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Let f be defined on T^n and have an absolutely convergent Fourier series

$$f(e^{i\sigma}) = \sum f_m e^{im \cdot \sigma}.$$

Set $\|f\| = \sum |f_m|$. In this paper sufficient conditions for $\|f^k\| = o(1)$, as $k \rightarrow \infty$, are obtained.

THEOREM. *Let f be defined on T^n , have an absolutely convergent Fourier series and satisfy*

$$(1) \quad |f(e^{i\sigma})| \leq 1 \text{ for all } \sigma.$$

If for each σ such that $|f(e^{i\sigma})| = 1$ there exists a rotation λ of R^n , a polynomial ρ such that $\operatorname{Re} \rho(\tau) > 0$ for all $\tau \neq \vec{0}$, an n -tuple p of positive integers such that $\rho(\langle r^{1/p_i} \tau_i \rangle) = r \rho(\tau)$ for all $r > 0$, and a function γ in $C^m(R^n)$, $m = \max(n + 1, p_1, p_2, \dots, p_n)$, such that

$$(2) \quad \gamma(\tau) = o\left(\sum_{i=1}^n \tau_i^{p_i}\right), \quad \tau \longrightarrow \vec{0},$$

and if for all τ in some R^n -neighborhood of $\vec{0}$

$$(3) \quad f(e^{i(\sigma + \lambda(\tau))}) = c \exp(\beta \cdot \tau i - (\rho + \gamma)(\tau)),$$

where $|c| = 1$, $\beta \in R^n$, then

$$(4) \quad \|f^k\| = o(1), \text{ as } k \longrightarrow \infty.$$

It is shown in § 2 that this theorem extends a result obtained by B. M. Schreiber in 1970.

1. Introduction. Let B and D denote the open and closed unit discs respectively. The problem of characterizing those functions f of n complex variables which are analytic on D^n and for which (4) holds has been solved only for $n = 1$. See [1, 2]. For the general case, n arbitrary, sufficient conditions on f and necessary conditions on f have been given [9, 6] and related problems have been studied [4, 5]. Unsolved, even in the case $n = 1$, is the problem of determining all functions analytic on B^n for which (4) holds. This problem is equivalent to the problem of determining all endomorphisms of the Banach algebra of power series of n complex

variables which converge absolutely on T^n . See [8, 7].

2. **Remarks, definitions and notation.** To see that the theorem in this paper extends Schreiber's result, Theorem 7.5 in [9], let $n = 2$ and consider $f(z) = h(z)g(z_1)1(z_2)$ where

$$\begin{aligned}
 h(z) &= z_1^2 z_2^2 - \left(\frac{z_1 - 1}{2}\right)^4 \left(\frac{z_2 - 1}{2}\right)^4, \\
 g(z_1) &= z_1^3 + \left(\frac{z_1 - 1}{2}\right)^6, \\
 1(z_2) &= z_2^6 + \left(\frac{z_2 - 1}{2}\right)^{12}.
 \end{aligned}$$

The function $f(e^{i\tau})$ has one absolute maximum, namely at $\tau = (0, 0)$, and about this maximum has a local expansion of the form

$$f(e^{i\tau}) = c \exp(\beta \cdot \tau i - a\tau_1^6 - b\tau_1^4 \tau_2^4 - d\tau_2^{12} + o(\tau_1^6 + \tau_2^{12})),$$

$\tau \rightarrow (0, 0)$, where $\text{Re}(a\tau_1^6 + b\tau_1^4 \tau_2^4 + d\tau_2^{12}) > 0$ for $\tau \neq (0, 0)$. (For details see [5, pp. 141-3].) Schreiber's result [9, p. 426] holds only for functions having about each absolute maximum a local expansion of the form

$$\begin{aligned}
 f(e^{i\tau}) &= c \exp(\beta \cdot \tau i - \sum_{j \cong m_1} a_j \tau_1^j - \sum_{j \cong m_2} b_j \tau_2^j - \sum_{j, k \cong M} d_{j, k} \tau_1^j \tau_2^k \\
 &\quad + O(\|\tau\|^M)),
 \end{aligned}$$

$\tau \rightarrow (0, 0)$, where $M = \max(m_1, m_2)$.

Define a polynomial ρ in n variables to be positive definite if $\rho(\tau) > 0$ for all $\tau \neq \bar{0}$ and to be q -homogeneous of index I if there exists an n -tuple q of positive integers and a positive number I such that for all $r > 0$ and all σ in R^n , $\rho(\langle r^{1/q_i} \sigma_i \rangle) = r^I \rho(\sigma)$.

