], $O_2(\tilde{X}) \neq 1$ and so $\tilde{X}$ has a regular normal subgroup $\tilde{Y}$. As $1 \neq \tilde{K}_1 \leqq \widetilde{C_X(O(C))} \lhd \tilde{X}$ and $\tilde{Y}$ is a self-centralizing minimal normal subgroup of $\tilde{X}$, it follows that $\tilde{Y} \leqq \widetilde{C_X(O(C))}$. This implies that the preimage Y of $\tilde{Y}$ in X is written as $Y = C_Y(O(C))O(C)$. Hence Y is 2-closed and if $D \in \mathrm{Syl}_2(Y)$, then $D = O_2(N(D_1))$, $N(D_1) = N(B_1)D$, $N(B_1) \cap D = D_1$, and D/D_1 is elementary. Furthermore, the irreducible action of $\overline{N(B_1)}$ on $\bar{A}_1$ yields that $\bar{A}_1 = Z(\bar{D})$ and so commutation by $\bar{t}$ induces an $N(B_1)$-isomorphism $\bar{D}/\bar{D}_1 \to \bar{A}_1$. Thus (3) holds.

Assume, therefore, that $N(B_1) < N(D_0)$ Recall that $W \cong Z_4$ in this case. The $\tilde{X}^{\Omega}$ is a 2-transitive group of degree 10, and the point-stabilizer $C_{\tilde{X}}(\tilde{t}) = \widetilde{N(D_0)}$ has a normal subgroup $O_3(\widetilde{N(B_1)}) = \widetilde{O_3(K_1)}\tilde{H}$ which is isomorphic to $Z_3 \times Z_3$ and is regular on $\Omega - \{\tilde{t}\}$ (see Lemma (2C)). A theorem of [18] now shows that

$$PSL(2, 9) \hookrightarrow \tilde{X} \hookrightarrow P\Gamma L(2, 9) .$$

Now $|X: N(D_0)| = 10$, $|N(D_0): N(B_1)| = 2$, and $N(B_1)/D_1O(C) \cong \Sigma_3 \times Z_3$ or $\Sigma_3 \times \Sigma_3$. Furthermore, $C(D_1/W) \cap N(B_1) = D_1O(C)$ as remarked before. Therefore, $|\tilde{X}|_2 \leqq 16$ and equality holds only when $C(D_1/W) = D_1O(C)$ and $N(B_1)/D_1O(C) \cong \Sigma_3 \times \Sigma_3$. We argue that F/D_1 is elementary. Indeed, $F/D_1 \cong F \cap E_2/D_1 \cap E_2$. By Lemmas (6B) and (6D), the mapping which associates with each element of E_2 its square induces an $N_C(B_2)$-isomorphism $E_2/A_2 \to A_2$, and it maps $F \cap E_2$ onto $A_1 \cap A_2$ by the definition of F. Thus $(F \cap E_2)^2 = A_1 \cap A_2$ and consequently, F/D_1 is elementary. This implies that $m(X) \geqq 3$ as $F \cap C(D_1/W) = F \cap N(D_0) \cap C(D_1/W) = Q_1 \cap C(D_1/W) = D_1$ by (e). Thus $\tilde{X} = \Sigma_6$ is the only possibility. In particular, $|\tilde{X}|_2 = 16$ and hence $C(D_1/W) = D_1O(C)$ and $N(B_1)/D_1O(C) \cong \Sigma_3 \times \Sigma_3$. This occurs only if $C \neq LC_C(L)$ (see Lemmas (2C) and (2G)). Furthermore, $N(D_0)/D_1O(C) =$

$C_{\bar{X}}(\bar{t}) \cong \Sigma_3$ wreath Z_2 by the structure of Σ_6. Thus all parts of the lemma hold.

LEMMA (6F). *If* $N(B_1) < N(D_0)$, *then Case* (3) *of the main theorem occurs.*

Proof. We shall apply Lemma (1R) with $C(W)$, W, A_1W/W, and t in place of $\hat{G}$, $\hat{Z}$, A, and t, respectively. Recall from Lemma (6E) that

$$N(D_1) \leqq N(A_1W) \leqq N(W) .$$

$N(D_1) \cap C(A_1W/W)/C(D_1/W)$ is a normal 2-subgroup of $N(D_1)/C(D_1/W)$ and so by Lemma (6E),

$$\text{(a)} \qquad N(D_1) \cap C(A_1W/W) = D_1O(C) .$$

As a consequence, we have that

$$\text{(b)} \qquad D_1 \in \mathrm{Syl}_2(C(A_1W/W)) .$$

Moreover,

$$\text{(c)} \qquad N(A_1W) = N(D_1)C(A_1W/W)$$

by a Frattini argument, and hence

$$\text{(d)} \qquad N(A_1W)/C(A_1W/W) \cong \Sigma_6$$

by (a) and Lemma (6E). Now $C \neq LC_C(L)$ by Lemma (6E)(4.5), so there is an element $f \in N_C(Q) - Q$ such that $f^2 \in Q$. Then $f \in N(B_1) \cap N(B_2)$ by Lemma (3J) and so f normalizes $Q_2 = D_1D_2$ and $Q_2\langle f \rangle$ has order 2^{12}. Also, $f \in N(D_1) \leqq N(W)$ and $Q_2 = D_1E_2 \leqq N(W)$. Thus $Q_2\langle f \rangle \leqq N(W)$. Furthermore,

$$N(A_1W) \cap Q_2\langle f \rangle = (N(A_1W) \cap Q_2)\langle f \rangle = F\langle f \rangle$$

as $N(A_1W) \cap Q_2$ normalizes $A_1WB_2 = Q_1$. Now $|F\langle f \rangle| = 2^{11}$. Thus, $F\langle f \rangle \in \mathrm{Syl}_2(N(A_1W))$ by (b) and (d), and hence

$$\text{(e)} \qquad |N(W) : N(A_1W)| \text{ is even} .$$

Now $W \cong Z_4$ by Lemma (6E) and $t \notin C(W)$, so

$$N(W) = C(W)\langle t \rangle .$$

It is now clear that (d), (b), and (e) imply the conditions (1), (2), and (3) of Lemma (1R), respectively.

