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THE FUNDAMENTAL DIVISOR OF NORMAL DOUBLE POINTS OF SURFACES

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Let W be a surface with a normal singular point w. Consider the minimal resolution of that singularity, $\pi\colon W' \to W$. Let $\pi^{-1}(w) = Y = Y_1 \cdots Y_d$, where the Y_i are distinct irreducible curves on W'. We are interested in two divisors on W' both of which have support on Y. These divisors are Z, the fundamental divisor, and M, the divisor of the maximal ideal. In general $Z \leqq M$. In this thesis we show that if w is a double point singularity which satisfies certain conditions, then $Z = M$.

Introduction. Let A denote a normal, two-dimensional local ring. For simplicity assume that the residue field, k, of A is algebraically closed. Let $\pi\colon Y \to \operatorname{Spec}(A)$ be a birational proper map with Y regular, i.e., a resolution of the singularity $\operatorname{Spec}(A)$. Denote by m' the maximal ideal of A. Let $\pi^{-1}(m') = Y_1 \cup \cdots \cup Y_d$, where the Y_i are distinct irreducible curves on Y. Then, according to Artin [1, page 132] there is a unique smallest positive divisor Z, with support $\bigcup_{i=1}^{d} Y_i$, such that $Z \cdot Y_i \leqq 0$ for all i. Z is called the fundamental divisor. We also have the divisor of the maximal ideal, M, given by

$$M = \sum_{i=1}^{d} m_i Y_i \ ,$$

where $m_i = \min_{t \in m'} \{w_i(t)\}$ and w_i is the valuation determined by $Y_i \subseteqq Y$. In general $Z \leqq M$. Artin [1, Theorem 4] shows that if $\operatorname{Spec}(A)$ has a rational singularity, then $Z = M$ on every resolution. Laufer [4, Theorem 3.13] proves that if $\operatorname{Spec}(A)$ has a minimally elliptic double point singularity, then $Z = M$ on every resolution. Laufer also gives examples of double point singularities for which $Z < M$. His surfaces have defining equation $z^2 = f(x, y)$, where $f(x, y) \in k[[x, y]]$, $f(0, 0) = 0$, and $f(x, y)$ is reducible at $(0, 0)$.

In this paper we show that if $f(x, y)$ has even order or if $f(x, y)$ has odd order and is irreducible at $(0, 0)$, then $Z = M$ on the minimal resolution of $z^2 = f(x, y)$. In §1 we give a method for obtaining a specific resolution of $\operatorname{Spec}(A)$ [3]. In §2 we perform some necessary computations with Z and M, and in §3 we give the proofs of the theorems.

1. Methods for resolving double point singularities. Let A

be a noetherian, complete, two-dimensional, equicharacteristic (not two), normal, local domain of multiplicity two. Assume that the residue field, k, of A is algebraically closed. One has the following characterization of A.

PROPOSITION 1. *With A as above, we have that*

$$A \cong \frac{k[[x, y, T]]}{(T^2 - f(x, y))},$$

where $f(x, y) \in k[[x, y]]$, $f(0, 0) = 0$, and $f(x, y)$ has no multiple factors.

Proof. According to [9, Ch. VIII, Theorem 22 and Theorem 24, Corollary 2] A is a finite module over $k[[x, y]]$ and $[A: k[[x, y]]] = 2$, where $\{x, y\}$ is a system of parameters of A. Let L be the quotient field of A and K be the quotient field of $k[[x, y]]$. Then $[L: K] = 2$ and there exists an element $z \in K$ such that $L = K(z)$ and $z^2 = f(x, y) \in k[[x, y]]$. Without loss of generality we may assume that $f(x, y)$ has no multiple factors. It is easy to see that the integral closure of $k[[x, y]]$ in L is $k[[x, y, z]]$. In fact, let $\alpha + \beta z$ be an element of L which is integral over $k[[x, y]]$. Then Trace $(\alpha + \beta z) = 2\alpha \in k[[x, y]]$ and Norm $(\alpha + \beta z) = \alpha^2 + \beta^2 f(x, y) \in k[[x, y]]$, which imply that α and β are elements of $k[[x, y]]$. But the fact that A is normal and integral over $k[[x, y]]$ implies that A, too, is the integral closure of $k[[x, y]]$ in L. Also, since A is local, $f(0, 0) = 0$ [8, Ch. V, Theorem 34].

We wish to obtain a resolution of the singularity of the surface Spec (A). Thus we wish to find a nonsingular surface W and a proper map $\pi: W \to \operatorname{Spec}(A)$ such that π induces an isomorphism between $W - \pi^{-1}(m')$ and $\operatorname{Spec}(A) - m'$, where m' denotes the maximal ideal of A.

