PACIFIC JOURNAL OF MATHEMATICS
Vol. 79, No. 2, 1978

L^p-ESTIMATES FOR SOLUTIONS TO THE INSTATIONARY NAVIER-STOKES-EQUATIONS IN DIMENSION TWO

CLAUS GERHARDT

In this paper we derive L^p-estimates for solutions to the instationary nonlinear problem which are known to be valid for solutions to the linear problem. Since the estimates do not depend on t explicitly, they can be used to prove an exponential decay of the solutions if t goes to infinity.

0. Introduction. The regularity of the weak solutions[1] to the Navier-Stokes-equations is an outstanding problem in the mathematical theory of fluid dynamics. In the three-dimensional case the answer to this question is still unknown in general, though definite answers have been given in the case of *small* data for *arbitrary large* times, and in the case of *large* data and *small* time intervals, cf. the remarks in [4, Chap. 6].

In the two-dimensional case the problem is much easier to settle: it is well-known that the (unique) weak solution to the Navier-Stokes-equations is regular provided the data are smooth enough. However, the answer is not quite satisfactory since the results of the L^p-theory of the nonstationary hydrodynamic potentials have not been carried over to the Navier-Stokes-equations, e.g., to prove that the solution has square integrable second derivatives one has not only to assume that the external force is square integrable but also that it has a square integrable time derivative.

Recently, v. Wahl filled this gap in proving that in dimension two the solution of the Navier-Stokes-equations has r-summable second derivatives if the right-hand side of the system is r-summable for $2 \leqq r < \infty$. Actually, he gave a detailed proof in the case $r = 2$, and indicated the steps necessary to prove the general result.

The aim of this paper is to give a simple proof of v. Wahl's result. To prove the L^r-estimates for arbitrary $r \geqq 2$ we apply the results of Solonnikov [7, § 17] valid for the *linear* Stokes-equations.

In the interesting case $r = 2$ we shall give an elementary proof relying only on Gronwall's inequality and a well-known interpolation theorem of Nirenberg. In this case we shall obtain an a priori estimate which does not depend on time explicitly. From this result we deduce a number of interesting conclusions concerning the solu-

[1] In the sense of Hopf [2].

tion's behaviour at $t = \infty$.[2]

Section 1 is concerned with preliminaries and with the statement of the L^r-estimates in the case of a bounded domain. The estimates will be proved in §§ 2 and 3.

In § 4 we prove that the solutions of the Navier-Stokes-equations are uniformly bounded in x and t, and that an exponential decay is valid for both

$$\sup_x |\boldsymbol{u}(x, t)| \text{ and } \left(\int_\Omega |D\boldsymbol{u}(x, t)|^2 dx\right)^{1/2} .$$

In § 5 we prove corresponding results for the Cauchy problem with the only exception of the exponential decay.

In § 6 we show that the solutions vary continuously in the space $W_2^{2,1}(Q_T) \cap L^\infty(Q_T)$ if the data vary appropriately, under modest assumptions on the data.

Finally, in § 7 we prove that stationary solutions of the Navier-Stokes-equations can be obtained as limits of instationary solutions in the space $H^{1,2}(\Omega) \cap L^\infty(\Omega)$ provided the norm in $L^2(\Omega)$ of the external force is sufficiently small. The convergence in the respective norms is of exponential type.

1. Statement of the main results in the case of a bounded domain. The fluid under consideration will occupy a cylinder Q_T in space-time,

$$Q_T = \Omega \times (0, T) ,$$

where Ω is a bounded open set in $\mathbb{R}^2$ and T a positive real number. The motion of the fluid will be governed by the so-called Navier-Stokes-equations[3]

$$\begin{aligned} \dot{u}_j - \Delta u_j + u_i D^i u_j + D^j p &= f_j , \\ \operatorname{div} \boldsymbol{u} &= 0 , \\ \boldsymbol{u}|_{\partial\Omega} &= \boldsymbol{o} , \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0 , \end{aligned} \tag{1}$$

for $j = 1, 2$, where $\boldsymbol{u} = (u_1, u_2)$ is the velocity of the fluid, $\boldsymbol{f} = (f_1, f_2)$ the external force, $\boldsymbol{u}_0$ the initial velocity, and where p is the (unknown) pressure. We adopt the convention to sum over repeated indices from 1 to 2.

The linearized form of (1) looks like

[2] v. Wahl's estimate depends on time explicitly, so that he cannot control the solution's behavior at $t=\infty$.

[3] For simplicity we assume the kinetic viscosity to be equal to 1.

$$
(2)\qquad \begin{aligned} \dot{u}_j - \Delta u_j + D^j p &= f_j\ , \\ \operatorname{div} \boldsymbol{u} &= o\ , \\ \boldsymbol{u}|_{\partial\Omega} &= \boldsymbol{o}\ , \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0\ . \end{aligned}
$$

In order to describe our results appropriately we recall the following standard notations and definitions: $H^{m,r}(\Omega)$, $m \geqq 0$, $r \geqq 1$, are the usual Sobolev spaces where m indicates the order of differentiation; m is allowed to be an arbitrary nonnegative real number. For $m = 0$ we obtain the usual Lebesgue spaces $L^r(\Omega)$.

If V is a Banach space then $L^r(0, T; V)$ denotes the space of all Lebesgue measurable functions u from $(0, T)$ into V with finite norm

$$\left(\int_0^T \|u\|_V^r dt\right)^{1/r}$$

for $1 \leqq r < \infty$ and with the usual definition for $r = \infty$.

We denote with $W_r^{2,1}(Q_T)$ the space of all measurable functions $u = u(x, t)$ defined in Q_T having generalized derivatives up to order two with respect to x and up to the first order with respect to t such that the norm

$$\|u\|_{W_r^{2,1}(Q_T)} := \left\{\int_0^T\int_\Omega (|u|^r + |\dot{u}|^r) dx dt + \int_0^T \|u\|_{2,r}^r dt\right\}^{1/r}$$

is finite for $1 \leqq r < \infty$, where $\|\cdot\|_{m,r}$ indicates the norm in $H^{m,r}(\Omega)$.

Vector valued functions $\boldsymbol{u}$ have always two components u_1 and u_2. We remark that we also use the notations for spaces of real valued functions to indicate spaces of vector valued functions, e.g., $\boldsymbol{u} \in H^{m,r}(\Omega)$ means $u_j \in H^{m,r}(\Omega)$ for $j = 1, 2$.

Finally, let

$$J_{0,1}(\Omega) = \{\boldsymbol{u} \in H_0^{1,2}(\Omega) : \operatorname{div} \boldsymbol{u} = 0\}\ ,$$

let $J_0(\Omega)$ be the closure of $J_{0,1}(\Omega)$ in $L^2(\Omega)$, and let $G(\Omega)$ be the gradient fields of all real valued functions $\varphi \in L^2_{\text{loc}}(\Omega)$ such that $D\varphi \in L^2(\Omega)$.

If Ω is a bounded open set in $\mathbb{R}^2$ with $\partial\Omega \in C^2$ then $L^2(\Omega)$ is decomposed into the orthogonal complements $G(\Omega)$ and $J_0(\Omega)$, i.e.,

$$(3)\qquad L^2(\Omega) = G(\Omega) \oplus J_0(\Omega)$$

(cf. [4, Chap. 1]).

With these definitions in mind we can state the first theorem which is due to Solonnikov [7, § 17].[4)]

---

[4] Solonnikov proved this theorem in the case $n=3$. The corresponding result for $n=2$ can easily be deduced from it as we shall show in the Appendix.

THEOREM 1. *Let Ω be a bounded open set in $\mathbb{R}^2$ with boundary $\partial\Omega \in C^2$. Suppose that $\boldsymbol{f} \in L^r(Q_T)$, $\boldsymbol{u}_0 \in H^{2-2/r,r}(\Omega)$, $1 < r < \infty$, $r \neq 3/2$, such that $\boldsymbol{u}_0|_{\partial\Omega} = \boldsymbol{o}$ and* $\operatorname{div} \boldsymbol{u}_0 = 0$. *Then, there exists a unique solution $\boldsymbol{u}, p$ of the equations* (2) *satisfying*

$$\boldsymbol{u} \in W_r^{2,1}(Q_T), \quad Dp \in L^r(Q_T)$$

and

$$\begin{aligned} (4) \qquad & \|\boldsymbol{u}\|_{W_r^{2,1}(Q_T)} + \|Dp\|_{L^r(Q_T)} \\ & \leqq c\cdot\{\|\boldsymbol{f}\|_{L^r(Q_T)} + \|\boldsymbol{u}_0\|_{H^{2-2/r,r}(\Omega)}\}\ , \end{aligned}$$

where the constant c depends on $\partial\Omega$, r and on T.

