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THE LEBESGUE CONSTANTS FOR (f, d_n)-SUMMABILITY

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It is well-known that the Fourier series of a continuous periodic function need not be pointwise convergent. This fact is a consequence of the unboundedness of the Lebesgue constants, which are the norms of the partial sum operators. It is equally-known that the Fourier series of a continuous function is uniformly $(C, 1)$-summable to the value of the function. Thus, the question naturally arises as to which summability matrices are effective in the limitation of Fourier series of continuous functions. In this paper we consider a very general class of matrices, the (f, d_n) means, and show that their Lebesgue constants are unbounded. An interesting corollary is that the Fourier series of a continuous periodic function need not be everywhere almost convergent.

If $A = (a_{nk})$ is a regular summability matrix, the nth Lebesgue constant corresponding to A is defined by

$$L_n(A) = \frac{2}{\pi}\int_0^{\pi/2} \frac{\left|\sum_{k=0}^{\infty} a_{nk} \sin(2k+1)t\right|}{\sin t} dt . \tag{1.1}$$

The sequence $\{L_n(A)\}$ is of considerable importance in the theory of Fourier series in that the unboundedness of this sequence implies the existence of a continuous function whose Fourier series fails to be A-summable at a specified point [1, pp. 58-60]. Conversely, if

$$\sum_{k=0}^{\infty} k|a_{nk}| < \infty \quad (n = 0, 1, \cdots)$$

and if the sequence $\{L_n(A)\}$ is bounded, then the Fourier series of each function continuous on an interval $[a, b]$ is uniformly A-summable to the value of the function on $[a, b]$. Extensive study of the Lebesgue constants has been made by a number of authors including Livingston [4] for the Euler means, Ishiguro [2] and Newman [7] for Taylor summability, Lorch [5] for the Borel exponential and integral methods, Sledd [10] for Sonnenschein matrices, and Lorch and Newman for $[F, d_n]$ means [6], and for Hausdorff means [8].

The (f, d_n) means are defined as follows: Let f be a nonconstant function, analytic on the disc $|z| < R$ for some $R > 1$, and let $\{d_n\}$ be a sequence of complex numbers, such that for all n, $d_n \neq -f(1)$. The elements of the matrix A are then given by the relations

$$a_{00} = 1 , \quad a_{0k} = 0 \quad (k \geqq 1)$$

$$\prod_{j=1}^{n} \frac{f(z) + d_j}{f(1) + d_j} = \sum_{k=0}^{\infty} a_{nk} z^k . \tag{1.2}$$

This family of matrices was introduced by Smith [11] as a generalization of the $[F, d_n]$ means of Jakimovski [3], to which they reduce if $f(z) = z$. In case $d_j = 0$ for all j, (1.2) becomes

$$[f(z)]^n = \sum_{k=0}^{\infty} a_{nk} z^k ,$$

and A is called the Sonnenschein matrix generated by f [12]. The purpose of the paper shall be to derive an asymptotic expansion for the sequence $\{L_n(A)\}$ for a class of regular (f, d_n) matrices. In the final section, we shall demonstrate that the unboundedness of the Lebesgue constants for a particular (f, d_n) mean implies the existence of continuous functions whose Fourier series fail to be everywhere almost convergent.

2. Preliminaries. In addition to the assumptions made regarding the function f and the sequence $\{d_n\}$ we further assume that

(2.1) the Maclaurin coefficients of f are real and nonnegative;

$$|f(z)| < 1 \quad \text{for} \quad |z| \leqq 1 (z \neq 1) ; \tag{2.2}$$

$$f(1) = f'(1) = 1 , \quad \text{while} \quad f''(1) \neq 0 ; \tag{2.3}$$

$$d_n \geqq 0 \quad \text{for all} \quad n ; \tag{2.4}$$

$$\Sigma_n (1 + d_n)^{-1} = \infty . \tag{2.5}$$

Condition (2.5) is necessary for regularity of A, as is condition (2.2) in case $d_n = 0$ for all n. Moreover, conditions (2.1), (2.4), and (2.5) are sufficient for regularity [11]. The following two lemmas will be useful in §3. The first of these is due to Lorch and Newman [6].