Let T, Z, R , and C denote the unit circle, integers, real numbers and complex numbers respectively. For any set S let S^n denote the n -fold cartesian product of the set with itself. For any subset S of T^n let CS denote the complement of S in T^n . The letters σ, τ , and z denote points of R^n, R^n , and C^n respectively. Alternatively denote a point z of C^n by using vector notation $\langle z_i \rangle$, where z_i represents the i th component of z , and let e^z denote $\langle e^{z_i} \rangle$. Let $\bar{0}$ and $\bar{1}$ represent the origin of R^n and the identity of T^n respectively. The scalar product of two points σ and τ of R^n will be denoted $\sigma \cdot \tau$. Let δ_{ij} denote the Kronecker delta. Take products indexed by the empty set to be equal to one.

Let $B(\sigma, r)$ denote the set of points $e^{i\tau}$ of T^n such that the Euclidean distance of τ from σ is less than r . Let E_f denote the set of points $e^{i\sigma}$ of T^n such that $|f(e^{i\sigma})| = 1$. Let $L^1(R^n)$ and $A(R^n)$

denote the space of Lebesgue integrable functions on R^n and the space of their Fourier transforms respectively. Similarly define $L^1(Z^n)$ and $A(T^n)$. For any function ζ on T^n let $\text{supp } \zeta$ denote the support of ζ .

Finally, the letters a, b, c, d will denote absolute positive constants. The use of one such letter in two inequalities does not mean that the letter represents the same absolute constant in both inequalities. However, the use of such a letter in an inequality involving the indices k or m means that the constant represented by the letter is independent of k and m . The phrases "for k sufficiently large" and "for all m " will be omitted finitely many times from this paper.

3. Proof of theorem.

LEMMA 1. *Let ρ be a positive definite, q -homogeneous polynomial of index 1. Then q_j is an even integer for $1 \leq j \leq n$ and*

$$\sum_{j=1}^n \tau_j^{q_j} < c_n \rho(\tau), \quad \text{for all } \tau \neq 0.$$

Proof. Fix $j, 1 \leq j \leq n$. Let $h(t) = \rho(\langle t \delta_{ij} \rangle)$, for all t in R . Since ρ is q -homogeneous of index 1,

$$h(t) = t^{q_j} h(\langle \delta_{ij} \rangle)$$

for all $t > 0$ and hence for all t , h being a polynomial. Since h is positive definite, q_j is an even integer.

Assume that $\tau_j \neq 0$, let $r = \tau_j^{q_j}$ and define σ_i by $\tau = \langle r^{-1/q_i} \sigma_i \rangle$. Since ρ is q -homogeneous of index 1, $\rho(\tau) = r \rho(\sigma)$. Since $\sigma_j = 1$ and since ρ is positive definite there is a positive constant a_j such that $\rho(\sigma) > a_j$. Thus, $\rho(\tau) \geq a_j \tau_j^{q_j}$. Note that this inequality also holds if $\tau_j = 0$. Summing over j yields $n \rho(\tau) \geq \sum_{j=1}^n a_j \tau_j^{q_j}$, from which the lemma follows since each a_j is positive.

LEMMA 2. *Let f be defined on T^n and suppose that for some real number r in the interval $(0, \pi)$ f vanishes on $CB(\sigma, r)$. Let g be defined on R^n by*

$$\begin{aligned} g(\tau) &= f(e^{i(\sigma+\tau)}) \quad \text{if } |\tau| < r, \\ &= 0 \quad \quad \quad \text{if } |\tau| \geq r. \end{aligned}$$

Then $f \in A(T^n)$ if and only if $g \in A(R^n)$. Moreover, there are positive numbers depending only on r such that $a \|g\| \leq \|f\| < b \|g\|$.

Proof. Without loss of generality, assume that $\sigma = 0$. Then

Lemma 2 is the special case of Lemma 7.7 of [9] obtained by letting $H = \{0\}$.

The following lemma is well-known and therefore its proof is omitted.