We shall prove a corresponding result for the solution of the Navier-Stokes-equations, namely,

THEOREM 2. *Let the assumptions of the preceding theorem be satisfied for $2 \leqq r < \infty$. Then, the equations* (1) *have a unique solution $\boldsymbol{u}, p$ such that for $2 \leqq r < \infty$*

$$\boldsymbol{u} \in W_r^{2,1}(Q_T), \quad Dp \in L^r(Q_T)$$

and

$$\begin{aligned} (5) \qquad & \|\boldsymbol{u}\|_{W_r^{2,1}(Q_T)} + \|Dp\|_{L^r(Q_T)} \\ & \leqq c\cdot(\|\boldsymbol{f}\|_{L^r(Q_T)} + \|\boldsymbol{u}_0\|_{H^{2-2/r,r}(\Omega)})\ , \end{aligned}$$

where the constant C depends on $\partial\Omega$, r, T,

$$\int_0^T\Big(\int_\Omega |\boldsymbol{f}|^2 dx\Big)^{1/2} dt, \text{ and on } \int_\Omega |\boldsymbol{u}_0|^2 dx\ .$$

In the special case $r = 2$ we can prove

THEOREM 3. *Let $\boldsymbol{u}, p$ be a solution of the equations* (1). *Then, the following a priori estimate is valid*

$$\begin{aligned} (6) \qquad & \|\boldsymbol{u}\|_{W_2^{2,1}(Q_T)} + \|Dp\|_{L^2(Q_T)} \\ & \leqq 4c_1\cdot I_1(T)\{1 + c_0\cdot c_1\cdot I_2(T)\cdot\exp(c_0\cdot c_1^2\cdot I_2(T))\}\ , \end{aligned}$$

where the constants only depend on Ω, and where

$$I_1(t) = \int_\Omega |D\boldsymbol{u}_0|^2 dx + \int_0^t\int_\Omega |\boldsymbol{f}|^2 dx d\tau$$

and

$$I_2(t) = \sup_{0\leqq\tau\leqq t}\int_\Omega |\boldsymbol{u}|^2 dx\cdot\int_0^t\int_\Omega |D\boldsymbol{u}|^2 dx d\tau\ .$$

The proof of the theorems will be accomplished with the help of the following lemmata.

LEMMA 1. *Let* $\boldsymbol{f} \in L^1(0, T; L^2(\Omega))$ *and* $\boldsymbol{u}_0 \in J_0(\Omega)$. *Then the solution* $\boldsymbol{u}$ *of the equations* (1) *satisfies the estimate*

$$|\boldsymbol{u}|_{Q_T} \leqq c \cdot \left\{ \int_0^T \left(\int_\Omega |\boldsymbol{f}|^2 dx \right)^{1/2} dt + \left(\int_\Omega |\boldsymbol{u}_0|^2 dx \right)^{1/2} \right\} \tag{7}$$

with some numerical constant c, *where*

$$|\boldsymbol{u}|_{Q_T} := \sup_{0 \leqq t \leqq T} \left(\int_\Omega |\boldsymbol{u}|^2 dx \right)^{1/2} + \left(\int_0^T \int_\Omega |D\boldsymbol{u}|^2 dx dt \right)^{1/2} .$$

LEMMA 2. *Let* $\boldsymbol{u} \in L^2(0, T; H_0^{1,2}(\Omega))$, $\Omega \subset \mathbb{R}^2$. *Then*

$$\|\boldsymbol{u}\|_{L^4(Q_T)} \leqq c \cdot |\boldsymbol{u}|_{Q_T} \tag{8}$$

with some numerical constant c.

LEMMA 3. (i) *Let* $\boldsymbol{u} \in L^{4/3}(0, T; H^{2,4/3}(\Omega))$, *where* Ω *is a bounded open set in* $\mathbb{R}^2$ *with Lipschitz boundary. Then*

$$\int_0^T |\boldsymbol{u}(t)|^2 dt \leqq c \cdot \int_0^T \|\boldsymbol{u}\|_{2,4/3}^{4/3} dt \cdot \sup_{0 \leqq t \leqq T} \left(\int_\Omega |\boldsymbol{u}|^2 dx \right)^{1/3} , \tag{9}$$

where the constant c *depends on* Ω, *and where*

$$|\boldsymbol{u}(t)| := \sup_{x \in \Omega} |\boldsymbol{u}(x, t)| .$$

(ii) *Let* Ω *be as above and let* $\boldsymbol{u} \in L^2(0, T; H^{2,2}(\Omega))$. *Then*

$$\int_0^T |\boldsymbol{u}(t)|^4 dt \leqq c_0 \cdot \sup_{0 \leqq t \leqq T} \int_\Omega |\boldsymbol{u}|^2 dx \cdot \int_0^T \|\boldsymbol{u}\|_{2,2}^2 dt \tag{10}$$

where c_0 *depends on* Ω.

LEMMA 4. *Let* $\mathscr{P}$ *be the projection operator from* $L^2(\Omega)$ *onto* $J_0(\Omega)$, *where* $\Omega \subset \mathbb{R}^2$ *is a bounded open set with* $\partial\Omega \in C^2$. *Then for any vector ualued function* $\boldsymbol{u} \in J_{0,1}(\Omega)$ *we have the estimates*

$$\|\mathscr{P}\Delta\boldsymbol{u}\|_{L^2(\Omega)} \leqq \|\Delta\boldsymbol{u}\|_{L^2(\Omega)} \leqq c \cdot \|\mathscr{P}\Delta\boldsymbol{u}\|_{L^2(\Omega)} \tag{11}$$

where the constant c *only depends on* Ω.

Lemma 1 can easily be derived by multiplying the equations (1) with $\boldsymbol{u}$, integrating over the cylinder $\Omega \times (0, t)$, using the fact that the integral containing the nonlinear term vanishes, and applying Gronwall's inequality, finally.

Lemma 2 is an immediate consequence of a general interpolation theorem of Nirenberg (cf. [6, p. 126] and [1, Thm. 10.1]). The estimate (8) can be found in [5, p. 75, formula (3.4)].

Lemma 3 also follows directly from Nirenberg's interpolation theorem.

Lemma 4 will enable us to give a proof of Theorem 2 in the case $r = 2$ being independent of the general result of Solonnikov. The lemma is proved in [4, p. 67].

2. **Proof of Theorem 2.** We shall only give a priori estimates since we can always assume to work with a solution $\boldsymbol{u} \in W_2^{2,1}(Q_T)$ by assuming $\boldsymbol{u}_0$ and $\boldsymbol{f}$ to be sufficiently regular. As we shall show, this will imply $\boldsymbol{u} \in W_r^{2,1}(Q_T)$ for any $r \in (2, \infty)$ provided $\boldsymbol{f} \in L^r(Q_T)$ and $\boldsymbol{u}_0 \in H^{2-2/r,r}(\Omega) \cap J_{0,1}(\Omega)$.

We shall consider the cases $r = 2$, $2 < r < 4$, and $4 \leqq r < \infty$, separately.

First case. $r = 2$. We observe that the nonlinear term $\boldsymbol{u}_i D^i \boldsymbol{u}$ in (1) is summable to the power 4/3 in Q_T since

$$(12) \qquad \int_{Q_T} |\boldsymbol{u}|^{4/3}\, |D\boldsymbol{u}|^{4/3} dx dt \leqq \left(\int_{Q_T} |\boldsymbol{u}|^4 dx dt\right)^{1/3} \cdot \left(\int_{Q_T} |D\boldsymbol{u}|^2 dx dt\right)^{2/3} .$$

From Lemma 1 and Lemma 2, and from Theorem 1 we thus obtain

$$(13) \qquad \|\boldsymbol{u}\|_{W^{2,1}_{4/3}(Q_T)} \leqq c\cdot\{\|\boldsymbol{f}\|_{L^{4/3}(Q_T)} + \|\boldsymbol{u}_0\|_{H^{2-3/2,4/3}(\Omega)} + 1\}$$

where c depends on Ω, T, $\int_\Omega |\boldsymbol{u}_0|^2 dx$, and on $\int_0^T \left(\int_\Omega |\boldsymbol{f}|^2 dx\right)^{1/2} dt$. Applying Lemma 3(i) we then get the estimate

$$(14) \qquad \int_0^T |\boldsymbol{u}(t)|^2 dt \leqq c_1 ,$$

the constant c_1 depending on the same quantities as the right-hand side of (13).

Now, multiplying the equations (1) with $-\mathscr{P}\Delta\boldsymbol{u}$ and integrating over $Q_t = \Omega \times (0, t)$ we obtain

$$(15) \qquad \begin{aligned} &\int_\Omega |D\boldsymbol{u}(t)|^2 dx + \int_{Q_t} |\mathscr{P}\Delta\boldsymbol{u}|^2 dx d\tau \\ &\qquad \leqq c_2\cdot\left\{\int_{Q_t} |\boldsymbol{u}|^2 |D\boldsymbol{u}|^2 dx d\tau + \int_{Q_t} |\boldsymbol{f}|^2 dx d\tau + \int_\Omega |D\boldsymbol{u}_0|^2 dx\right\} \end{aligned}$$

with some numerical constant c_2, where we used an appropriate version of Cauchy's inequality and the fact that $\boldsymbol{u}(\cdot, t) \in J_{0,1}(\Omega)$ for a.e.t.