LEMMA 2.1. *Let $|a_k| \leqq 1$ and $|b_k| \leqq 1$ for $k = 1, \cdots, n$, and let A be a positive constant. If $|a_k - b_k| \leqq Ac_k$ for $k = 1, \cdots, n$ then*

$$\left| \prod_{k=1}^{n} a_k - \prod_{k=1}^{n} b_k \right| \leqq A \sum_{k=1}^{n} c_k .$$

LEMMA 2.2. *Let K be a positive constant and let $\alpha, \beta \in [0, \pi/2]$. If $|e^{i\alpha} - e^{i\beta}| \leqq K$, then $|\alpha - \beta| \leqq K\pi/2$.*

Proof. $e^{i\alpha} - e^{i\beta} = 2i \exp[i(\alpha + \beta)/2] \sin[(\alpha - \beta)/2]$, so

$$|e^{i\alpha} - e^{i\beta}| = 2|\sin[(\alpha - \beta)/2]| \geqq \frac{2}{\pi}|\alpha - \beta| ,$$

and the lemma follows.

3. The asymptotic behavior of $\{L_n(A)\}$. According to (1.1),

$$L_n(A) = \frac{2}{\pi}\int_0^{\pi/2} \frac{|K_n|}{\sin t} dt ,$$

where

$$K_n = \sum_{k=0}^{\infty} a_{nk} \sin(2k + 1)t .$$

From (1.2) and (2.3) it follows that

$$K_n = \frac{1}{2i}\left\{e^{it} \prod_{j=1}^{n} \frac{f(e^{2it}) + d_j}{1 + d_j}\right\} - e^{-it} \prod_{j=1}^{n} \frac{f(e^{-2it}) + d_j}{1 + d_j}\Big\} .$$

Define

$$R_j e^{i\theta_j} = f(e^{2it}) + d_j$$
$$\rho_j e^{i\varphi_j} = f(e^{-2it}) + d_j .$$

The assumptions made about f cause its Taylor expansion about $z_0 = 1$ to be of the form

$$f(z) = z + a_2(z - 1)^2 + O(z - 1)^3 , \tag{3.1}$$

where $a_2 = f''(1)/2 > 0$ by (2.1). It follows that

$$R_j e^{i\theta_j} = e^{2it} + d_j - 4a_2 t^2 + O(t^3) \tag{3.2}$$

$$\rho_j e^{i\varphi_j} = e^{-2it} + d_j - 4a_2 t^2 + O(t^3) . \tag{3.3}$$

Since $d_j \geqq 0$ for all j, these relations imply that

$$\rho_j = R_j + O(t^3) \tag{3.4}$$

and

$$\varphi_j = -\theta_j + O\left(\frac{t^2}{1 + d_j}\right) . \tag{3.5}$$

Now (3.4) implies that

$$\left|\frac{R_j}{1 + d_j} - \frac{\rho_j}{1 + d_j}\right| \leqq \frac{Kt^3}{1 + d_j} ,$$

so that

$$\left|\prod_{j=1}^{n} \frac{R_j}{1 + d_j} - \prod_{j=1}^{n} \frac{\rho_j}{1 + d_j}\right| \leqq Kt^3 \sum_{j=1}^{n} (1 + d_j)^{-1} \equiv KH_n t^3 ,$$

by Lemma 2.1. It follows that

$$\begin{aligned} K_n &= \frac{1}{2i}\Big\{\Big(\prod_{j=1}^{n}\frac{R_j}{1+d_j}\Big)\exp\Big[i\Big(t+\sum_{j=1}^{n}\theta_j\Big)\Big] \\ &\quad -\Big(\prod_{j=1}^{n}\frac{\rho_j}{1+d_j}\Big)\exp\Big[i\Big(-t+\sum_{j=1}^{n}\varphi_j\Big)\Big]\Big\} \\ &= \frac{1}{2i}\prod_{j=1}^{n}\frac{R_j}{1+d_j}\Big\{\exp\Big[i\Big(t+\sum_{j=1}^{n}\theta_j\Big)\Big] \\ &\quad -\exp\Big[i\Big(-t+\sum_{j=1}^{n}\varphi_j\Big)\Big]\Big\} + O(H_n t^3) \\ &= \Big(\prod_{j=1}^{n}\frac{R_j}{1+d_j}\Big)\sin\Big[\frac{1}{2}\sum_{j=1}^{n}(\theta_j-\varphi_j)+t\Big] \\ &\quad \times\exp\Big[\frac{i}{2}\sum_{j=1}^{n}(\theta_j+\varphi_j)\Big] + O(H_n t^3)\,. \end{aligned}$$