Now notice that $\langle t, W \rangle = D_0$, and recall from Lemma (6E) that

$$N(D_0) \leqq N(D_1) \text{ and } N(D_0)/D_1O(C) \cong \Sigma_3 \text{ wreath } Z_2 .$$

Thus

$$(f) \qquad A_1 W \leqq N(D_0) \leqq N(A_1 W)\ ,$$

and using (a), we have

$$(\mathrm{g}) \qquad N(D_0)C(A_1 W/W)/C(A_1 W/W) \cong \Sigma_3 \text{ wreath } Z_2\ .$$

Noticing that $\langle t, A_1 W\rangle = D_1$, we can now derive conditions (5), (6), and (7) of Lemma (1R) from (f), (g), and (c), respectively. We know that conditions (4) and (8) are satisfied. Thus by Lemma (1R), $C(W)$ has a quasisimple characteristic subgroup K containing W such that

$$(\mathrm{h}) \qquad C(K) = WO(C(W))$$

and either $K/O(K) \cong SU(4, 3)$ or $K/Z(K)$ has an S_2-subgroup of type $PSL(6, q)$, $q \equiv 3 \bmod 4$. Now $N(W) \leqq C(Z)$, $K \lhd N(W)$, and $W/Z \in \mathrm{Syl}_2(C(K/Z))$ by (h). Thus K/Z is a standard subgroup of $C(Z)/Z$. The fours group D_0/Z acts on $X = O(C(Z))$. Let $x \in N(D_0) - N(B_1)$. Then $V^x \neq V$ as $N(V) = N(B_1)$ and so $X = \langle N_X(V), N_X(V^x), N_X(W)\rangle \leqq O(N(W))$. Hence $[K, X] = 1$. We have proved that Case (3) of the main theorem occurs.

In view of Lemma (6F), we assume from now on that G satisfies the following.

Hypothesis (6.2). $N(B_1) = N(D_0)$.

Furthermore, we make the following definition.

DEFINITION (6.4). Let $D = O_2(N(D_1))$ and $R_1 = Q_1 D$.

Then by Lemma (6E)(3), $N(D_1) = N(B_1)D$, $N(B_1) \cap D = D_1$, D/D_1 is elementary, and $D/D_1 \cong A_1/Z$ as $N(B_1)$-modules.

LEMMA (6G). *The following conditions hold.*
(1) $R_1 \cap Q_2 = F$.
(2) $R_1 \leqq N(Q_2)$.
(3) E_2 *is elementary abelian.*
(4) $N(D_2) = N(B_2) \leqq N(E_2)$.
(5) $N(Q_2) \leqq N(E_2)$.

Proof. By Lemma (6E)(2), $N(D_1) \cap Q_2 = F$. Hence (1) will follow once we show $F \leqq R_1$. To see this, notice first that $|N(D_1)/D|_2 \leqq 4$ by Lemmas (6C)(3) and (6E)(3). Next, $F \leqq N(R_1)$ as $F \leqq N(Q_1) \cap N(D_1)$. Hence $Q_1 < R_1 \cap F \leqq F$. As H acts irreducibly on F/Q_1 by

Lemma (6B) and $H \leqq N(R_1 \cap F)$, we have that $F = R_1 \cap F$, proving (1).

Now Lemma (6E)(3) in particular implies that $|N_{R_1}(Q_1)/D_1| = 8$, so $F = N_{R_1}(Q_1)$ and consequently, $F \lhd R_1$ by Lemma (1C).

We show that $F \cap E_2$ is the only A_{128}-subgroup of F. Suppose X is an A_{128}-subgroup of F. If $X \leqq F \cap D_2$, then as $F \cap E_2$ is an abelian maximal subgroup of $F \cap D_2$ and as $Z(F \cap D_2) \leqq B_2$, it follows that $X = F \cap E_2$. Assume, therefore, that $X \not\leqq F \cap D_2$. Then $F \neq X(F \cap E_2)$. For otherwise, $Y = X \cap F \cap E_2$ has order 16 and $Y \leqq Z(F)$. However, $Z(F) \leqq Z(C_F(t)) = Z(Q) = V$, a contradiction. Thus $|Y| \geqq 32$ and so if $x \in X - D_2$, then $|C_{E_2}(x)| \geqq 32$. However, on the other hand, Lemma (6B) shows that $|C_{E_2/A_2}(x)| = 4 = |C_{A_2}(x)|$ if $x \in Q_2 - D_2$. This contradiction shows that $F \cap E_2$ is the only A_{128}-subgroup of F.

A similar argument shows that E_2 is the only A_{256}-subgroup of Q_2. Therefore, $N(F) \leqq N(F \cap E_2)$ and $N(Q_2) \leqq N(E_2)$.

Now $R_1 \leqq N(F) \leqq N(F \cap E_2)$. However, $R_1 \not\leqq N(A_2)$ as $N_{R_1}(A_2) \leqq N_{R_1}(A_2D_1) = N_{R_1}(Q_1) = F$. These and Lemma (6D) imply that $F \cap E_2$ is elementary abelian, and hence (3) follows. The statement (4) now follows from Lemma (1C). By the same lemma, $C(F/F \cap E_2) \leqq N(F \cap D_2) \leqq N(B_2)$. Also, $Q_2 \leqq C(F/F \cap E_2)$ as $Q_2/F \cap E_2 = F/F \cap F_2 \times E_2/F \cap E_2$ and $F/F \cap E_2 \cong Q/A_2$. Therefore, Q_2 is the only S_2-subgroup of $C(F/F \cap E_2)$ by the structure of $N(B_2)/B_2$ discussed in Lemma (6B). Thus $Q_2 \lhd N(F)$ as $C(F/F \cap E_2) \lhd N(F)$. In particular, $R_1 \leqq N(Q_2)$. The proof is complete.

DEFINITION (6.5). Let $T = R_1Q_2$, $S = C_T(W)$, and $E_1 = C_D(W)$.

Because of Lemma (6G)(2), T is a subgroup.

LEMMA (6H). *The following conditions hold.*

(1) $T \leqq N(E_2)$.

(2) $T = S\langle t\rangle$.

(3) $D = E_1\langle t\rangle$.

(4) $W^{s_2} = (E_2 \cap E_2^{s_1})^{s_2} = ((E_1 \cap E_2) \cap (E_1 \cap E_2)^{s_1})^{s_2}$ *is a complement for* E_1 *in* S.

(5) $((E_1 \cap E_2) \cap (E_1 \cap E_2)^{s_2})^{s_1}$ *is a complement for* E_2 *in* S.

(6) E_1/W *is elementary abelian.*

(7) $N(Q) \leqq N(S)$.

Proof. The assertion (1) follows from Lemma (6G)(5). By Lemma (6E)(1), $R_1 \leqq N(D_1) \leqq N(W)$. Also, $Q_2 = D_1E_2$ normalizes W. Therefore, $T \leqq N(W)$ and hence (2) and (3) follow.

Let $X = E_2 \cap E_2^{s_1}$. Then as $B_2 \cap B_2^{s_1} = V$ and $N(V) = N(B_1)$ by

Lemma (6C)(2), we have that $X \leqq N_{Q_2}(B_1) = Q_1$. Thus $X \leqq Q_1 \cap Q_1^{s_1} = D_1$. By Lemma (6C), $D_1 \cap D_2 = (B_1 \cap D_2)D_0 = (B_1 \cap B_2)D_0$ and then $X \leqq (B_1 \cap B_2)D_0 \cap ((B_1 \cap B_2)D_0)^{s_1} = D_0$. Thus $X \leqq D_0 \cap E_2 = W$. As $W = W^{s_1} \leqq X$ by Lemma (6C)(8), we conclude that $W = E_2 \cap E_2^{s_1} = (E_1 \cap E_2) \cap (E_2 \cap E_2)^{s_1}$. Furthermore, as $|E_1| = 2^{10}$ by (3), we have $E_1 = (E_1 \cap E_2)(E_1 \cap E_2)^{s_1}$ by order consideration. As $E_1 \cap E_2 \lhd E_1$ by (1), (6) holds by Lemma (6G)(3).