LEMMA 3. Let λ be a rotation of R^n and g be a function defined on R^n . For each $\mu \in L^1(R^n)$

$$(5) \quad \widehat{\mu \circ \lambda} = \widehat{\mu} \circ \lambda$$

so that $g \in A(R^n)$ if and only if $g \circ \lambda \in A(R^n)$. Moreover,

$$\|g\| = \|g \circ \lambda\|.$$

LEMMA 4. There exists a constant $M = M(n)$ so that for any sequence $\{a_q\}_{q \in Z^n}$ of complex numbers and any constant b in C^n

$$(6) \quad \Sigma |a_u| \leq M(\pi \Sigma \nu(q, l) |a_q|^2)^{2^{-n-1}}, \quad n \geq 1,$$

where the sums are over all q in Z^n , the product is over all $l \in A^n \equiv \{0, 1\}^n$ and

$$\nu(q, l) = \prod_{j=1}^n (1 + |q_j - b_j|^2)^{1_j}.$$

Proof. See Lemma 3 of [6].

LEMMA 5. Let u and w be q -homogeneous polynomials with indices I_u and I_w respectively. Let σ be a point such that $u \cdot w(\sigma) \neq 0$. Then

$$|u/w|(\langle r^{1/q_i} \sigma_i \rangle) = 0(1), \text{ as } r \longrightarrow 0,$$

if and only if $I_u > I_w$.

Proof. Applying the definition of q -homogeneity to u and to w yields

$$|u/w|(\langle r^{1/q_i} \sigma_i \rangle) = r^{I_u - I_w} u(\sigma)/w(\sigma)$$

from which the lemma follows directly.

LEMMA 6. Let u be a q -homogeneous, positive definite polynomial of index I_u and v a polynomial in n variables such that

$$(7) \quad |v(\tau)| = 0(|u(\tau)|), \quad \tau \longrightarrow \vec{0}.$$

Then the index of q -homogeneity of each term of v is greater than

or equal to I_u .

Proof. It has been implicitly assumed in the statement of the lemma that each term of v is q -homogeneous. It is straightforward to prove that in fact each term of any polynomial is q -homogeneous and that a sum of terms having a given index of q -homogeneity is q -homogeneous with that same index. Therefore, if w is defined to be the sum of those terms of v having smallest index of q -homogeneity and if I_w denotes the index of w , it suffices to prove that $I_w \geq I_u$.

Fix a point σ such that $w(\sigma) \neq 0$. Since w is q -homogeneous, $\sigma \neq \bar{0}$. From (7) it follows that

$$(8) \quad |v/u|(\langle r^{1/q_i} \sigma_i \rangle) = 0(1), \text{ as } r \longrightarrow 0.$$

Since $v - w$ is a sum of terms each of index exceeding I_v and since $w(\sigma) \neq 0$ it follows that each term of $v - w$ upon division by w and evaluation at $\langle r^{1/q_i} \sigma_i \rangle$ is either zero or by Lemma 5 approaches 0 as $r \rightarrow 0$. Thus,

$$(9) \quad \left| \frac{v - w}{w} \right|(\langle r^{1/q_i} \sigma_i \rangle) = o(1), \text{ as } r \longrightarrow 0.$$

Since

$$\frac{w}{u} = \frac{v}{u} (1 + (v - w)/w)^{-1}$$

it follows from (8) and (9) that

$$\left| \frac{w}{u} \right|(\langle r^{1/q_i} \sigma_i \rangle) = 0(1), \text{ as } r \longrightarrow 0,$$

which by Lemma 5 implies that $I_u \leq I_w$ since $w(\sigma) \neq 0$ and $u(\sigma) \neq 0$, u being positive definite.

LEMMA 7. Given any finite, pairwise-disjoint collection $\{B(\sigma, 2r_\sigma)\}_{\sigma \in S}$ of open balls in T^n there exists a family $\{\zeta_\sigma\}_{\sigma \in S}$ of nonnegative functions defined on T^n such that for each $\sigma \in S$

(i) $\text{supp } \zeta_\sigma = \overline{B(\sigma, 2r_\sigma)}$,

(ii) $\zeta_\sigma \equiv 1$ on $B(\sigma, r)$,

(iii) $\zeta_\sigma(e^{i\tau}) \in C^\infty(\mathbb{R}^n)$

and such that

(iv) $\sum_{\sigma \in S} \zeta_\sigma \leq 1$ on T^n .

Proof. This lemma follows directly from Proposition 29 on page 254 of [3].