Hence,

$$|K_n| = \Big(\prod_{j=1}^{n}\frac{R_j}{1+d_j}\Big)\Big|\sin\Big[\frac{1}{2}\sum(\theta_j-\varphi_j)+t\Big]\Big| + O(H_n t^3)\,.$$

Suppose that $0 < \xi < \pi/2$. Then

$$\begin{aligned} &\int_0^{\xi}\frac{|K_n|}{\sin t}dt \\ &= \int_0^{\xi}\prod_{j=1}^{n}\frac{R_j}{1+d_j}\frac{\Big|\sin\Big[\frac{1}{2}\sum(\theta_j-\varphi_j)+t\Big]\Big|}{\sin t}dt + O(H_n\xi^3)\,. \end{aligned} \tag{3.6}$$

We may replace $\sin t$ by t in the integral on the right of (3.6), introducing an error of $O(\xi)$.
Thus,

$$\begin{aligned} &\int_0^{\xi}\frac{|K_n|}{\sin t}dt \\ &= \int_0^{\xi}\prod_{j=1}^{n}\frac{R_j}{1+d_j}\frac{\Big|\sin\Big[\frac{1}{2}\sum(\theta_j-\varphi_j)+t\Big]\Big|}{t}dt + O(\xi) + O(H_n\xi^3)\,. \end{aligned}$$

Using the expansion

$$\begin{aligned} &\sin\Big[\frac{1}{2}\sum(\theta_j-\varphi_j)+t\Big] \\ &= \sin\Big[\frac{1}{2}\sum(\theta_j-\varphi_j)\Big]\cos t + \cos\Big[\frac{1}{2}\sum(\theta_j-\varphi_j)\Big]\sin t\,, \end{aligned}$$

we obtain

$$\int_0^\xi \prod_{j=1}^n \frac{R_j}{1+d_j} \cdot \frac{1}{t}\left|\sin\left[\frac{1}{2}\sum(\theta_j-\varphi_j)+t\right]-\sin\left[\frac{1}{2}\sum(\theta_j-\varphi_j)\right]\right|dt$$
$$\leqq \int_0^\xi \frac{1-\cos t}{t}dt + \int_0^\xi \frac{\sin t}{t}dt = O(\xi)\ .$$

Therefore,

$$\int_0^\xi \frac{|K_n|}{\sin t}dt$$
$$= \int_0^\xi \prod_{j=1}^n \frac{R_j}{1+d_j} \cdot \frac{\left|\sin \frac{1}{2}\sum(\theta_j-\varphi_j)\right|}{t}dt + O(\xi) + O(H_n\xi^3)\ . \tag{3.7}$$

We now estimate $\left(\int_\xi^{\pi/2} |K_n|/\sin t\right)dt$. To this end, define $\mathrm{Re}^{i\theta} = f(e^{2it})$. From (3.1) it follows that

$$R = 1 - 4a_2t^2 + O(t^3)\ ,$$

or

$$R = 1 - 4a_2t^2(1 + t\psi(t))\ , \tag{3.8}$$

where ψ is bounded in a neighborhood of $t = 0$. Now

$$\begin{aligned} R_j^2 &= |f(e^{2it}) + d_j|^2 \\ &= R^2 + 2Rd_j\cos\theta + d_j^2 \\ &\leqq R^2 + 2Rd_j + d_j^2\ . \end{aligned}$$

Substitution of right-hand side of (3.8) for R yields

$$\begin{aligned} R_j^2 \leqq\ & 1 + 16a_2^2t^4(1+t\psi(t))^2 - 8a_2t^2(1+t\psi(t)) \\ & + d_j^2 + 2d_j - 8a_2d_jt^2(1+t\psi(t))\ . \end{aligned} \tag{3.9}$$