Now by Lemma (6B), commutation by t induces an $N_G(B_2)$-isomorphism $E_2/A_2 \to A_2$, which maps WA_2/A_2 onto Z and $F \cap E_2/A_2$ onto $A_1 \cap A_2$. Hence $(F \cap E_2) \cap W^{s_2}A_2 = A_2$ as $(A_1 \cap A_2) \cap Z^{s_2} = 1$. Notice that $E_1 \cap E_2 \leqq F \cap E_2$ by Lemma (6G)(1) and that $E_1 \cap A_2 = A_1 \cap A_2$ by Lemmas (6C)(3) and (6E)(3). Therefore, $E_1 \cap W^{s_2} \leqq (A_1 \cap A_2) \cap Z^{s_2} = 1$. As $|S: E_1| = 4$ by (2) and (3), we conclude that W^{s_2} is a complement for E_1 in S, proving (4). In particular, $S = E_1E_2$.

As a consequence of (4), we have that $(E_1 \cap E_2)^{s_2} = ((E_1 \cap E_2) \cap (E_1 \cap E_2)^{s_2}) \times W^{s_2}$ and so

$$(E_1 \cap E_2)^{s_1} = ((E_1 \cap E_2) \cap (E_1 \cap E_2)^{s_2})^{s_1} \times W .$$

Hence

$$\begin{aligned} S &= E_1E_2 \\ &= (E_1 \cap E_2)(E_1 \cap E_2)^{s_1}E_2 \\ &= (E_1 \cap E_2)^{s_1}E_2 \\ &= ((E_1 \cap E_2) \cap (E_1 \cap E_2)^{s_2})^{s_1} WE_2 \\ &= ((E_1 \cap E_2) \cap (E_1 \cap E_2)^{s_2})^{s_1}E_2 . \end{aligned}$$

Furthermore,

$$\begin{aligned} &((E_1 \cap E_2) \cap (E_1 \cap E_2)^{s_2})^{s_1} \cap E_2 \\ &\qquad \leqq (E_1 \cap E_2) \cap (E_1 \cap E_2)^{s_1} \\ &\qquad = W. \end{aligned}$$

Therefore (5) holds.

Finally, $N(Q) \leqq N(B_1) \cap N(B_2)$ by Lemma (3J). Hence subgroups used to define S are all normalized by $N(Q)$ (see Definitions (6.1)-(6.5)). Thus $N(Q) \leqq N(S)$.

DEFINITION (6.6). Let $K = K_2E_2$ and $L_2 = \langle K^{N(E_2)} \rangle$.

LEMMA (6I). *The following conditions hold.*

(1) $L_2/E_2 \cong SL(2, 4) \times SL(2, 4)$ *and* t *interchanges two components of* L_2/E_2.

(2) $S \in \mathrm{Syl}_2(L_2)$.

(3) $O(N(E_2) \bmod E_2) = C(L_2/E_2)$.
(4) $C(E_2) \leqq O(N(E_2) \bmod E_2)$.
(5) $Z(S) = W$.

Proof. Let bars denote images in $N(E_2)/E_2$. Then by Lemma (6G)(4) and Lemma (6B), $C(\bar{t}) = \overline{N(B_2)} = \overline{N_C(B_2)}$. Therefore, $\bar{K} \lhd C(\bar{t})$ and $\langle \bar{t} \rangle \in \mathrm{Syl}_2(C(\bar{K}) \cap C(\bar{t}))$. Furthermore, $\bar{S}$ is an E_{16}-subgroup of $\overline{N(E_2)}$ and is invariant under $N(Q_2) \cap N(B_2) = N(Q)E_2$ by Lemma (6H). Thus (2) and (3) hold and either (1) holds or $L_2/E_2 \cong SL(2, 16)$ by Lemma (1N). As a consequence, we have that $C(E_2) \cap L_2 = E_2$ since $K \not\leqq C(A_2)$. Thus (4) follows from (3). Hence $Z(S) \leqq N_{E_2}(P) \leqq Q_1 \cap E_2 = A_2 W$, and then $Z(S) \leqq Z(PW) = W$. As W centralizes $S = E_1 E_2$ by Lemma (6H)(4), (5), (5) holds.

Now $\bar{P} \in \mathrm{Syl}_2(\bar{K})$, $\bar{P} \leqq \bar{S} \in \mathrm{Syl}_2(\bar{L}_2)$, and $C_{E_2}(\bar{S}) = Z(S) = W$. Furthermore, A_2 is a $\bar{K}$-invariant subgroup of E_2 and $C_{A_2}(\bar{P}) = Z < W$. Thus $\bar{L}_2 \not\cong SL(2, 16)$ by Lemma (1K). The proof is complete.

In view of Lemma (6I), we make the following definition.

DEFINITION (6.7). Let $L_2/E_2 = M_2/E_2 \times M_2^t/E_2$ with $M_2/E_2 \cong SL(2, 4)$, and set $S_2 = S \cap M_2$.

LEMMA (6J). *Assume that* $C_{E_2}(M_2) = 1$. *Then* $\langle L^G \rangle \cong PSL(4, 4)$.

Proof. Let $N = N(E_2)$ and let bars denote images in $N/C(E_2)$. Our aim is to use Lemma (1L) to E_2 and $\bar{N}$. By Lemma (6G)(3), E_2 is elementary abelian of order 256. By Lemma (6I)(4), $C(E_2) = E_2 O(N)$ and so Definition (6.7) and Lemma (6I)(3) imply that $\bar{N}$ satisfies the conditions (1) and (2) of Hypothesis (1.1). Also, $C_{E_2}(\bar{S}_2 \bar{S}_2^t) = C_{E_2}(\bar{S}) = Z(S) = W$ by Lemma (6I)(5), so $\bar{N}$ satisfies the condition (3) of Hypothesis (1.1) as well. Our assumption implies that $C_{E_2}(\bar{M}_2) = 1$, so that $\bar{N}$ satisfies the condition (4) of Lemma (1L). Now $\bar{K} = C_{\bar{L}_2}(\bar{t}) = \{\overline{xtxt} \mid \bar{x} \in \bar{L}_2\}$ and $\bar{H}$ is a complement for $\bar{P} = C_{\bar{S}}(\bar{t})$ in $N_{\bar{K}}(\bar{P})$ as $\bar{K} = \bar{K}_2$. Hence $\bar{H} = \{\overline{htht} \mid \bar{h} \in \bar{H}^*\}$ for some complement $\bar{H}^*$ for $\bar{S}_2$ in $N_{\bar{M}_2}(\bar{S}_2)$. Since $[W, H] = 1$ by Lemma (6C)(8), $\bar{N}$ satisfies the condition (5) Lemma (1L) as well. Thus we can apply Lemma (1L) to determine the structure of $\bar{N}$ and the action of $\bar{N}$ on E_2. As for the structure of $\bar{N}$, we have

$$\langle L^*, t^* \rangle \hookrightarrow \bar{N} \hookrightarrow \langle L^*, t^*, f^*, D^* \rangle .$$

In this embedding, $\bar{L}_2$, $\bar{M}_2$, $\bar{S}$, and $\bar{t}$ correspond to L^*, M^*, $R^* R^{*t^*}$, and t^*, respectively.