We conclude that for any $t \in [0, T]$ the inequality

$$\begin{aligned}\int_\Omega |Du(t)|^2 dx \leqq c_2\cdot \Big\{ \int_0^t |u(\tau)|^2 \cdot \int_\Omega |Du|^2 dx d\tau \\ + \int_\Omega |Du_0|^2 dx + \int_{Q_T} |f|^2 dx d\tau \Big\}\end{aligned} \tag{16}$$

is valid. Gronwall's inequality then gives

$$\begin{aligned}\int_\Omega |Du(t)|^2 dx \leqq c_2\cdot \Big\{ \int_\Omega |Du_0|^2 dx + \int_{Q_T} |f^2 dx d\tau \Big\} \\ \cdot \exp\Big(c_2 \cdot \int_0^t \int_\Omega |Du|^2 dx d\tau \Big) .\end{aligned} \tag{17}$$

Inserting this estimate into the right-hand side of (15) and using (11) and (14) we obtain a bound for

$$\int_0^T ||u||_{2,2}^2 dt . \tag{18}$$

To get the final estimate, we multiply (1) with $\dot{u}$, thus deriving a bound for

$$\int_{Q_T} |\dot{u}|^2 dx dt$$

in view of (16) since $u(\cdot, t) \in J_{0,1}(\Omega)$ for a.e.t.

The estimate for $||Dp||_{L^2(Q_T)}$ then follows directly from (1).

Second case. $2 < r < 4$. As $u \in W_2^{2,1}(Q_T)$ and $\Omega \subset \mathbb{R}^2$, we know that $u \in L^q(Q_T)$ for any finite q, and that

$$||u||_{L^q(Q_T)} \leqq c \cdot ||u||_{W_2^{2,1}(Q_T)} , \tag{19}$$

where the constant c depends on q and on vol Q_T. This follows either from a general result in [3, p. 186, Thm. 3.4], or can easily be proved directly by using Stampacchia's version of De Giorgi's truncation method taking into account that $u_0 \in H^{2-2/r,r}(\Omega)$ for $r > 2$ implies $u_0 \in L^\infty(\Omega)$.

Moreover, from Nirenberg's interpolation theorem [6, p. 126] we deduce

$$\int_\Omega |Du|^4 dx \leqq c \cdot ||u||_{2,2}^2 \cdot \int_\Omega |Du|^2 dx ,$$

where c depends on Ω, thus getting an a priori bound for

$$||Du||_{L^4(Q_T)} \tag{20}$$

in view of (17) and (18).

Hölder's inequality then shows that the nonlinear term $u_i D^i u$ belongs to $L^r(Q_T)$ for any $2 < r < 4$ with an a priori bound for the norm depending on $\|\boldsymbol{u}\|_{W_2^{2,1}(Q_T)}$, vol Q_T, and on r.

Hence, applying Theorem 1 we conclude that $\boldsymbol{u} \in W_{r_i}^{2,1}(Q_T)$ and that an estimate of the kind (5) is valid.

Third case. $4 \leqq r < \infty$. We already know that $\boldsymbol{u} \in W_q^{2,1}(Q_T)$ for any $2 \leqq q < 4$. Therefore, $\boldsymbol{u}$ is bounded in Q_T and $D\boldsymbol{u} \in L^s(Q_T)$ for any $s \in [1, \infty)$, where the respective norms can be estimated by s and known quantities (cf. [3, p. 186, Thm. 3.4]). Thus, we know that $u_i D^i \boldsymbol{u} \in L^r(Q_T)$ for any $r \in [4, \infty)$ with a known a priori bound for the norm. The final result then follows from applying Theorem 1 once more.

3. **Proof of Theorem 3.** From the considerations in the first part of the proof of Theorem 2 we immediately conclude

$$\|\boldsymbol{u}\|_{W_2^{2,1}(Q_t)}^2 \leqq c_1 \cdot \left\{\int_\Omega |D\boldsymbol{u}_0|^2 dx + \int_0^t\int_\Omega |\boldsymbol{f}|^2 dx d\tau + \int_0^t\int_\Omega |\boldsymbol{u}|^2 \cdot |D\boldsymbol{u}|^2 dx d\tau\right\} \tag{21}$$

where the constant c_1 depends on Ω only. The last integral on the right-hand side of this inequality can be estimated from above by

$$\int_0^t |\boldsymbol{u}(\tau)|^2 \cdot \int_\Omega |D\boldsymbol{u}|^2 dx d\tau \leqq \frac{\varepsilon}{2} \cdot \int_0^t |\boldsymbol{u}(\tau)|^4 d\tau + \frac{1}{2\varepsilon} \cdot \left\{\int_0^t \left(\int_\Omega |D\boldsymbol{u}|^2 dx\right)^2 d\tau\right\}$$

where ε is any positive number.

Now, we apply (10) and choose ε equal to

$$\left\{c_0 \cdot c_1 \sup_{0 \leqq \tau \leqq t} \int_\Omega |\boldsymbol{u}|^2 dx\right\}^{-1}$$

to obtain

$$\|\boldsymbol{u}\|_{W_2^{2,1}(Q_t)}^2 \leqq c_1 \left\{2 \cdot I_1(t) + c_0 \cdot c_1 \sup_{0 \leqq \tau \leqq t} \int_\Omega |\boldsymbol{u}|^2 dx \cdot \int_0^t \left(\int_\Omega |D\boldsymbol{u}|^2 dx\right)^2 d\tau\right\} \tag{22}$$

for any $0 \leqq t \leqq T$. $I_1(t)$ is defined as in Theorem 3.

On the other hand, we have the trivial estimate

$$\int_\Omega |D\boldsymbol{u}(x, t)|^2 dx \leqq \int_\Omega |D\boldsymbol{u}_0|^2 dx + \|\boldsymbol{u}\|_{W_2^{2,1}(Q_t)}^2 .$$

Assuming c_1 to be greater than 1 we therefore deduce from (22)

$$\begin{aligned}\int_\Omega |Du(x, t)|^2 dx \leqq c_1 \cdot & \Big\{ 3 \cdot I_1(t) \\ & + c_0 \cdot c_1 \sup_{0 \leqq \tau \leqq t} \int_\Omega |u|^2 dx \cdot \int_0^t \Big(\int_\Omega |Du|^2 dx \Big)^2 d\tau \Big\} .\end{aligned} \tag{23}$$

Gronwall's inequality and (22) yield to the desired estimate (6).

REMARK 1. There is still another variant of estimating the nonlinear term

$$\int_0^t \int_\Omega |u|^2 \cdot |Du|^2 dx d\tau$$

in (21). Namely, from Nirenberg's interpolation theorem it follows that

$$\begin{aligned}\int_\Omega |u|^2 |Du|^2 dx & \leqq \Big(\int_\Omega |u|^4 dx \Big)^{1/2} \cdot \Big(\int_\Omega |Du|^4 dx \Big)^{1/2} \\ & \leqq c \cdot \|u\|_{2,2} \Big(\int_\Omega |Du|^2 dx \Big)^{1/2} \cdot \Big(\int_\Omega |u|^4 dx \Big)^{1/2} ,\end{aligned} \tag{22}$$

where $c = c(\Omega)$. Hence, we obtain from (21)

$$\begin{aligned}\|u\|^2_{W_2^{2,1}(Q_t)} \leqq c_1' \cdot & \Big\{ \int_\Omega |Du_0|^2 dx + \int_0^t \int_\Omega |f|^2 dx d\tau \Big\} \\ & + \int_0^t \int_\Omega |Du|^2 dx \cdot \int_\Omega |u|^4 dx d\tau\end{aligned} \tag{23}$$

from which we deduce an a priori bound as before, since

$$\Big(\int_0^T \int_\Omega |u|^4 dx d\tau \Big)^{1/4} \leqq c \cdot |u|_{Q_T} .$$

4. Boundedness of the solutions in the case of a bounded domain. Assuming the conditions in Theorem 2, the boundedness of a solution $\boldsymbol{u}$ of the Navier-Stokes-equations in any finite cylinder Q_T, $0 < T < \infty$, would be guaranteed provided $r > 2$. But, unfortunately, the bound will depend on T since the L^r-estimate for $\boldsymbol{u}$ depends on T in general. Though we are convinced that one must be able to prove (4) with a constant independent of T, the estimates in [7] are not of this kind.

Nevertheless, we shall be able to prove uniform boundedness of $\boldsymbol{u}$ with respect to x and t for all t, $0 \leqq t \leqq \infty$, and even an exponential decay with respect to t assuming some further restrictions on $\boldsymbol{f}$. The proceeding is as follows:

First we shall prove that

$$\int_0^\infty \int_\Omega |\dot{u}|^r dx dt \leqq \text{const} \tag{24}$$

for all $2 \leqq r \leqq 4$. Then, in the *second* step, we look at the stationary equations

$$-\Delta \boldsymbol{u} + Dp = \boldsymbol{F} \tag{25}$$

obtained from the instationary's by shifting the nonlinear term and $\dot{\boldsymbol{u}}$ to the right-hand side.

For solutions of (25) it is known (cf. [5, p. 67]) that

$$\|\boldsymbol{u}\|_{H^{2,r}(\Omega)} + \|Dp\|_{L^r(\Omega)} \leqq c\cdot\|\boldsymbol{F}\|_{L^r(\Omega)} \tag{26}$$

for all $r, 1 < r < \infty$, where the constant depends on Ω and r. Combining (24), (26), the condition on $\boldsymbol{f}$, and the a priori estimates for the nonlinear term, we conclude

$$\|\boldsymbol{u}\|_{W_r^{2,1}(Q_\infty)} + \|Dp\|_{L^r(Q_\infty)} \leqq \text{const} \tag{27}$$

for some $r \geqq 2$, where

$$Q_\infty = \Omega \times (0, \infty) .$$

From this estimate the boundedness of $\boldsymbol{u}$ in Q_∞ follows immediately.