LEMMA 8. Let f be defined on T^n , have an absolutely convergent Fourier series and satisfy (1). Further assume that E_f is nonempty and finite. If $\{r_\sigma\}_{\sigma \in E_f}$ and $\{\zeta_\sigma\}_{\sigma \in E_f}$ satisfy (i) – (iv) of Lemma 7 and if, for each $\sigma \in E_f$, λ_σ is a rotation of R^n followed by translation by σ , then

$$\|f^k\| = 0(1), \quad k \longrightarrow \infty, \quad \text{if } \|(\zeta_\sigma \cdot f^k) \circ e^{i\lambda_\sigma}\| = 0(1), \quad k \longrightarrow \infty, \\ \text{for each } \sigma \in E_f.$$

Proof. This lemma is a direct consequence of Lemmas 2 and 3 of this paper and Theorem 3.3 of [9].

Proof of theorem. From (2), (3), and Lemma 1 it follows that for each point $e^{i\sigma}$ of the set E_f , which was defined in § 2,

$$(10) \quad |f(e^{i(\sigma+\lambda(\tau))})| \leq \exp\left(-c \sum_{j=1}^n \tau_j^{p_j}\right)$$

for all τ in some neighborhood of $\vec{0}$ and hence that E_f has no limit points. Since T^n is compact, E_f is finite. If $E_f = \emptyset$, then by the spectral radius formula $\|f^k\| \rightarrow 0$, as $k \rightarrow \infty$. So assume that $E_f \neq \emptyset$. Also, by Lemma 8 it suffices to assume that $E_f = \{\vec{0}\}$ and that λ is the identity. Thus, it suffices to prove that

$$\sum_{q \in \mathbb{Z}^n} |a_{q,k}| = 0(1), \quad k \longrightarrow \infty,$$

where the coefficients $a_{q,k}$ are defined by

$$f^k(e^{i\tau}) = \sum_{q \in \mathbb{Z}^n} a_{q,k} e^{i\tau \cdot q}.$$

For this purpose let $b = k\beta$ and $a_q = a_{q,k}$, where β is defined by (3), and show that the right side of (6) remains bounded, as k tends to ∞ , by showing for each l in \mathcal{L}^n that

$$(11) \quad \sum_{q \in \mathbb{Z}^n} \nu(q, l) |a_{q,k}|^2 = 0(k^{(\sum_{j \in J} 2 - \sum_{j=1}^n) (1/p_j)}),$$

as $k \rightarrow \infty$, where J denotes the set of integers j for which $l_j \neq 0$.

Choose $r_0 > 0$ so that (3) and (10) hold for all τ such that $|\tau| < r_0$. Let B denote the set $\{\sigma \in R^n: e^{i\sigma} \in B(1, r_0)\}$. Let \mathcal{D} denote the differential operator

$$\prod_{j \in J} \left(-i \frac{\partial}{\partial t_j} - b_j\right).$$

Recall that products indexed by the empty set are taken to be 1 so that if $J = \emptyset$, \mathcal{D} is taken to be the identity. Use Parseval's

equality to write

$$c \sum_{q \in \mathbb{Z}^n} \left(\prod_{j \in J} |q_j - b_j|^2 \right) |a_{q,k}|^2 = \int_{T^n} |\mathcal{D} f^k(e^{i\tau})|^2 d\tau$$

and conclude that (11) will follow from

$$(12) \quad \int_B |f^k(e^{i\tau})|^2 d\tau = O(k^{(2 \sum_{j \in J} - \sum_{j=1}^n)(1/p_j)}),$$

$k \rightarrow \infty$, and

$$\int_{T^n \setminus B} |f^k(e^{i\tau})|^2 d\tau = O(e^{-ck}), \quad k \rightarrow \infty.$$

The last estimate is easy since for some δ in $(0, 1)$, $|f(e^{i\tau})| < \delta$ on $T^n \setminus B$. Thus, to prove the theorem it suffices to establish the estimate given in (12). Since, by (3),

$$f^k(e^{i\tau}) = c \exp k(\beta \cdot \tau i - (\rho + \gamma)(\tau))$$

for all $\tau \in B$ and since $b = k\beta$ it follows that

$$(13) \quad |\mathcal{D} f^k(e^{i\tau})| = |\mathcal{D}_J \exp k(-\rho - \gamma)(\tau)|,$$

for all $\tau \in B$, where for any subset S of $\{1, 2, \dots, n\}$ the differential operator \mathcal{D}_S is defined by $\mathcal{D}_S = \prod_{j \in S} \partial / \partial t_j$.