Let $S_0 = ((E_1 \cap E_2) \cap (E_1 \cap E_2)^{s_2})^{s_1}$. Then by Lemma (6H)(5) $\langle S_0, t \rangle = S_0 \langle t \rangle$ is a complement for E_2 in T. Since $S \in \mathrm{Syl}_2(L_2)$ by Lemma

(6I)(2), $T \in \mathrm{Syl}_2(\langle L_2, t\rangle)$ and hence E_2 has a complement in $\langle L_2, t\rangle$ by Gaschütz's theorem [19, Hauptsatz 17.4]. Therefore, the structure of $\langle L_2, t\rangle$ is uniquely determined by Lemma (1L). There is an isomorphism

$$\sigma: \langle L_2, t\rangle \longrightarrow \langle L^*E^*, t^*\rangle .$$

Here $L_2^\sigma = L^*E^*$, $(tE_2)^\sigma = t^*E^*$, and σ maps S onto the group S^* of matrices

$$\begin{pmatrix} 1 & & & \\ a & 1 & & \\ b & c & 1 & \\ d & e & f & 1 \end{pmatrix}$$

with entries in $\boldsymbol{F}_4$. We know that each S^* and $S^*/Z(S^*)$ has precisely one E_{256}-subgroup, E_2^* and $E_1^*/Z(S^*)$. Since E_2 and E_1/W are elementary and $Z(S) = W$ (see Lemmas (6G)-(6I)), it follows that E_1 and E_2 are characteristic subgroups of S and that $E_i^\sigma = E_i^*$ for $i \in \{1, 2\}$.

Now consider the case where $\bar{N}$ does not contain an element that corresponds to f^*. Then $T = \langle S, t\rangle \in \mathrm{Syl}_2(N)$. Since $\langle S, t\rangle^\sigma = \langle S^*, t^*\rangle$, we see that E_2 is the only E_{256}-subgroup of T. Hence $N(T) \leqq N$, which implies that $T \in \mathrm{Syl}_2(\mathrm{G})$. Next, since $S^\sigma = S^*$ and $I(S^*) = I(E_1^*) \cup I(E_2^*)$, we have $I(S) = I(E_1) \cup I(E_2)$. Hence if $x \in t^G \cap S$, then $x \in E_i$ for some $i \in \{1, 2\}$. Since $|C_{E_i}(x)| \geqq 256$ by Lemma (1D) and $|C|_2 \leqq 256$, we have $C_{E_i}(x) \in \mathrm{Syl}_2(C(x))$. But class of $C_{E_i}(x) \leqq 2$ and class of $P = 3$, a contradiction. Therefore, $t^G \cap S = \emptyset$. Then $t \notin G'$ by Lemma (1E), and since $L_2^\sigma = L^*E^*$ is perfect, $S \in \mathrm{Syl}_2(G')$. We now appeal to [22] to conclude that $O^{2'}(G'/O(G')) \cong O^{2'}(X)$ for some parabolic subgroup X of $PSL(4, 4)$. By Lemma (1H), $L(G) = \langle L^G\rangle$ and $[\langle L^G\rangle, O(G)] = 1$. Therefore, $\langle L^G\rangle \cong PSL(4, 4)$.

Assume, therefore, that $\bar{N}$ contains an element $\bar{f}$ that corresponds to f^*. Let f' be a preimage of $\bar{f}$ in N. Since $\bar{f} \in N(\bar{T})$, we may choose $f' \in N_N(T)$. Then as $\bar{f} \in C(\bar{t})$ and $\langle \bar{t}\rangle = \bar{D}_2$, $f' \in N(D_2) = N(B_2)$ by Lemma (6G)(4). Also, since $\bar{f}$ normalizes $\bar{Q}_2 = C_{\bar{T}}(\bar{t})$, $f' \in N(Q_2)$. Recall that $N(B_2) = N_C(B_2)E_2$ and $N_C(B_2) \cap E_2 = A_2$. Hence we may choose $f' \in N_C(B_2)$. Then f' normalizes $Q_2 \cap C = Q$, but $f' \notin Q$. Thus $f' \in C - LC_C(L)$. Also, we may choose f' so that $f'^2 \in E_2$. Then $f'^2 \in C \cap E_2 = A_2 \leqq L$. Therefore, $L\langle f'\rangle \cong \mathrm{Aut}(L)$. We can now choose $f \in I(Lf')$ so that the action of f on L is induced by the involutive automorphism of $\boldsymbol{F}_4$. Then $f \in C(s_1) \cap C(s_2)$ and $f \in N(S)$ by Lemma (6H)(7), hence $f \in N(S_0)$. Thus, $\langle S_0, t, f\rangle$ is a complement for E_2 in $\langle S, t, f\rangle$. As $\langle S, t, f\rangle \in \mathrm{Syl}_2(\langle L_2, t, f\rangle)$, E_2 has a complement in $\langle L_2, t, f\rangle$ by Gaschütz's theorem, and the structure of $\langle L_2, t, f\rangle$ is uniquely determined by Lemma (1L). Notice that $f \in Pf'^h$ for some $h \in H$, hence $f \in N_N(M_2)$. Hence by Lemma (1L), there is an isomorphism

$$\sigma\colon \langle L_2, t, f\rangle \longrightarrow \langle L^*, E^*, t^*, f^*\rangle$$

such that $L_2^\sigma = L^*E^*$, $S^\sigma = S^*$, $(tE_2)^\sigma = t^*E^*$, and $(fE_2)^\sigma = f^*E^*$. As $I(t^*E^*) = t^{*E^*}$, we may assume that $t^\sigma = t^*$. Replacing f by $f^{*\sigma^{-1}}$, we may also assume that $f^\sigma = f^*$. Thus f is an involution of C normalizing $P = C_S(t)$.