In a *third* step we shall repeat these proceedings showing that the same results are also valid for $\boldsymbol{v} = \boldsymbol{u}e^{\lambda t}$ where λ is some (small) positive constant depending on Ω. Thus, we shall have proved the exponential decay of the solution, and not only the exponential decay of the supremum's norm but also the exponential decay of

$$\int_\Omega |D\boldsymbol{u}(x, t)|^2 dx .$$

The first precise result is the following

THEOREM 4. *Let* $\boldsymbol{u}_0 \in H^{2-2/r,r}(\Omega)$, $\boldsymbol{f}$, $\dot{\boldsymbol{f}} \in L^1(0, \infty; L^2(\Omega))$, *and* $\boldsymbol{f} \in L^2(Q_\infty) \cap L^r(Q_\infty)$ *for some* $r > 2$. *Then, the solution* $\boldsymbol{u}$, p *of the equations* (1) *satisfies the estimates*

$$\|\boldsymbol{u}\|_{W_s^{2,1}(Q_\infty)} + \|Dp\|_{L^s(Q_\infty)} \leqq \text{const} \tag{28}$$

and

$$\|\boldsymbol{u}\|_{L^\infty(Q_\infty)} \leqq \text{const} \tag{29}$$

for $s = \min(r, 4)$, *where the constants depend on the data.*

Proof. According to what was said above it will be sufficient to prove the estimates not for $\boldsymbol{u}$ and p but for $\boldsymbol{w}=\boldsymbol{u}\cdot\eta$, and $q=p\cdot\eta$, where $0 \leqq \eta(t) \leqq 1$ is a smooth real valued function vanishing in the interval $[0, 1]$ and being identically equal to 1 for values of

$t \geqq 2$. $\boldsymbol{w}$ satisfies the equations

(30) $$\dot{\boldsymbol{w}} - \Delta \boldsymbol{w} + \boldsymbol{u} D \boldsymbol{w} + Dq = \boldsymbol{f} \cdot \eta + \boldsymbol{u} \cdot \dot{\eta} \equiv \boldsymbol{g} .$$

We note that $\boldsymbol{w} \in W_2^{2,1}(Q_\infty)$ since $\boldsymbol{u}$ has this property, that

(31) $$\boldsymbol{w}(0) = \dot{\boldsymbol{w}}(0) = \boldsymbol{0}$$

and

(32) $$\int_0^\infty \left(\int_\Omega |\dot{\boldsymbol{g}}|^2 \right)^{1/2} dx dt \leqq \text{const}$$

in view of the assumptions on $\boldsymbol{f}$, and due to the fact that $\dot{\eta}$ has compact support.

Thus, multiplying the differentiated version of (30) with $\dot{\boldsymbol{w}}$ and integrating over Q_t we obtain

(33) $$\begin{aligned} &\int_\Omega |\dot{\boldsymbol{w}}(x, t)|^2 dx + \int_0^t \int_\Omega |D\dot{\boldsymbol{w}}|^2 dx d\tau \\ &\leqq c \cdot \left\{ \int_0^t \left(\int_\Omega |\dot{\boldsymbol{g}}|^2 dx \right)^{1/2} \left(\int_\Omega |\dot{\boldsymbol{w}}|^2 dx \right)^{1/2} d\tau + \int_0^t \int_\Omega |\boldsymbol{u}|^4 \cdot dx d\tau \right. \\ &\left. \quad + \int_0^t \int_\Omega |\dot{\boldsymbol{w}}|^2 |\boldsymbol{u}|^2 dx d\tau \right\} , \end{aligned}$$

where we used the relation

$$\dot{\boldsymbol{u}} D \boldsymbol{w} = \dot{\boldsymbol{w}} D \boldsymbol{u} - \boldsymbol{u} \cdot \dot{\eta} D \boldsymbol{u}$$

and an appropriate version of Cauchy's inequality after integrating by parts in a couple of terms.

Moreover, increasing the constant by the factor two we see that we can replace the left-hand side of (33) by $|\dot{\boldsymbol{w}}|_{Q_t}$. Then, using the interpolation inequality for L^p-spaces we estimate

$$\begin{aligned} \int_0^t \int_\Omega |\dot{\boldsymbol{w}}|^2 \cdot |\boldsymbol{u}|^2 dx d\tau &\leqq \|\dot{\boldsymbol{w}}\|_{L^3(Q_t)}^2 \cdot \|\boldsymbol{w}\|_{L^6(Q_\Omega)}^2 \\ &\leqq \|\dot{\boldsymbol{w}}\|_{L^4(Q_t)}^{2a} \cdot \|\dot{\boldsymbol{w}}\|_{L^2(Q_t)}^{2(1-a)} \cdot \|\boldsymbol{u}\|_{L^6(Q_t)}^2 \end{aligned}$$

for some $a \in (0, 1)$. Young's inequality and (8) now yields

$$\begin{aligned} |\dot{\boldsymbol{w}}|_{Q_t}^2 \leqq c \cdot &\left\{ \int_0^t \left(\int_\Omega |\dot{\boldsymbol{g}}|^2 dx \right)^{1/2} \cdot \left(\int_\Omega |\dot{\boldsymbol{w}}|^2 dx \right)^{1/2} d\tau \right. \\ &\left. + \int_0^\infty \int_\Omega |\boldsymbol{u}|^4 dx d\tau + \int_0^\infty \int_\Omega |\dot{\boldsymbol{w}}|^2 dx d\tau \left(\int_0^\infty \int_\Omega |\boldsymbol{u}|^6 dx d\tau \right)^{1/3(1-a)} \right\} . \end{aligned}$$

Hence, we conclude

(34) $$|\dot{\boldsymbol{w}}|_{Q_\infty} \leqq \text{const}$$

and

$$\dot{\boldsymbol{w}} \in L^4(Q_\infty) \,. \tag{35}$$

Thus, we settled the first step of the announced proceeding.

For the second step we first note that in view of (34)

$$\sup_t \int_\Omega |D\boldsymbol{w}(x, t)|^3 dx \leqq 3\Big(\int_{Q_\infty} |D\dot{\boldsymbol{w}}|^2 dxdt\Big)^{1/2} \cdot \Big(\int_{Q_\infty} |D\boldsymbol{w}|^4 dxd\tau\Big)^{1/2} \leqq \text{const} \,.$$

Nirenberg's interpolation theorem then shows

$$\begin{aligned} \int_0^\infty \int_\Omega |D\boldsymbol{w}|^5 dxdt \leqq c\cdot\Big(\int_0^\infty \|\boldsymbol{w}\|_{2,2}^2 dt\Big)^{2/5} \cdot \Big(\int_0^\infty \int_\Omega |D\boldsymbol{w}|^3 dxd\tau\Big)^{3/5} \\ \cdot \sup_t \Big(\int_\Omega |D\boldsymbol{w}|^3\Big)^{4/5} \leqq \text{const} \,. \end{aligned} \tag{36}$$

From (35), (36) and in view of the assumptions on $\boldsymbol{f}$ we therefore conclude

$$-\Delta \boldsymbol{w} + Dq = \boldsymbol{F} \tag{37}$$

where

$$\boldsymbol{F} \in L^s(Q_\infty)$$

for $s = \min(r, 4)$.

Using (26) we then obtain the desired estimate (28) for $\boldsymbol{w}$ and hence for $\boldsymbol{u}$.

$\boldsymbol{u}$ is therefore a solution of the equations

$$\begin{aligned} \dot{\boldsymbol{u}} - \Delta \boldsymbol{u} + \boldsymbol{u} D\boldsymbol{u} = \boldsymbol{g} \,, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \,, \end{aligned} \tag{38}$$

where $\boldsymbol{g} \in L^s(Q_\infty)$ and $s > 2$.

If s is strictly less than 4 then the nonlinear term could be absorbed by $\boldsymbol{g}$ but we do not need this.

We shall prove the final result as an extra lemma

LEMMA 5. *Let $\boldsymbol{u} \in L^2(0, T; J_{0,1}(\Omega))$ be a weak solution of* (38), *where $\boldsymbol{u}_0 \in J_0(\Omega) \cap L^\infty(\Omega)$. Then*

$$\sup_{Q_T} |\boldsymbol{u}| \leqq \sqrt{2}k_0 + c\cdot \|\boldsymbol{g}\|_{L^s(Q_T)} \cdot \Big(\int_{Q_T} |\boldsymbol{u}|^4 dxdt\Big)^{1/2-1/s} \cdot k_0^{-4(1/2-1/s)} \,, \tag{39}$$

where k_0 is any number greater than $\sup_\Omega |\boldsymbol{u}_0|$. The constant c only depends on s. T can be any positive number the value plus infinity not excluded.