If t is sufficiently close to zero, then $|t\psi(t)| < 1/2$, and the right-hand side of (3.9) is dominated by

$$(1+d_j)^2 - (1+d_j)[4a_2t^2 - 36a_2^2t^4]\ . \tag{3.10}$$

If we further insist that t be less than $(18a_2)^{-1/2}$, then it follows that $36a_2^2t^4 < 2a_2t^2$, so (3.9) and (3.10) combine to yield

$$R_j^2 \leqq (1+d_j)^2 - 2a_2t^2(1+d_j)\ . \tag{3.11}$$

Using (3.11) and the familiar inequality $1 + x < e^x$, valid for real x, we obtain

$$\left(\frac{R_j}{1+d_j}\right)^2 \leqq 1 - \frac{2a_2t^2}{1+d_j} \leqq \exp\left(-\frac{2a_2t^2}{1+d_j}\right),$$

so that

$$\prod_{j=1}^{n} \frac{R_j}{1+d_j} \leqq \exp[-a_2 H_n t^2] .$$

Analogously, one shows that

$$\prod_{j=1}^{n} \frac{\rho_j}{1+d_j} \leqq \exp[-a_2 H_n t^2] .$$

Hence,

$$|K_n| \leqq \frac{1}{2}\left[\prod \frac{R_j}{1+d_j} + \prod \frac{\rho_j}{1+d_j}\right] \leqq \exp[-a_2 H_n t^2] ,$$

and

$$\int_{\xi}^{\pi/2} \frac{|K_n|}{\sin t} dt \leqq \frac{\pi}{2} \int_{\xi}^{t} \frac{\exp[-a_2 H_n t^2]}{t} dt = O[\xi^{-1} \exp(-H_n \xi^2)] .$$

This estimate, together with (3.7) gives

$$\begin{aligned} \int_0^{\pi/2} \frac{|K_n|}{\sin t} dt &= \int_0^{\xi} \prod_{j=1}^{n} \frac{R_j}{1+d_j} \frac{\left|\sin \frac{1}{2} \sum (\theta_j - \varphi_j)\right|}{t} dt \\ &\quad + O(\xi) + O(H_n \xi^3) + O[\xi^{-1} \exp(-H_n \xi^2)] . \end{aligned} \tag{3.12}$$

We now replace the product appearing in the integral on the right of (3.12) by a more managable expression. By equation (3.2)

$$f(e^{2it}) + d_j = 1 + d_j + 2it - (2 + 4a_2)t^2 + O(t^3) ,$$

so that

$$\begin{aligned} \frac{f(e^{2it}) + d_j}{1+d_j} &= 1 + \frac{2it}{1+d_j} - \frac{2+4a_2}{1+d_j} t^2 + O\left(\frac{t^3}{1+d_j}\right) \\ &= \exp\left\{\frac{2it}{1+d_j} - \left[\frac{4a_2}{1+d_j} + \frac{2d_j}{(1+d_j)^2}\right] t^2\right\} + O\left(\frac{t^3}{1+d_j}\right) . \end{aligned}$$

By Lemma 2.1, it follows that

$$\prod_{j=1}^{n} \frac{f(e^{2it}) + d_j}{1+d_j} = \exp\{2iH_n t - 4a_2 H_n t^2 - S_n t^2\} + O(H_n t^3) , \tag{3.13}$$

where $S_n = 2 \sum_{j=1}^{n} d_j (1+d_j)^{-2}$. Hence,

$$\prod_{j=1}^{n} \frac{R_j}{1+d_j} = \exp\{-(4a_2 H_n + S_n) t^2\} + O(H_n t^3) ,$$

and (3.12) becomes

$$\int_0^{\pi/2} \frac{|K_n|}{\sin t} dt = \int_0^{\xi} \exp\left[-(4a_2H_n + S_n)t^2\right] \frac{\left|\sin \frac{1}{2} \sum (\theta_j - \varphi_j)\right|}{t} dt$$
$$+ O(\xi) + O(H_n\xi^3) + O[\xi^{-1} \exp(-H_n\xi^2)] .$$