Now let $X = C(tf)$, $Y = C_L(f)$, and $M = C_{L_2}(tf)$. As $C(f) \cap N_L(A_2) = C(f) \cap C(t) \cap L_2 \cong C(f^*) \cap C(t^*) \cap L^*E^*$, $C(f) \cap N_L(A_2)$ is an extension of E_8 by $SL(2, 2)$. Thus f acts on L as a field automorphism by Lemma (2K)(4), hence $Y \cong Sp(4, 2)$. Also, $M \cong C_{L^*E^*}(t^*f^*)$ is isomorphic to the commutator subgroup of a maximal parabolic subgroup of $Sp(4, 4)$, and as $x^t = x^f$ for $x \in M$, the action of t on M is induced by a field automorphism of $Sp(4, 4)$. As C is a semidirect product of $\langle L, t, f\rangle$ and $O(C)$, we have

$$C_X(t) = C(f) \cap C(t) = \langle Y, t, f, C_{O(C)}(f)\rangle .$$

We argue that $t \not\sim f$. Indeed, $C_{L_2}(f)\langle f\rangle \cong C_{L^*E^*}(f^*)\langle f^*\rangle$ is an extension of an elementary abelian group of order 32 by $SL(2, 2) \times SL(2, 2)$, while C does not contain such a group by Lemma (3J). Let bars denote images in $X/\langle tf\rangle$. Then $\bar{t} \in I(\bar{X})$ and since $t \not\sim f$,

$$C_{\bar{X}}(\bar{t}) = \overline{N_X(\langle t, tf\rangle)} = \overline{C_X(t)} .$$

Therefore,

$$C_{\bar{X}}(\bar{t}) = \bar{Y} \times \langle \bar{t}\rangle \times O(C_{\bar{X}}(\bar{t}))$$

with $\bar{Y} \cong Sp(4, 2)$. We can now apply Lemma (1P) to conclude that $E(\bar{X}) \cong Sp(4, 4)$ and $C_{\bar{X}}(E(\bar{X})) = O(\bar{X})$. Consequently, $|X|_2 \leqq 2^{11}$. As the Schur multiplier of $Sp(4, 4)$ is trivial, it follows that $E(X) \cong Sp(4, 4)$ and $C_X(E(X)) = \langle tf, O(X)\rangle$. Thus $E(X)$ is a standard subgroup of G and $C(E(X))$ has a cyclic S_2-subgroup. Also, as $|G\colon X|$ is even, $tf \notin Z^*(G)$ and so $E(X)O(G) \ntriangleleft G$ by Lemma (1H). Appealing to [11], we conclude that $\langle E(X)^G\rangle \cong PSU(4, 4)$, $PSU(5, 4)$, $PSL(4, 4)$, $PSL(5, 4)$, $PSp(4, 16)$ or $Sp(4, 4) \times Sp(4, 4)$. Since $C(t)$ has a component of type $PSU(4, 2)$, we must have that $\langle E(X)^G\rangle \cong PSL(4, 4)$ (see [3, § 19]). Thus by Lemma (1H), $\langle L^G\rangle \cong PSL(4, 4)$. The proof is complete.

In view of Lemma (6J), we now study the following situation.

Hypothesis (6.3). $C_{E_2}(M_2) \neq 1$.

LEMMA (6K). *$L_2 = N_2 \times N_2^t$, where N_2 is isomorphic to the semidirect product of the natural A_5-module by A_5.*

Proof. By Lemma (6H)(5) and Gaschütz's theorem, E_2 has a complement N in $L_2\langle t\rangle$. As in the proof of Lemma (6J), E_2 and N satisfy Hypothesis (1.1) and $C_{E_2}(S_2S_2^t) = W$. Also, $C_{E_2}(M_2) \neq 1$ by our hypothesis. As $W \cap W^{s_2} = 1$ by Lemma (6H)(4), the assertion follows from Lemma (1M).

DEFINITION (6.8). Let $R = S \cap N_2$, $F_2 = O_2(N_2)$, and $U = Z(R)$. Let F_1/U be an element of $\mathscr{E}^*(R/U)$ different from F_2/U.

REMARK. $N_2 \cong K_2A_2$ and $R \in \mathrm{Syl}_2(N_2)$, hence $R \cong P$. Thus $\mathscr{E}^*(R/U) = \{F_1/U, F_2/U\}$ and F_1 is extra-special of order 32. Also, $W = U \times U^t$ by Lemma (6I).

LEMMA (6L). *For $i \in \{1, 2\}$, the following holds.*
(1) *$E_i = F_i \times F_i^t$.*
(2) *$s_i \in N(F_i)$.*

Proof. For $i = 2$, the assertion is obvious, so consider the case $i = 1$. As $S/W = RW/W \times R^tW/W$ and $RW/W \cong R/U$, we have

$$\mathscr{E}^*(S/W) = \{F_1F_1^t/W, F_2F_2^t/W, F_1F_2^t/W, F_1^tF_2/W\} .$$

Therefore, $F_1F_1^t/W$ is the only member of $\mathscr{E}^*(S/W)$ of order greater than or equal to 2^8. As E_1/W is elementary of order 2^8 by Lemma (6H), (1) holds.

Now $s_1 \in C(W) \leqq C(U)$ by Lemma (6C)(8), and hence s_1 acts on $Z(E_1/U) = U^tF_1/U$. Now $K_2A_2 = C_{L_2}(t) = \{xx^t \mid x \in N_2\}$ and H is a complement for $P = C_S(t)$ in $N_{K_2A_2}(P)$, so $H = \{xx^t \mid x \in H^*\}$ for some complement H^* for R in $N_{N_2}(R)$. As H^* acts fixed-point-freely on F_1/U by the structure of N_2, so also does H. Hence it follows that $[U^tF_1/U, H] = F_1/U$ since H centralizes U^t by Lemma (6C)(8). Therefore, $s_1 \in N(F_1)$.

DEFINITION (6.9). Let $L_1 = \langle S, S^{s_1}\rangle$, $N_1 = \langle R, R^{s_1}\rangle$, $G_0 = \langle L_1, L_2\rangle$, and $G_1 = \langle N_1, N_2\rangle$. Notice that $N_2 = \langle R, R^{s_2}\rangle$.

LEMMA (6M). *G_0 is a central product of G_1 and G_1^t.*

Proof. It is clear that $G_0 = \langle G_1, G_1^t\rangle$, so we shall prove $[G_1, G_1^t] = 1$. The structure of $N(E_2)/E_2$ shows $S \cap S^{s_2} = E_2$ (see Lemma (6I)). In particular, $E_1 \cap E_1^{s_2} \leqq E_2$ so $(E_1 \cap E_1^{s_2})^{s_1}$ is a complement for E_2

in S by Lemma (6H)(5). Thus

$$S = E_2(E_1 \cap E_1^{s_2})^{s_1} .$$

Now, $E_1^{s_1} = F_1^{s_1}F_1^{s_1t}$ and $E_1^{s_2s_1} = F_1^{s_2s_1}F_1^{s_2s_1t}$ by Lemma (6L). As $F_1^{s_1}$, $F_1^{s_2s_1} \leqq N_2^{s_1}$ and $L_2^{s_1} = N_2^{s_1} \times N_2^{s_1t}$, we have that

$$(E_1 \cap E_1^{s_2})^{s_1} = (F_1 \cap F_1^{s_2})^{s_1} \times (F_1 \cap F_1^{s_2})^{s_1t} .$$