Proof of the lemma. Let $\boldsymbol{\phi}$ be the vector with components

$$\varphi_i = \operatorname{sign} u_i \cdot \max(|u_i| - k, 0)$$

for $i = 1, 2$ and for $k \geqq k_0 > \sup_\Omega |\boldsymbol{u}_0|$. Then

$$\int_\Omega \boldsymbol{u} D\boldsymbol{u}\boldsymbol{\phi} dx = \int_\Omega \boldsymbol{u} D\boldsymbol{\phi}\boldsymbol{\phi} dx = 0 ,$$

so that we get from (38)

$$\tag{40} \begin{aligned} &\frac{1}{2}\int_\Omega |\boldsymbol{\phi}(x, t)|^2 dx + \int_0^t\int_\Omega |D\boldsymbol{\phi}|^2 dx d\tau \\ &\leqq ||\boldsymbol{g}||_{L^s(Q_T)} \cdot \left(\int_0^t\int_\Omega |\boldsymbol{\phi}|^{s/s-1} dx d\tau\right)^{(s-1)/s} . \end{aligned}$$

Let

$$A(k, t) = \{(x, \tau) \in Q_t : |u_1(x, \tau)| > k \text{ or } |u_2(x, \tau)| > k\}$$

and denote with $|A(k, t)|$ the Lebesgue measure of this set.

Since the integration in the last integral on the right-hand side of (40) is only performed over $A(k, t)$ we conclude with the help of Hölder's inequality

$$\begin{aligned} &\frac{1}{2} \cdot \int_\Omega |\boldsymbol{\phi}(x, t)|^2 dx + \int_0^t\int_\Omega |D\boldsymbol{\phi}|^2 dx d\tau \\ &\leqq ||\boldsymbol{g}||_{L^s(Q_T)} \cdot \left(\int_0^t\int_\Omega |\boldsymbol{\phi}|^4 dx d\tau\right)^{1/4} \cdot |A(k, t)|^{(3s-4)/4s} , \end{aligned}$$

hence

$$|\boldsymbol{\phi}|_{Q_t}^2 \leqq 4 \cdot ||\boldsymbol{g}||_{L^s(Q_T)} \left(\int_0^t\int_\Omega |\boldsymbol{\phi}|^4 dx d\tau\right)^{1/4} \cdot |A(k, t)|^{(3s-4)/4s} .$$

Using (8) twice and Hölder's inequality we obtain for $h > k \geqq k_0$

$$\tag{41} \begin{aligned} |h - k| \cdot |A(h, t)| &\leqq \int_{A(h,t)} |\boldsymbol{\phi}| dx d\tau \leqq \int_{A(k,t)} |\boldsymbol{\phi}| dx d\tau \\ &\leqq |\boldsymbol{\phi}|_{L^4(Q_t)} \cdot |A(k, t)|^{3/4} \leqq 4 \cdot c^2 \cdot ||\boldsymbol{g}||_{L^s(Q_T)} \cdot |A(k, t)|^{3/2-1/s} , \end{aligned}$$

where c is the constant in (8).

Now, we can apply a lemma due to Stampacchia [8, Lemma 4.1] to deduce

$$\tag{42} \sup_{Q_t} |\boldsymbol{u}| \leqq \sqrt{2}k_0 + c_1 \cdot c^2 \cdot ||\boldsymbol{g}||_{L^s(Q_T)} \cdot |A(k_0, t)|^{1/2-1/s}$$

where the constant c_1 depends on s.

On the other hand, it is evident that

$$|A(k_0, t)| \leqq k_0^{-4} \cdot \int_{Q_t} |\boldsymbol{u}|^4 dx d\tau .$$

Thus, (39) is proved.

For the prove of the exponential decay, we start with the equations (30). Let

$$v = we^{\lambda t} ,$$

for $\lambda > 0$. Then, v satisfies

$$\dot{v} - \Delta v - \lambda v + uDv + D\tilde{q} = \tilde{g} , \quad v(0) = o , \tag{43}$$

where we have set

$$\tilde{q} = qe^{\lambda t} \text{ and } \tilde{g} = ge^{\lambda t} .$$

Multiplying (43) with v we deduce

$$\begin{aligned} \frac{1}{2} \cdot \int_\Omega |v(x, t)|^2 dx + \int_0^t\int_\Omega |Dv|^2 dx d\tau - \lambda \int_0^t\int_\Omega |v|^2 dx d\tau \\ \leqq \int_0^t \Big(\int_\Omega |\tilde{g}|^2 dx\Big)^{1/2} \Big(\int_\Omega |v|^2 dx\Big)^{1/2} d\tau . \end{aligned} \tag{44}$$

Since Ω is a bounded domain, it follows that for small λ the estimate

$$\begin{aligned} \int_\Omega |v(x, t)|^2 dx + \int_0^t\int_\Omega |Dv|^2 dx d\tau \\ \leqq c \cdot \int_0^t \Big(\int_\Omega |\tilde{g}|^2 dx\Big)^{1/2} \Big(\int_\Omega |v|^2 dx\Big)^{1/2} d\tau , \end{aligned} \tag{45}$$

is valid, where $c = c(\lambda)$.

We therefore conclude

$$|v|_{Q_t} \leqq c \cdot \int_0^t \Big(\int_\Omega |\tilde{g}|^2 dx\Big)^{1/2} d\tau . \tag{46}$$

The right-hand side of this inequality is bounded provided

$$\int_0^t \Big(\int_\Omega |f|^2 dx\Big)^{1/2} \cdot e^{\lambda\tau} d\tau \leqq \text{const} \tag{47}$$

in view of the definition of g.

If (47) holds uniformly for all $t \geqq 0$, we obtain from (46)

$$|v|_{Q_\infty} \leqq \text{const} \tag{48}$$

and

$$\int_0^\infty \int_\Omega |v|^2 dx d\tau \leqq \text{const} , \tag{49}$$

since Ω is bounded.

Thus, we deduce from (43) with the help of Theorem 3

$$\|\boldsymbol{v}\|_{W_2^{2,1}(Q_\infty)} + \|D\tilde{q}\|_{L^2(Q_\infty)} \leqq \text{const} \tag{50}$$

having in mind e.g.. that

$$\boldsymbol{u} \in L^r(Q_\infty) \quad \text{for all} \quad 2 \leqq r \leqq \infty \,. \tag{51}$$

We then proceed as in the proof of Theorem 4. Differentiating (43) with respect to t yields

$$\dot{\boldsymbol{v}} - \Delta\dot{\boldsymbol{v}} - \lambda\dot{\boldsymbol{v}} + \boldsymbol{v}D\dot{\boldsymbol{v}} + \dot{\boldsymbol{u}}D\boldsymbol{v} + D\dot{\tilde{q}} = \dot{\tilde{\boldsymbol{g}}} \tag{52}$$

from which we obtain

$$\begin{aligned}&\int_\Omega |\dot{\boldsymbol{v}}(x, t)|^2 dx + \int_0^t\int_\Omega |D\dot{\boldsymbol{v}}|^2 dx d\tau \\ &\leqq c\left\{\int_0^t\int_\Omega |\dot{\boldsymbol{u}}|^2 \cdot |\boldsymbol{v}|^2 dx d\tau + \int_0^t\left(\int_\Omega |\dot{\tilde{\boldsymbol{g}}}|^2 dx\right)^{1/2} dx \cdot \left(\int_\Omega |\dot{\boldsymbol{v}}|^2 dx\right)^{1/2} d\tau\right\} ,\end{aligned} \tag{53}$$

where $c = c(\lambda)$.

Since $\dot{\boldsymbol{u}} \in L^s(Q_\infty)$ for some $s > 2$ and $\boldsymbol{v} \in L^r(Q_\infty)$ for all $2 \leqq r < \infty$ the first integral on the righ-thand side of this inequality is bounded uniformly in t. The integral involving $\dot{\tilde{\boldsymbol{g}}}$ is bounded provided

$$\boldsymbol{f}\cdot e^{\lambda t}, \; \dot{\boldsymbol{f}}e^{\lambda t} \in L^1(0, \infty; L^2(\Omega)) \tag{54}$$

in view of the definition of $\tilde{\boldsymbol{g}}$.

Proceeding then in the same way as in the proof of Theorem 4 we have thus proved,

THEOREM 5. *Suppose, that besides of the assumptions in Theorem 4, the conditions* (54) *are satisfied. Then, for small values of* λ *the estimates*

$$\sup_x |\boldsymbol{u}(x, t)| \leqq c\cdot e^{-\lambda t} \tag{55}$$

and

$$\left(\int_\Omega |D\boldsymbol{u}(x, t)|^2 dx\right)^{1/2} \leqq c\cdot e^{-\lambda t} \tag{56}$$

are valid for all $0 \leqq t < \infty$, *where the constant c depends on* λ, Ω, *and on the data.*

5. **The Cauchy problem.** The results of the preceding sections except that of the exponential decay are also valid for the

Cauchy problem,[5] where the equations (1) are to be satisfied in the whole plane $\mathbb{R}^2$. The reason is that the estimates for the corresponding linear problem hold as follows:

Let for $r \geqq 2$ the expressions $|\boldsymbol{u}|_{W_r^{2,1}(Q_T)}$ and $|\boldsymbol{u}_0|_{2-2/r,r}$ be defined through the assignments

$$|\boldsymbol{u}|^r_{W_r^{2,1}(Q_T)} = \int_0^T\int_\Omega |\boldsymbol{u}|^r dxdt + \int_0^T\int_\Omega \sum_{i,j=1}^2 |D^iD^j\boldsymbol{u}|^r dxdt$$

and

$$|\boldsymbol{u}_0|^r_{2-2/r,r} = \int_\Omega\int_\Omega \frac{|D\boldsymbol{u}_0(x) - D\boldsymbol{u}_0(y)|^r}{|x-y|^r} dxdy$$

if $r > 2$, and

$$|\boldsymbol{u}_0|^2_{1,2} = \int_\Omega |D\boldsymbol{u}_0|^2 dx\ .$$

Note that in our case $\Omega = \mathbb{R}^2$.