Expanding the expression on the right side of (13) by applying \mathcal{D}_J yields

$$(14) \quad |f^k(e^{i\tau})| = \left| \sum_{\mathcal{P}} k^{|\mathcal{P}|} \prod_{H \in \mathcal{P}} (-\mathcal{D}_H(\rho + \gamma)(\tau)) \exp k(-\rho - \gamma)(\tau) \right|,$$

where the sum is over all partitions \mathcal{P} of J , $|\mathcal{P}|$ denotes the number of members in \mathcal{P} and the product is over all subsets H of J belonging to \mathcal{P} .

By hypothesis $\gamma \in C^m(R^n)$ where $m = \max_{1 \leq i \leq n} p_i$. Applying Taylor's formula yields

$$(15) \quad \gamma(\tau) = v(\tau) + O(|\tau|^m), \quad \tau \longrightarrow \vec{0},$$

where $v(\tau)$ is a polynomial in n variables. Since $|\gamma(\tau)|$ and $|\tau|^m$ are both $O(\sum \tau_i^{p_i})$, $\tau \rightarrow \vec{0}$, it follows from (15) that $|v(\tau)|$ is also. This implies by Lemma 6 that the index of p -homogeneity of each term of $v(\tau)$ is at least 1 since $\sum \tau_i^{p_i}$ is a p -homogeneous polynomial with index of p -homogeneity equal to 1 and since by Lemma 1 $\sum \tau_i^{p_i}$ is positive definite. Thus, the polynomial $\mu \equiv \rho + v$ has no term with index of p -homogeneity less than 1 and

$$(\rho + \gamma)(\tau) = \mu(\tau) + O(|\tau|^m), \quad \tau \longrightarrow \vec{0}.$$

Since $\rho, \gamma, \mu \in C^m(R^n)$, $m \geq n + 1$, Taylor's formula and the last

equality imply that

$$\mathcal{D}_H(\rho + \gamma)(\tau) = \mathcal{D}_H\mu(\tau) + 0(|\tau|^{m-|H|}),$$

$\tau \rightarrow \bar{0}$, where $|H|$ denotes the number of elements in H . Since (3) and (10) hold for all $\tau \in B$,

$$|\exp(-\rho-\gamma)(\tau)| \leq \exp(-c \sum \tau_j^{p_j}).$$

This, (14) and the preceding estimate imply that

$$(16) \quad |\mathcal{D}f^k(e^{i\tau})| \leq \sum_{\mathcal{S}} k^{|\mathcal{S}|} \prod_{H \in \mathcal{S}} (|\mathcal{D}_H\mu(\tau)| + c|\tau|^{m-|H|}) \exp(-c \sum \tau_j^{p_j})$$

where the sum and product are indexed as in (14).

For any n -tuple q of nonnegative integers,

$$\begin{aligned} \mathcal{D}_H(\prod_j \tau_j^{q_j}) &= 0 \text{ if } q_j = 0 \text{ for some } j \text{ in } H, \\ &= \prod_j \tau_j^{r_j} \text{ otherwise,} \end{aligned}$$

where

$$(17) \quad \begin{aligned} r_j &= q_j - 1 \text{ if } j \in H, \\ &= q_j \text{ otherwise.} \end{aligned}$$

Also, the index of p -homogeneity of any term $a \prod_j \tau_j^{q_j}$ of μ is at least 1, so that

$$1 \leq \sum q_j/p_j = \sum r_j/p_j + \sum_{j \in H} 1/p_j.$$

Thus,

$$(18) \quad |\mathcal{D}_H\mu(\langle k^{-1/p_j} \tau_j \rangle)| \leq ck^{-1+ \sum_{j \in H} 1/p_j}$$

for all $\tau \in B$. Also, since $m \geq p_j$ for $1 \leq j \leq n$,

$$\begin{aligned} |\langle k^{-1/p_j} \tau_j \rangle|^{m-|H|} &\leq |\langle k^{-1/m} \tau_j \rangle|^{m-|H|} \\ &\leq ck^{-1+|H|/m} \\ &\leq ck^{-1+ \sum_{j \in H} 1/p_j}. \end{aligned}$$

This estimate, (18) and (16) imply upon substitution of $k^{-1/p_j} \tau_j$ for τ_j in (12) that

$$\int_B |\mathcal{D}f^k(e^{i\tau})|^2 d\tau = 0((\sum_{\mathcal{S}} k^{|\mathcal{S}|} \prod_{H \in \mathcal{S}} k^{-1+ \sum_{j \in H} 1/p_j})^2 k^{-\sum_{j=1}^n 1/p_j}),$$

as $k \rightarrow \infty$, where the sum and product are indexed as in (14). From this estimate (12) follows and the theorem is proved.

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