Also, $E_2 = F_2 \times F_2^t$. As F_2, $(F_1 \cap F_1^{s_2})^{s_1} \leqq R$, the above factorization of S yields that

$$R = F_2(F_1 \cap F_1^{s_2})^{s_1} .$$

This shows that $R = F_2F_1$ and $R^{s_2} = F_2(F_1 \cap F_1^{s_2})^{s_1s_2}$ as $s_i \in N(F_i)$ by Lemma (6L). Hence if $X = \langle F_1, (F_1 \cap F_1^{s_2})^{s_1s_2}\rangle$, then $N_2 = F_2X$ and so $F_2 \cap F_1 \leqq F_2 \cap X \lhd N_2$. As N_2 acts irreducibly on F_2, $F_2 \cap X = F_2$. Thus

$$N_2 = \langle F_1, (F_1 \cap F_1^{s_2})^{s_1s_2}\rangle .$$

Now

$$[F_1, F_1^t] \leqq [N_2, N_2^t] = 1 .$$

Since $s_1 \in N(F_1)$,

$$[F_1, (F_1 \cap F_1^{s_2})^{s_1s_2t}] \leqq [F_1, F_1^{s_2t}] \leqq [N_2, N_2^t] = 1 .$$

Conjugating this by s_1t, we have

$$[(F_1 \cap F_1^{s_2})^{s_1s_2s_1}, F_1^t] = 1 .$$

Also, since $(s_2s_1)^2 = (s_1s_2)^2$,

$$\begin{aligned} [(F_1 \cap F_1^{s_2})^{s_1s_2s_1},\ (F_1 \cap F_1^{s_2})^{s_1s_2t}] &\leqq [F_1^{s_2s_1s_2s_1}, F_1^{s_2s_1s_2t}] \\ &= [F_1^{s_2s_1s_2}, F_1^{s_2s_1s_2t}] \\ &= [F_1, F_1^t]^{s_2s_1s_2} = 1 . \end{aligned}$$

Since $N_2^{s_1} = \langle F_1, (F_1 \cap F_1^{s_2})^{s_1s_2s_1}\rangle$ and $N_2^t = \langle F_1^t, (F_1 \cap F_1^{s_2})^{s_1s_2t}\rangle$, we conclude that

(1) $$[N_2^{s_1}, N_2^t] = 1 .$$

In particular, $[R^{s_1}, R^t] = 1$, and since $[R, R^t] = 1$ and $N_1 = \langle R, R^{s_1}\rangle$, it follows that

(2) $$[N_1, N_1^t] = 1 .$$

Also, $[R^{s_1t}, N_2] \leqq [N_2^{s_1}, N_2^t]^t = 1$. As $[R^t, N_2] \leqq [N_2^t, N_2] = 1$, it follows that

(3) $$[N_1^t, N_2] = 1 .$$

The equations (1), (2), and (3) show $[G_1, G_1^t] = 1$, as desired.

LEMMA (6N). *The following conditions hold.*
(1) $G_1 \cong PSU(4, 2)$.
(2) $G_0 = G_1 \times G_1^t$.
(3) $L = C_{G_0}(t) = \{xx^t \mid x \in G_1\}$.
(4) $C(G_0) = O(N(G_0))$.
(5) $R \in \mathrm{Syl}_2(G_1)$.

Proof. By Lemma (6K), N_2 is perfect. Therefore, $R \leqq N_2 \leqq G_1'$ and then $R^{s_1} \leqq (G_1')^{s_1} = G_1'$ as $s_1 \in G_0 \leqq N(G_1)$. Thus $N_1 = \langle R, R^{s_1}\rangle \leqq G_1'$ and $G_1 = G_1'$.

Let $L_0 = \{xx^t \mid x \in G_1\}$ and $Z_0 = G_1 \cap G_1^t$. Then, as $G_0 = G_1 * G_1^t$ by Lemma (6M), it follows that $C_{G_0}(t) = L_0 C_{Z_0}(t)$. By the same reason, the mapping $x \to xx^t$ is a homomorphism from G_1 onto L_0 with the kernel contained in $Z(G_1)$. In particular, L_0 is perfect by the first paragraph and so $C_{G_0}(t)' = C_{G_0}(t)^\infty = L_0$. On the other hand, $L = \langle P, s_1, s_2\rangle \leqq C_{G_0}(t)$ and so $C_{G_0}(t)^\infty = L$ as $C^\infty = L$. Thus $L = L_0$, and consequently $G_1/Z(G_1) \cong PSU(4, 2)$.

Now $C(G_0) \lhd C(L) \cap N(G_0)$ as $L \leqq G_0$. Since $\langle t\rangle \in \mathrm{Syl}_2(C(L) \cap N(G_0))$ and $t \notin C(G_0)$, it follows that $C(G_0)$ has odd order. This proves (4) as G_0 is semisimple. Now $Z(G_1)$ has odd order, so as the Schur multiplier of $PSU(4, 2)$ has order 2, we have that $Z(G_1) = 1$. Hence (1), (2), and (3) follow. Finally, (5) is obvious by (1).

LEMMA (6O). *If $t \in N(G_0)^g$ for $g \in G$, then $g \in N(G_0)$.*

Proof. We first show that $N(Q) \leqq N(G_0)$. By Lemma (3J), $N(Q) \leqq N(B_1)$, hence $N(Q) = D_1 N_C(Q) = A_1 W N_C(Q)$ (see Lemma (6C) and a remark after Definition (6.3)). $A_1 W$ and $N_L(P) \leqq L_2 \leqq G_0$, and $N_C(Q) = \langle N_L(P), t, O(C)\rangle$ or $\langle N_L(P), t, O(C), f\rangle$, where f is an element of C acting on L as a field automorphism. Thus it is enough to show $t, O(C)$, and $f \in N(G_0)$. Clearly, $t, O(C)$, and f normalize Q and centralize s_1, s_2. By Lemma (6H)(7), $N(Q) \leqq N(S)$. Hence $t, O(C)$, and f normalize $L_i = \langle S, S^{s_i}\rangle$ for $i \in \{1, 2\}$, and hence normalize $G_0 = \langle L_1, L_2\rangle$. Thus $N(Q) \leqq N(G_0)$.

Now assume that $t \in N(G_0)^g$. Then t acts, by conjugation, on the set $\{G_1^g, G_1^{tg}\}$. Suppose that t normalizes G_1^g and G_1^{tg}. Then both $G_1^g \cap C(t)$ and $G_1^{tg} \cap C(t)$ have 2-rank at least 3 by Lemmas (2E) and (2K), so $m(G_0^g \cap C(t)) \geqq 6$. This is a contradiction because $m(C) = 5$ by Lemma (3J). Therefore, t interchanges G_1^g and G_1^{tg}. As a consequence, we have $L = G_0^g \cap C(t) = \{xx^t \mid x \in G_1^g\}$ since $G_0^g = G_1^g \times G_1^{tg}$.