Then, for the solution of the linear Cauchy problem the estimate

$$(57)\qquad ||\boldsymbol{u}|_{W_r^{2,1}(Q_T)} + ||Dp||_{L^r(Q_T)} \leqq c\cdot\{|\boldsymbol{u}_0|_{2-2/r,r} + ||\boldsymbol{f}||_{L^r(Q_T)}\}$$

is valid with some constant depending only on r. It is the same estimate as for solutions to the heat equation (cf. [7]). The proof of (57) is rather simple since the pressure can easily be expressed with the help of a Newton potential in this case. The Calderon-Zygmund inequalities and the estimates for solutions to the heat equation then yield the result.

Moreover, the Nirenberg-interpolation-theorem which we used so extensively above also holds in $\Omega = \mathbb{R}^2$ involving only derivatives of the highest order in the respective norms, e.g., the estimate (10) is valid with $||\boldsymbol{u}||^2_{2,2}$ replaced by $\sum_{i,j=1}^2 \int_\Omega |D^iD^j\boldsymbol{u}|^2 dx$.

Therefore, the Theorems 2 and 3 are also valid in this case without any change in the proofs, if we observe that the estimates should be read as indicated in (57).

Since the estimates hold uniformly in t we conclude from Lemma 5 that the following theorem is valid.

THEOREM 6. *Let $\boldsymbol{u}_0 \in L^\infty(\mathbb{R}^2) \cap J_0(\mathbb{R}^2)$, $D\boldsymbol{u}_0 \in L^2(\mathbb{R}^2)$, and let $|\boldsymbol{u}_0|_{2-2/r,r}$ be finite for some $r>2$. Assume moreover that $\boldsymbol{f} \in L^1(0, \infty; L^2(\mathbb{R}^2)) \cap L^2(Q_\infty) \cap L^r(Q_\infty)$. Then, the solution $\boldsymbol{u}$, p of the equations* (1) *satisfies the relations*

[5] We shall only prove a priori estimates. For the existence of a solution we refer to a forthcoming paper treating the Cauchy problem in arbitrary dimension.

(58) $$|\boldsymbol{u}|_{Q_\infty} \leqq \text{const},$$

(59) $$|\boldsymbol{u}|_{W_s^{2,1}(Q_\infty)} + \|Dp\|_{L^s(Q_\infty)} \leqq \text{const}$$

for all $2 \leqq s \leqq r$, *and*

(60) $$\|\boldsymbol{u}\|_{L^\infty(Q_\infty)} \leqq \text{const} .$$

6. Continuous dependence on the data. We shall show that the solutions of the Navier-Stokes-equations depend continuously on the data in the norms of the spaces $W_2^{2,1}(Q_T)$ and $L^\infty(Q_T)$. The value plus infinity for T is allowed.

THEOREM 7. *Let* $\boldsymbol{u}_i$, p_i, $i = 1, 2$, *be solutions of the Navier-Stokes-equations to the data* $\boldsymbol{u}_{0i}$ *and* $\boldsymbol{f}_i$, *and let* $\boldsymbol{u}$, p, $\boldsymbol{u}_0$, *and* $\boldsymbol{f}$ *be the differences of the corresponding terms.*

Then, the following estimates are valid

(61) $$\begin{aligned}|\boldsymbol{u}|_{Q_T}^2 \leqq c_1\Big\{\int_\Omega |\boldsymbol{u}_0|^2 dx + \int_0^T\Big(\int_\Omega |\boldsymbol{f}|^2 dx\Big)^{1/2} dt \,. \\ \cdot\Big(\int_\Omega |\boldsymbol{u}_0|^2 dx + \int_0^T\Big(\int_\Omega |\boldsymbol{f}|^2 dx\Big)^{1/2} dt \cdot \exp\Big(\int_0^T\int_\Omega |\boldsymbol{u}_2|^4\Big)\Big)\Big\} \,,\end{aligned}$$

(62) $$\|\boldsymbol{u}\|_{W_2^{2,1}(Q_T)}^2 + \|Dp\|_{L^2(Q_T)} \leqq c_2 \cdot \Big\{|\boldsymbol{u}|_{Q_T}^2 + \int_\Omega |D\boldsymbol{u}_0|^2 dx + \int_0^T\int_\Omega |\boldsymbol{f}|^2 dxdt\Big\},$$

and

(63) $$\sup_{Q_T} |\boldsymbol{u}| \leqq \sqrt{2}k_0 + c_3 \cdot k_0^{-\alpha} \cdot |\boldsymbol{u}|_{Q_T}^\beta \,,$$

where c_1 *is a numerical constant,* c_2 *depends on* Ω *and* $\|\boldsymbol{u}_i\|_{W_2^{2,1}(Q_T)}$, $i = 1, 2$, c_3 *depends on* $\|\boldsymbol{u}_i\|_{W_r^{2,1}(Q_T)}$ *and* $\|Dp_i\|_{L^r(Q_T)}$, k_0 *is any positive number greater than* $\|\boldsymbol{u}_0\|_{L^\infty(\Omega)}$, *and* α *and* β *are positive numbers depending on* r.

We omit the proof of the theorem since the estimates either follow directly from the preceding theorems and their proofs, or can easily be deduced with the help of similar techniques. We only note that to prove (62) one has to estimate an integral like $\int_0^t\int_\Omega |\boldsymbol{u}_1|^2 \cdot |D\boldsymbol{u}|^2 dxd\tau$ as follows

(64) $$\begin{aligned}\int_0^t\int_\Omega |\boldsymbol{u}_1|^2 \cdot |D\boldsymbol{u}|^2 dxd\tau &\leqq \|\boldsymbol{u}_1\|_{L^6(Q_t)}^2 \cdot \|D\boldsymbol{u}\|_{L^3(Q_t)}^2 \\ &\leqq \|\boldsymbol{u}_1\|_{L^6(Q_t)}^2 \cdot \|D\boldsymbol{u}\|_{L^4(Q_t)}^{2a} \cdot \|D\boldsymbol{u}\|_{L^2(Q_t)}^{2(1-a)}\end{aligned}$$

with some appropriate number $a \in (0, 1)$.

As a corollary we obtain

THEOREM 8. *Let the data u_{0i} and f_i, $i = 1, 2$, be such that the constants in the preceding theorem are finite. Then, the expressions on the left-hand side of the estimates* (61), (62), *and* (63) *tend to zero if u_{20} converges to u_{01} in $L^\infty(\Omega) \cap H_0^{1,2}(\Omega)$ and f_2 to f_1 in $L^2(Q_T) \cap L^1(0, T; L^2(\Omega))$. T might be infinite.*

Proof. We only need to prove the third assertion: Since in view of (61) $|u|_{Q_T}$ tends to zero we conclude from (63)

$$\limsup \,(\sup_{Q_T} |u|) \leqq \sqrt{2}\, k_0 \,, \tag{65}$$

where k_0 can be an arbitrary positive number for $\lim \|u_0\|_{L^\infty(\Omega)} = 0$. Letting k_0 go to zero we obtain the result.

7. **On the attainability of stationary solutions.** It is well-known that solutions of the stationary Navier-Stokes-equations

$$\begin{aligned} -\Delta v + vDv + Dp &= f \,, \\ \operatorname{div} v &= 0 \,, \\ v|_{\partial\Omega} &= o \end{aligned} \tag{66}$$

exist and are of class $H^{2,2}(\Omega)$ provided that $f \in L^2(\Omega)$ and that Ω is a bounded open set with $\partial\Omega \in C^2$. Physically, stationary solutions are only of interest if they are obtained as the limit of instationary solutions if t goes to infinity. We shall show in the following that this is always the case if only $\left(\int_\Omega |f|^2 dx\right)^{1/2}$ is sufficiently small depending on Ω and the viscosity.

THEOREM 9. *Let u be a solution of the instationary Navier-Stokes-equations corresponding to the data u_0 and f, where $u_0 \in H^{2-2/r,r}(\Omega)$ for some $r > 2$, and where $f = f(x) \in L^2(\Omega)$. Then, if t goes to infinity u tends to a solution v of* (66), *which will therefore be unique in this case, provided $\left(\int_\Omega |f|^2 dx\right)^{1/2}$ is sufficiently small. For the difference $w = u - v$ the estimates*

$$\left(\int_\Omega |Dw(x, t)|^2 dx\right)^{1/2} \leqq c\cdot e^{-\lambda t} \tag{67}$$

and

$$\sup_\Omega |w(x, t)| \leqq c\cdot e^{-\lambda t} \tag{68}$$

are valid, where the constants depend on Ω and the data.