From (3.13) it follows that

$$\exp(i \textstyle\sum \theta_j) = \exp(2iH_nt) + O(H_nt^3) .$$

Lemma 2.2 now implies that

$$\textstyle\sum \theta_j = 2H_nt + O(H_nt^3) .$$

In similar fashion it is shown that

$$\textstyle\sum \varphi_j = -2H_nt + O(H_nt^3) .$$

Hence, $\sin 1/2 \sum (\theta_j - \varphi_j) = \sin 2H_nt + O(H_nt^3)$, and

$$\int_0^{\pi/2} \frac{|K_n|}{\sin t} dt = \int_0^{\xi} \exp\left[-(4a_2H_n + S_n)t^2\right] \frac{|\sin 2H_nt|}{t} dt \tag{3.14}$$
$$+ O(\xi) + O(H_n\xi^3) + O[\xi^{-1} \exp(-H_n\xi^2)] .$$

Here, the interval of integration may be extended from $[0, \xi]$ to $[0, \pi/2]$ with an error of $O[\xi^{-1} \exp(-H_n\xi^2)]$. This having been done, we now let $\xi = H_n^{-3/8}$. Since $H_n \to \infty$ as $n \to \infty$, all of the error terms in (3.14) become $o(1)$. We now make the substitution $u_n = 2H_n$ and $s_n = 4a_2H_n + S_n$. Thus

$$L_n(A) = \frac{2}{\pi} \int_0^{\pi/2} \exp(-s_nt^2) \frac{|\sin u_nt|}{t} dt + o(1) .$$

Since, for our choice of u_n and s_n, $s_n \to \infty$ and $u_n^2/s_n \to \infty$, the derivation of [6; §5] may be applied, yielding

$$L_n(A) = \frac{2}{\pi^2} \log \frac{u_n^2}{s_n} + \alpha + o(1) , \tag{3.15}$$

where

$$\alpha = -\frac{2}{\pi^2} C + \frac{2}{\pi} \int_0^1 \frac{\sin t}{t} dt - \frac{2}{\pi} \int_1^{\infty} \frac{1}{t} \left\{ \frac{2}{\pi} - |\sin t| \right\} dt ,$$

and C denotes Euler's constant. In terms of H_n and S_n, our expansion takes the form

$$L_n(A) = \frac{2}{\pi^2} \log\left(\frac{4H_n^2}{4a_2H_n + S_n}\right) + \alpha + O(1) ,$$

from which it is clear that $\{L_n(A)\}$ is an unbounded sequence. We note in conclusion that if $d_j = 0$ for all j, then $S_n = 0$ and $H_n = n$, so that for Sonnenschein methods we have

$$L_n(A) = \frac{2}{\pi^2} \log \frac{n}{a_2} + \alpha + o(1) ,$$

which is the result obtained by Sledd [10].

4. **A special case.** A regular matrix A is said to be strongly regular provided every almost convergent sequence is A-summable. Lorentz [9] has shown that a necessary and sufficient condition for strong regularity of a regular matrix $A = (a_{nk})$ is that

$$\lim_n \sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}| = 0 . \tag{4.1}$$

If we take $f(z) = e^{z-1}$ and $d_n = 0$ for all n, then the resulting (f, d_n) mean is the Borel matrix:

$$a_{nk} = e^{-n}\frac{n^k}{k!} .$$

Now

$$\begin{aligned}\sum_{k=1}^{\infty} |a_{nk} - a_{n,k+1}| &= e^{-n}\left\{\sum_{k=1}^{n-1} \frac{n^k}{(k+1)!}[n-(k+1)] + \sum_{k=n}^{\infty} \frac{n^k}{(k+1)!}[(k+1) - n]\right\} \\ &= e^{-n}\left[\sum_{k=1}^{n-1} \frac{n^{k+1}}{(k+1)!} - \sum_{k=1}^{n-1} \frac{n^k}{k!} + \sum_{k=n}^{\infty} \frac{n^{k+1}}{(k+1)!}\right] \\ &= e^{-n}\left[2\left(\frac{n^n}{n!}\right) - n\right] .\end{aligned}$$

By Stirlings formula, $n^n/n!e^n = O(n^{-1/2})$, so that $e^{-n}[2(n^n/n!) - n] \to 0$ as $n \to \infty$, and A is strongly regular. It follows that there exist continuous functions whose Fourier series fail to be almost convergent; for if this were not the case, then the Borel matrix would sum the Fourier series of each continuous function, contrary to the unboundedness of the Borel-Lebesgue constants.

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