Hence if $Y \in \mathrm{Syl}_2(G_1^g)$, then $\hat{P} = \{yy^t \mid y \in Y\}$ is an S_2-subgroup of L. As Q and $\langle \hat{P}, t\rangle$ are conjugate by an element of $L \leqq G_0$, $N(\langle \hat{P}, t\rangle) \leqq N(G_0)$ by the first paragraph. Let $z \in Z(Y)^{\#}$. Then as $z^2=1$, $z^{-1}tz = ztzt \cdot t \in \hat{P}t$, so that $z \in N(\langle \hat{P}, t\rangle)$. As $z \notin L$, we conclude that $L < N(G_0) \cap G_0^g$. Then [1, Lemma 2.5] shows that $G_0^g \leqq N(G_0)$, hence $G_0^g = N(G_0)^{\infty} = G_0$. The proof is complete.

DEFINITION (6.10). Let $T \leqq S_1 \in \mathrm{Syl}_2(N(G_0))$, $S_0 = N_{S_1}(G_1)$, and $R_0 = C_{S_0}(G_1^t)$. Notice that $S_0 = N_{S_1}(G_1^t)$ by Lemma (6N), and that $R \leqq R_0$ and $S \leqq S_0$.

LEMMA (6P). $S_1 \in \mathrm{Syl}_2(G)$.

Proof. Let $g \in N(S_1)$. Then $t^g \in S_1 \leqq N(G_0)$, so that $g \in N(G_0)$ by Lemma (6O). Thus $N(S_1) \leqq N(G_0)$, and the assertion follows.

LEMMA (6Q). $S \in \mathrm{Syl}_2(G^{\infty})$.

Proof. There are three cases to consider:
1. $R_0 \neq R$.
2. $R_0 = R$ but $S_0 \neq S$.
3. $R_0 = R$ and $S_0 = S$.

Let $N = N(G_0)$. Then Lemma (6N)(4) shows that $R_0 \cap R_0^t = 1$ and that $C_N(G_1^t)/O(N) \hookrightarrow \mathrm{Aut}\,(G_1)$. Hence $R_0S \cap R_0^tS = S$ and $|R_0S/S| = |R_0/R| \leqq 2$ as $S \cap R_0 = R$. Also, $N_N(G_1)/C_N(G_1) \hookrightarrow \mathrm{Aut}\,(G_1)$, hence $|S_0/R_0^tS| \leqq 2$. Therefore in Case 1, $|R_0S/S| = |R_0/R| = 2$ and $S_0/S = R_0S/S \times R_0^tS/S$. Similarly, $|S_0: S| = 2$ in Case 2.

Suppose $t^g \in N_N(G_1)$. Then $t^g \in N$ and so $g \in N$ by Lemma (6O). But then $t^g \notin N_N(G_1)$ as $N_N(G_1) \lhd N$, a contradiction. Therefore,

$$t^G \cap S_0 = \varnothing .$$

In Case 3, $T = S_1 \in \mathrm{Syl}_2(G)$ by Lemma (6P) and $t^G \cap S = \varnothing$ by the above. Therefore, $t \notin G'$ by Lemma (1E). Since

$$S \leqq G_0 \leqq G^{\infty} ,$$

it follows that $S \in \mathrm{Syl}_2(G^{\infty})$. Therefore, we assume that

$$S < S_0 .$$

Then $S < N_{S_0}(T)$. Also, $N_{S_0}(T) = C_{S_0}(t)S$ as $I(T - S) = t^S$ by Lemma (1B). Thus $C_{S_0}(t) > C_S(t) = P$. As $t \notin C_{S_0}(t)$, $C_{S_0}(t)$ is isomorphic to an S_2-subgroup of $\mathrm{Aut}\,(L)$. Therefore, we can choose an involution $a \in C_{S_0}(t) - S$.

We compute $|C_{S_1}(x)|$ for $x \in I(N_{S_0}(T) - S)$. In Case 1, $S_0 = R_0 \times$

R_0^t, so that $x = yz$ with $y \in I(R_0 - R)$ and $z \in I(R_0^t - R^t)$. Hence $C_{S_0}(x) = C_{R_0}(y) \times C_{R_0^t}(z)$. As y induces an outer automorphism on G_1, $|C_{R_0}(y)| \leqq 32$, and similarly $|C_{R_0^t}(z)| \leqq 32$ (see Lemma (2E)). Thus $|C_{S_0}(x)| \leqq 1024$ and $|C_{S_1}(x)| \leqq 2048$. In Case 2, x induces outer automorphisms on G_1 and G_1^t, so $|C_{S_0}(x)| \leqq 512$ and $|C_{S_1}(x)| \leqq 1024$.

We show that

$$a^G \cap (R_0S \cup R_0^tS) = \varnothing .$$

Suppose that $a^g \in R_0S \cup R_0^tS$ for some $g \in G$. Choose a^g so that $|C_{S_1}(a^g)|$ is maximal. As $R_0S = R_0 \times R^t$, we may write $a^g = uv$ with $u \in R_0$ and $v \in R^t$. Assume Case 1. Then conjugating in $N(G_0)$, we may assume that $|C_{R_0}(u)| \geqq 32$ and that $|C_{R_0^t}(v)| \geqq 64$ (see Lemmas (2E) and (2K)), so $|C_{S_0}(a^g)| \geqq 2048$. Similarly in Case 2, we may assume that $|C_R(u)|$ and $|C_{R^t}(v)| \geqq 32$, so that $|C_S(a^g)| \geqq 1024$. Thus in any case, we may assume that $|C_{S_1}(a^g)| \geqq |C_{S_1}(x)|$ for all $x \in N_{S_0}(T) - S$. Also, if $w \in I(S_1 - S_0)$, then w interchanges R_0 and R_0^t, and so $|C_{S_1}(w)| \leqq 256 < |C_{S_1}(a^g)|$. Thus we may assume that a^g is an extremal conjugate of a in S_1. Then we may also assume that $C_{S_1}(a)^g \leqq S_1$, since $S_1 \in \mathrm{Syl}_2(G)$. Then $t^g \in S_1 \leqq N$, and Lemma (6O) yields that $g \in N$. But now $a \notin X = G_1C_N(G_1) \cup G_1^tC_N(G_1^t)$ and $a^g \in X$, which is a contradiction because X is a normal subset of $N(G_0)$. Thus we have proved that $a^G \cap (R_0S \cup R_0^tS) = \varnothing$.

Consider Case 1. Then $S_1/S \cong D_8$, and S_0/S and $\langle t, a, S\rangle/S$ are the fours subgroups of S_1/S. Since $S_1 \in \mathrm{Syl}_2(G)$ and since $a^G \cap S_0 \leqq aS$ and $t^G \cap S_0 = \varnothing$, Lemma (1G) shows that $S \in \mathrm{Syl}_2(G^\infty)$.