Proof. Let v *be a solution of* (66) *and let* $w = u - v$. w satis-

fies the equations

$$\begin{aligned} \dot{w} - \Delta w + wDw + vDw + wDv + Dp &= o \\ w|_{\partial\Omega} &= o\ , \\ w(0) = u_0 - w &= w_0\ . \end{aligned} \tag{69}$$

Multiplying (69) with w we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega |w(x,t)|^2 dx + \int_0^t\int_\Omega |Dw|^2 dx d\tau \\ &\leqq \frac{1}{2}\int_\Omega |w_0|^2 dx + \int_0^t \left(\int_\Omega |w|^4 dx\right)^{1/2} \cdot \left(\int_\Omega |Dv|^2 dx\right)^{1/2} d\tau\ . \end{aligned} \tag{70}$$

But, from the Sobolev imbedding theorem we obtain

$$\left(\int_\Omega |w|^4 dx\right)^{1/2} \leqq c_1 \cdot \int_\Omega |Dw|^2 dx\ , \tag{71}$$

and from (66)

$$\begin{aligned} \int_\Omega |Dv|^2 dx &\leqq \left(\int_\Omega |f|^2 dx\right)^{1/2} \cdot \left(\int_\Omega |v|^2 dx\right)^{1/2} \\ &\leqq c_2 \cdot \left(\int_\Omega |f|^2 dx\right)^{1/2} \cdot \left(\int_\Omega |Dv|^2 dx\right)^{1/2} \end{aligned} \tag{72}$$

or

$$\left(\int_\Omega |Dv|^2 dx\right)^{1/2} \leqq c_2 \cdot \left(\int_\Omega |f|^2 dx\right)^{1/2}\ . \tag{73}$$

Assuming therefore

$$\left(\int_\Omega |f|^2 dx\right)^{1/2} \leqq \frac{1}{2}(c_1 \cdot c_2)^{-1} \tag{74}$$

we conclude from (70)

$$\int_\Omega |w(x,t)|^2 dx + \int_0^t\int_\Omega |Dw|^2 dx d\tau \leqq \int_\Omega |w_0|^2 dx \tag{75}$$

from which we obtain

$$|w|^2_{Q\infty} \leqq 2 \cdot \int_\Omega |w_0|^2 dx\ . \tag{76}$$

We can now argue as in the proof of Theorem 3 to deduce

$$w \in W_2^{2,1}(Q_\infty), \quad Dp \in L^2(Q_\infty)\ , \tag{77}$$

namely,

$$\begin{aligned}
&\|\boldsymbol{w}\|^2_{W_2^{2,1}(Q_t)} + \|Dp\|^2_{L^2(Q_t)} \\
(78)\qquad &\leqq c\cdot\left\{\int_\Omega |D\boldsymbol{w}_0|^2 dx + \int_0^t\int_\Omega |\boldsymbol{w}|^2 |D\boldsymbol{w}|^2 dx d\tau + \int_0^t\int_\Omega |\boldsymbol{v}|^2\cdot|D\boldsymbol{w}|^2 dx d\tau \right.\\
&\left. + \int_0^t\int_\Omega |\boldsymbol{w}|^2\cdot|D\boldsymbol{v}|^2 dx d\tau\right\} .
\end{aligned}$$

Since $\boldsymbol{v}$ is bounded the only difficulty arises from the last integral on the right-hand side. But, using the Sobolev imbedding theorem twice we conclude

$$\begin{aligned}
&\int_0^t\int_\Omega |\boldsymbol{w}|^2\cdot|D\boldsymbol{v}|^2 dx d\tau \leqq \int_0^t\left(\int_\Omega |\boldsymbol{w}|^4 dx\right)^{1/2}\cdot\left(\int_\Omega |D\boldsymbol{v}|^4 dx\right)^{1/2} d\tau \\
(79)\qquad &\leqq c\cdot\|\boldsymbol{v}\|^2_{2,2}\cdot\int_0^t\int_\Omega |D\boldsymbol{w}|^2 dx d\tau .
\end{aligned}$$

Following now the arguments of the proof of Theorem 3 we get (77).

Moreover, arguing as in the proofs of the Theorems 4 and 5, it can easily be checked that the results of those theorems are also valid for $\boldsymbol{w}$ and $\boldsymbol{w}e^{\lambda t}$ if λ is sufficiently small. The estimates (67) and (68) now follow immediately.

REMARK 2. If p and q are the pressures corresponding to $\boldsymbol{u}$ and $\boldsymbol{v}$ then $p' = p - q$ is the pressure corresponding to $\boldsymbol{w}$. In view of (77) we know $Dp' \in L^2(Q_\infty)$, and the same result holds also for $D(p'e^{\lambda t})$. $p'e^{\lambda t}$ is the pressure corresponding to $\boldsymbol{w}e^{\lambda t}$. Moreover, let $\eta = \eta(t)$ be a smooth function vanishing in neighborhood of zero and being identically equal to one for t greater than two.

Then,

$$\tilde{\boldsymbol{w}} = \boldsymbol{w}e^{\lambda t}\cdot\eta \text{ and } \tilde{p} = p'e^{\lambda t}\cdot\eta$$

satisfy an equation from which we can rather easily deduce, after having differentiated it with respect to t, an a priori bound for

$$\|\dot{\tilde{\boldsymbol{w}}}|_{W_2^{2,1}(Q_\infty)} + \|D\dot{\tilde{p}}\|_{L^2(Q_\infty)} .$$

For the proof we have only to use the already known estimates for $|\tilde{\boldsymbol{w}}|_{Q_\infty}$ and $\|\boldsymbol{w}\|_{W_2^{2,1}(Q_\infty)}$.

We therefore conclude

$$\begin{aligned}
&\sup_t \int_\Omega |D\tilde{p}(x,t)|^2 dx \leqq 2\cdot\left(\int_0^\infty\int_\Omega |D\dot{\tilde{p}}|^2 dx dt\right)^{1/2} \cdot \\
(80)\qquad &\cdot\left(\int_0^\infty\int_\Omega |D\tilde{p}|^2 dx dt\right)^{1/2} \leqq c^2
\end{aligned}$$

hence

$$\left(\int_\Omega |Dp'(x, t)|^2 dx\right)^{1/2} \leqq c \cdot e^{-\lambda t} \tag{81}$$

for all $t \geqq 2$.

Thus, we have an exponential decay not only for the velocities but also for the pressures.

Similar arguments are also applicable in the case of Theorem 5 provided the external force $\boldsymbol{f}$ is such that in addition to the old assumptions $\partial/\partial t(\boldsymbol{f}e^{\lambda t}\eta)$ is square integrable over Q_∞.

Appendix. Here, we shall show, how the L^p-estimates of Solonnikov valid for solutions of the three-dimensional Stokes-equations can be used to derive the same estimates in the two-dimensional case.

Let $\boldsymbol{u} = (u_1, u_2)$ be a solution of the equation

$$\begin{aligned} \dot{\boldsymbol{u}} - \Delta \boldsymbol{u} + Dp &= \boldsymbol{f}\,, \\ \operatorname{div} \boldsymbol{u} &= 0\,, \\ \boldsymbol{u}|_{\partial\Omega} &= \boldsymbol{o}\,, \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0 \end{aligned} \tag{A1}$$

in a cylinder $Q_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^2$ is a bounded open set with C^2-boundary. p is the corresponding pressure.

We extend (A1) to a three-dimensional problem by setting

$$\begin{aligned} \hat{\boldsymbol{u}}(x^1, x^2, x^3, t) &= (u_1(x^1, x^2, t),\ u_2(x^1, x^2, t), O), \\ \hat{p}(x^1, x^2, x^3, t) &= p(x^1, x^2, t)\,. \end{aligned}$$

$\hat{\boldsymbol{u}}_0$ and $\hat{\boldsymbol{f}}$ are similarly defined.

Then $\hat{\boldsymbol{u}}$ solves in $\hat{Q}_T = \hat{\Omega} \times (O, T)$, $\hat{\Omega} = \Omega \times \mathbb{R}$ the equation

$$\begin{aligned} \hat{\boldsymbol{u}} - \Delta \hat{\boldsymbol{u}} + D\hat{p} &= \hat{\boldsymbol{f}}\,, \\ \operatorname{div} \hat{\boldsymbol{u}} &= 0\,, \\ \hat{\boldsymbol{u}}|_{\partial\hat{\Omega}} &= \boldsymbol{o} \\ \hat{\boldsymbol{u}}(0) &= \hat{\boldsymbol{u}}_0\,. \end{aligned} \tag{A2}$$

Unfortunately, $\hat{\Omega}$ is unbounded so that the results of Solonnikov cannot be applied directly. Truncating the domain would yield a nonsmooth boundary, also.

Therefore, let ζ be a cut-off function, $0 \leqq \zeta \leqq 1$, $\zeta(0) = 1$, and set

$$\hat{\boldsymbol{v}}(x^1, x^2, x^3, t) = \hat{\boldsymbol{u}}(x^1, x^2, x^3, t) \cdot \zeta(x^3)\,.$$

Then, $\hat{\boldsymbol{v}}$ satisfies

$$\operatorname{div}\hat{\boldsymbol{v}} = D^i(\hat{u}_i\zeta) = D^i\hat{u}_i\zeta + \hat{u}_iD^i\zeta = 0 + u_3\cdot D^3\zeta = 0\ ,$$

since $u_3 = 0$, and

$$\begin{aligned}\Delta\hat{\boldsymbol{v}} &= \Delta\hat{\boldsymbol{u}}\cdot\zeta + 2D\hat{\boldsymbol{u}}D\zeta + \hat{\boldsymbol{u}}\Delta\zeta \\ &= \Delta\hat{\boldsymbol{u}}\zeta + \hat{\boldsymbol{u}}\Delta\zeta\ ,\end{aligned}$$

since $D\hat{\boldsymbol{u}}D\zeta = 0$.

Hence, $\hat{\boldsymbol{v}}$ solves

$$\text{(A3)}\qquad \begin{aligned}&\dot{\hat{\boldsymbol{v}}} - \Delta\hat{\boldsymbol{v}} + D(\hat{p}\zeta) = \hat{\boldsymbol{f}}\cdot\zeta + \hat{\boldsymbol{v}}\Delta\zeta + \hat{p}D\zeta\ ,\\ &\operatorname{div}\hat{\boldsymbol{v}} = 0\ ,\\ &\hat{\boldsymbol{v}}|_{\partial\hat{\Omega}} = \boldsymbol{o}\ ,\\ &\hat{\boldsymbol{v}}(0) = \hat{\boldsymbol{u}}_0\zeta\end{aligned}$$

and has compact support with respect to x^3.

Applying the estimates of Solonnikov [7, § 17] we obtain for $1 < r < \infty$, $r \neq 3/2$, and for all t, $0 \leqq t \leqq T$,

$$\text{(A4)}\qquad \begin{aligned}&\int_0^t\int_{\hat{\Omega}}|\hat{\boldsymbol{v}}|^r\,dxd\tau + \int_0^t\int_{\hat{\Omega}}|\Delta\hat{\boldsymbol{v}}|^r dxd\tau + \int_0^t\int_{\hat{\Omega}}|D(\hat{p}\zeta)|^r dxd\tau\\ &\leqq c\cdot\Big\{\|\hat{\boldsymbol{u}}_0\zeta\|^r_{2-2/r,r,\hat{\Omega}} + \int_0^t\int_{\hat{\Omega}}|\hat{\boldsymbol{f}}|^r|\zeta|^r dxd\tau\\ &\quad + \int_0^t\int_{\hat{\Omega}}|\hat{\boldsymbol{u}}|^r|\zeta|^r dxd\tau + \int_0^t\int_{\hat{\Omega}}|\hat{p}|^r|D\zeta|^r dxd\tau\Big\}\ ,\end{aligned}$$

where $c = c(r, T, \partial\Omega, \zeta)$.

To simplify the estimates we observe that e.g., $\int_{\hat{\Omega}}|\hat{\boldsymbol{v}}|^r dx$ is equal to

$$\int_\Omega|\boldsymbol{u}|^r dx\cdot\int_{-\infty}^{\infty}|\zeta|^r d\tau\ .$$

Thus,

$$\text{(A5)}\qquad \begin{aligned}&\int_0^t\int_\Omega|\boldsymbol{u}|^r dxd\tau + \int_0^t\int_\Omega|\Delta\boldsymbol{u}|^r dxd\tau\\ &\quad + \int_0^t\int_\Omega|Dp|^r dxd\tau \leqq c\cdot\Big\{\|\boldsymbol{u}_0\|^r_{2-2/r,r,\Omega}\\ &\quad + \int_0^t\int_\Omega|\boldsymbol{f}|^r dxd\tau + \int_0^t\int_\Omega|\boldsymbol{u}|^r dxd\tau + \int_0^t\int_\Omega|p|^r dxd\tau\Big\}\end{aligned}$$

with some new constant c.

To estimate the integral involving p on the right-hand side we go back to (A1) taking the divergence to obtain

$$\begin{aligned}&\Delta p = \operatorname{div}\boldsymbol{f} &&\text{in } \Omega\\ &\frac{\partial p}{\partial\nu} = \boldsymbol{f}\cdot\nu + \Delta\boldsymbol{u}\cdot\nu &&\text{on } \partial\Omega\ .\end{aligned}$$

Fixing p by requiring that $\int_{\partial\Omega} p dH_1 = 0$
we see that

(A6) $$p(x, t) = \int_\Omega N(x, y) \operatorname{div} \boldsymbol{f} dy + \int_{\partial\Omega} N(x, y)\{\boldsymbol{f}\cdot\nu + \Delta\boldsymbol{u}\cdot\nu\} dH_1 ,$$

where N is the Neumannkernel, e.g., if $\partial\Omega$ is straight $N(x, y) = c\cdot \log(|x - y|)$. In any case N satisfies

$$|DN(x, y)| \leqq \frac{c}{|x - y|} \quad \text{and} \quad |D^2N(x, y)| \leqq \frac{c}{|x - y|^2} ,$$

provided $\partial\Omega$ is of class C^2. The symbol D^2N means the second derivatives of N.

Integrating by parts in (A6) we conclude

$$p(x, t) = -\int_\Omega D_y N(x, y) \Delta \boldsymbol{u} dy - \int_\Omega D_y N(x, y) \boldsymbol{f} dy$$

from which we derive (cf. [7, Lemma 9] and observe that $n = 2$ in our case)

(A7) $$\begin{aligned} \int_\Omega |p|^r dx &\leqq \varepsilon \int_\Omega |\Delta\boldsymbol{u}|^r dx + c_\varepsilon \int_\Omega |D\boldsymbol{u}|^r dx \\ &+ c\cdot \int_\Omega |\boldsymbol{f}|^r dx \leqq \varepsilon \int_\Omega |\Delta\boldsymbol{u}|^r dx + \varepsilon \int_\Omega |\Delta\boldsymbol{u}|^r dx \\ &+ c'_\varepsilon \int_\Omega |\boldsymbol{u}|^r dx + c\cdot \int_\Omega |\boldsymbol{f}|^r dx , \end{aligned}$$

where we used Nirenberg's interpolation lemma e.g., to deduce the second inequality. ε is any positive number.

Inserting this estimate in (A5), where ε is appropriately chosen, we obtain

(A8) $$\begin{aligned} &\int_0^t\int_\Omega |\boldsymbol{u}|^r dx d\tau + \int_0^t\int_\Omega |\Delta\boldsymbol{u}|^r dx d\tau + \int_0^t\int_\Omega |Dp|^r dx d\tau \\ &\leqq c\cdot \left\{ ||\boldsymbol{u}_0||^r_{2-2/r,r,\Omega} + \int_0^t\int_\Omega |\boldsymbol{f}|^r + \int_0^t\int_\Omega |\boldsymbol{u}|^r dx d\tau \right\} . \end{aligned}$$

Now, using the simple estimate

$$\begin{aligned} \int_\Omega |\boldsymbol{u}(x, t)|^r dx &\leqq \int_\Omega |\boldsymbol{u}_0|^r + c\cdot \int_0^t\int_\Omega |\dot{\boldsymbol{u}}|\cdot|\boldsymbol{u}|^{r-1} dx d\tau \\ &\leqq \int_\Omega |\boldsymbol{u}_0|^r + \varepsilon \int_0^t\int_\Omega |\dot{\boldsymbol{u}}|^r dx d\tau + c_\varepsilon \int_0^t\int_\Omega |\boldsymbol{u}|^r dx d\tau \end{aligned}$$

and taking

$$||\boldsymbol{u}_0||_{L^r} \leqq c\cdot ||\boldsymbol{u}_0||_{2-2/r,r}$$

into account, we conclude from (A8)

$$\int_\Omega |\boldsymbol{u}(x, t)|^r dx \leqq c \cdot \left\{ \|\boldsymbol{u}_0\|^r_{2-2/r,r} + \int_0^t \int_\Omega |\boldsymbol{f}|^r dx d\tau + \int_0^t \int_\Omega |\boldsymbol{u}^r dx d\tau \right\} .$$

Gronwall's lemma then yields to

$$\int_\Omega |\boldsymbol{u}(x, \tau)|^r dx \leqq c \cdot \left\{ \int_0^t \int_\Omega |\boldsymbol{f}|^r dx d\tau + \|\boldsymbol{u}_0\|^r_{2-2/r,r} \right\} \cdot e^{ct}$$

for all $0 \leqq \tau \leqq t$.

Going back to (A8) with this estimate we obtain the final result.

REFERENCES

1. A. Friedman, *Partial differential equations*, New York: Holt, Reinhart and Winston, 1969.
2. E. Hopf, *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachrichten, **4** (1950/51), 213-231.
3. V. P. Ill'in, *The properties of some classes of differentiable functions of several variables defined in an n-dimensional region*, Amer. Math. Soc. Translations, Ser. 2, Vol. **81** (1969), 91-256 (translated from Trudy Mat. Inst. Steklov, **66** (1962), 227-363).
4. O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Second Edition, New York-London-Paris: Gordon and Breach, 1969.
5. O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and quasi-linear equations of parabolic type*, Providence, Rhode Island Math. Soc., Amer. 1968.
6. L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa, Ser. 3, **13** (1959), 115-162.
7. V. A. Solonnikov, *Estimates of the solutions of a nonstationary linearized system of Navier-Stokes equations*, Amer. Math. Soc. Translations, Ser. 2, Vol. **75** (1968), 1-116. (translated from Trudy Mat. Inst. Steklov, **70** (1964), 213-317).
8. G. Stampacchia, *Equations elliptiques du second ordre à coefficients discontinus*, Sém. Math. Sup. Université de Montréal, 1966.
9. W. von Wahl, *Instationary Navier-Stokes equations and parabolic systems*, Pacific J. Math., **72** (1977), 557-569.

Received August 15, 1977.

UNIVERSITÄT HEIDELBERG
69 HEIDELBERG, GERMANY