Therefore, assume that Case 2 holds. We show

$$(ta)^G \cap S = \varnothing .$$

Suppose $b \in (ta)^G \cap S$. As before, we may choose b so that $|C_{S_1}(b)| \geqq 1024$. Since $|C_{S_1}(x)| \leqq 1024$ for $x \in I(S_0 - S)$ and since $|C_{S_1}(y)| \leqq 256$ for any $y \in I(S_1 - S_0)$, we may assume that b is an extremal conjugate of ta in S_1. Then we may assume $b = (ta)^g$ and $C_{S_1}(ta)^g \leqq S_1$ for some $g \in G$. But then Lemma (6O) yields a contradiction just as before. Therefore, $(ta)^G \cap S = \varnothing$. Since $t^G \cap \langle a, S\rangle = \varnothing$ and $a^G \cap S = \varnothing$, Lemma (1F) shows that $S \in \mathrm{Syl}_2(G^\infty)$. The proof is complete.

LEMMA (6R). $\langle L^G\rangle \cong PSU(4, 2) \times PSU(4, 2)$.

Proof. We argue that R is strongly involution closed in S with respect to G^∞ (see [25]). By way of contradiction, let $x \in I(R)$ and assume $x^g \in S - R$ with $g \in G^\infty$. By conjugating in G_0, we may choose $x \in F_2$ and $x^g \in F_2 \times F_2^t - F_2$. Since E_2 is the unique E_{256}-

subgroup of S and $S \in \mathrm{Syl}_2(G^\infty)$ by Lemma (6Q), we may also choose $g \in N(E_2) \cap G^\infty$. Now $Y = N(E_2) \cap G^\infty$ acts, by conjugation, on $\{F_2, F_2^t\}$ since $F_2 = O_2(N_2)$. Hence $|Y: N_Y(F_2)| \leqq 2$. Since $S \in \mathrm{Syl}_2(Y)$ by Lemma (6Q) and since $S \leqq N(F_2)$, it follows that $Y \leqq N(F_2)$. Thus $g \in N(F_2)$. But then $x^g \in F_2$, which is a contradiction proving the assertion.

We can now apply Corollary 2 of [25] to get that

$$[\langle I(R)^{G^\infty}\rangle, \langle I(R^t)^{G^\infty}\rangle] \leqq O(G^\infty) .$$

Set $X = \langle I(R)^{G^\infty}\rangle$ and let bars denote images in $G/O(G)$. Then $[\bar{X}, \bar{X}^t] = 1$ so $F^*(\bar{G})$ can not be simple. Thus Lemma (1H) shows $\langle L^G\rangle \cong PSU(4, 2) \times PSU(4, 2)$.

Lemma (6R) completes the proof of Theorem (6A). The main theorem follows from Lemmas (3H), (3G), Theorems (4A), (5A), and (6A).

ACKNOWLEDGMENT. Most of this work was done while the author stayed at the Department of Mathematics of the Ohio State University during 1976/77 academic year. Thanks are due to the department for giving me a visiting professorship, and to my colleagues in the department for helpful suggestions.

REFERENCES

1. M. Aschbacher, *On finite groups of component type*, Illinois J. Math., **19** (1975), 87-115.
2. M. Aschbacher and G. Seitz, *On groups with a standard component of known type*, Osaka J, Math., **13** (1976), 439-482.
3. ———, *Involutions in Chevalley groups over fields of even order*, Nagoya Math., J., **63** (1976), 1-91.
4. R. Brauer, *On simple groups of order* $5 \cdot 3^a \cdot 2^b$, Bull. Amer. Math. Soc., **74** (1968), 900-903.
5. W. Feit, *The current situation in the theory of finite simple groups*, Actes, Congrès Intern. Math., **1** (1970), 55-93.
6. W. Feit and J. Thompson, *Solvability of groups of odd order*, Pacific J. Math., **13** (1963), 755-1027.
7. L. Finkelstein, *Finite groups with a standard component isomorphic to* M_{23}, J. Algebra, **40** (1976), 541-555.
8. P. Fong and G. Seitz, *Groups with a* (B, N)*-pair of rank* 2, Inventiones Math., **21** (1973), 1-57; 24 (1974), 191-240.
9. R. Gilman and D. Gorenstein, *Finite groups with Sylow* 2*-subgroups of class two*, Trans. Amer. Math. Soc., **207** (1975), 1-126.
10. G. Glauberman, *Central elements in core-free groups*, J. Algebra, **4** (1966), 403-420.
11. K. Gomi, *Finite groups with a standard subgroup isomorphic to* $Sp(4, 2^n)$, Japanese J. Math., **4** (1978), 1-76.
12. D. Gorenstein, *Finite Groups*, Harper and Row, New York, 1968.

13. D. Gorenstein and K. Harada, *On finite groups with Sylow* 2-*subgroups of type* $\hat{A}_n$, $n=8$, 9, 10, 11, J. Algebra, **19** (1971), 185-227.
14. ———, *Finite Groups Whose* 2-*Subgroups are Generated by at Most* 4 *Elements*, Memoir 147, American Mathematical Society, Providence, R. I., 1974.
15. D. Gorenstein and J. Walter, *The characterization of finite groups with dihedral Sylow* 2-*subgroups*, J. Algebra, **2** (1965), 88-151, 218-270, 354-393.
16. R. Griess, D. Mason, and G. Seitz, *Bender groups as standard subgroups*, Trans. Amer. Math. Soc., **238** (1978), 179-212.
17. K. Harada, *On finite groups having self-centralizing* 2-*subgroups of small order*, J. Algebra, **33** (1975), 144-160.
18. C. Hering, W. Kantor, and G. Seitz, *Finite groups having a split BN-pair of rank* 1, J. Algebra, **20** (1972), 435-475.
19. B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin Heidelberg, New York, 1967.
20. S. Lang, *Algebraic groups over finite fields*, Amer. J. Math., **18** (1956), 555-563.
21. D. Mason, *Finite simple groups with Sylow* 2-*subgroups of type* $PSL(4, q)$, q *odd*, J. Algebra, **26** (1973), 75-97.
22. P. McBride, *A classification of groups of type* $A_n(q)$ *for* $n\geqq 3$ *and* $q=2^k\geqq 4$, Ph. D Thesis, University of Illinois at Chicago Circle, 1975.
23. I. Miyamoto, *Finite groups with a standard subgroup isomorphic to* $U_4(2^n)$, Japanese J. Math., to appear.
24. A. Prince, *On* 2-*groups admitting* A_5 *or* A_6 *with an element of order* 5 *acting fixed point freely*, J. Algebra, **49** (1977), 374-386.
25. E. Shult, *Disjoint triangular sets*, to appear.
26. G. Stroth, *Endliche einfache Gruppen mit einer* 2-*lokalen Untergruppe* $E_{16}\Sigma_6$, J. Algebra, **47** (1977), 455-479.
27. T. Yoshida, *Character theoretic transfers*, J. Algebra, **52** (1978), 1-38.

Received November 10, 1977 and in revised form May 8, 1978.

UNIVERSITY OF TOKYO
KOMABA, MEGURO
TOKYO, JAPAN