Let $R = k[[x, y]]$ and let m denote the maximal ideal of R. Let $\phi: V \to \operatorname{Spec}(R)$ be a proper birational map obtained by successively belonging up closed points. Let $\phi^{-1}(m) = X = X_1 \cup \cdots \cup X_n$, where the X_i are distinct irreducible curves on V. Let D be the divisor of $f(x, y)$ on V. Then $D = D_1 + D_2$, where D_1 has support in X and D_2 does not involve any X_i. It is well known that we can find V so that $(D_1)_{\text{red}} = \sum_{i=1}^n X_i$ has only normal crossings and D_2 is nonsingular. Each $X_i \subseteq V$ gives rise to a valuation x_i on the function field of V. Call X_i an odd (even) curve if $v_i(f(x, y))$ is odd (even). Suppose X_i and $X_j (i \neq j)$ are both odd curves such that $X_i \cdot X_j = 1$. Let us blow up the point of intersection of X_i and X_j. Then we obtain an even curve E such that $E \cdot \bar{X}_i = E \cdot \bar{X}_j = 1$ and $\bar{X}_i \cdot \bar{X}_j = 0$, where $\bar{X}_i$ and $\bar{X}_j$ are the proper transforms of X_i and X_j. Thus we may assume that no two odd curves meet.

Now let V' be the normalization of V in L. Then we get the following commutative diagram:

$$(*)\qquad \begin{array}{ccc} \mathrm{Spec}\,(A) & \xleftarrow{\pi} & V' \\ \downarrow & & \downarrow g \\ \mathrm{Spec}\,(A) & \xleftarrow[\phi]{} & V \end{array}$$

We claim that π is a resolution of spec(A), i.e., that V' is non-singular. This follows easily from Proposition 1. In fact, let S be the local ring of a point on V. Let $f(x, y)S = \alpha u^a v^b$, where $\{u, v\}$ is a regular system of parameters for S and α is a unit. Then S', the integral closure of S, is also the integral closure of $S[z]$, where $z^2 = f(x, y) = \alpha u^a v^b$. Hence $S' = S[z']$, where $(z')^2 = \alpha u^{a'} v^{b'}$, $0 \leqq a'$, $b' \leqq 1$, $a \equiv a' \bmod 2$, and $b \equiv b' \bmod 2$. Thus S' is regular.

Let m' denote the maximal ideal of A. Note that $\pi^{-1}(m') = g^{-1}\phi^{-1}(m) = g^{-1}(X)$. Thus, to find the irreducible components of $\pi^{-1}(m')$ we must see how the curves $X_i \subseteqq V$ behave under normalization. The rules are as follows and are easily deduced from the above description of S'.

(1) If X_i is and odd curve, then its reduced inverse image in V' is an isomorphic copy of X_i. This is because each point of X_i has just one point lying above it in V' (check locally).

(2) If X_i is an even curve meeting no odd curves, then in V', X_i splits into two disjoint copies of itself. This follows because $X_i \cong \boldsymbol{P}'$ and the ramification points of X_i are precisely the points of intersection of X_i with odd curves. Note that $N = 2g + 2$, where N is the number of ramification points of X_i and g is the genus of the inverse image of X_i in V'.

(3) If X_i is an even curve meeting some odd curves, then the inverse image of X_i in V' is a two fold branched cover of X_i. This again follows from the local algebra. In this case, each even curve must meet an even number of odd curves. This follows from the formula $N = 2g + 2$.

Note that if X_i is an even curve in X meeting at most three other curves, then the inverse image of X_i in V' is rational.

We wish to determine the self-intersection numbers of the inverse images of the X_i from the numbers (X_i^2). The rules are as follows.

(1) If X_i is an odd curve, then the self-intersection number of the inverse image of X_i in V' is $(X_i^2)/2$.

(2) If X_i is an even curve meeting no odd curves, then in V' each component of the inverse image of X_i has self-intersection

number equal to (X_i^2).

(3) If X_i is an even curve which meets some odd curves, then the self-intersection number of the inverse image of X_i in V' is $2(X_i^2)$.

Let us prove rule one (the proofs of the other two rules are similar). Let Z_i denote the inverse image of X_i. Let g be as in diagram (*), g_{Z_i} be g restricted to Z_i, $i_{X_i}: X_i \to V$ and $i_{Z_i}: Z_i \to V'$ be inclusions, and let $\mathcal{O}_V$ and $\mathcal{O}_{V'}$ denote structure sheaves. Then

$$\begin{aligned} 2(Z_i \cdot Z_i) &= (2Z_i \cdot Z_i) = \deg i_{Z_i}^*(\mathcal{O}_{V'}(2Z_i)) \\ &= \deg i_{Z_i}^* g^*(\mathcal{O}_V(X_i)) = \deg g_{Z_i}^* i_{X_i}^*(\mathcal{O}_V(X_i)) \\ &= \deg i_{X_i}^*(\mathcal{O}_V(X_i)) = (X_i^2) \ . \end{aligned}$$

See [5, Ch. IV, §13] for details.

Note that $m'\mathcal{O}_{V'}$ is locally principal.

2. Definitions and computations. Let $\pi: V' \to \operatorname{Spec}(A)$ be as before and let $\pi^{-1}(m') = X_1' \cup \cdots \cup X_s'$, where the X_i' are distinct irreducible curves on V'. Let $a_i = \min_{t \in m}\{v_i(t)\}$ and let $a_i' = \min_{u \in m'}\{v_i'(u)\}$, where v_i and v_i' are the valuations determined by $X_i \subseteqq V$ and $X_i' \subseteqq V'$. Define a divisor M on V' by:

$$M = \sum_{i=1}^{s} a_i' X_i' \ .$$

M is called the divisor of the maximal ideal. The a_i' can be computed from the a_i as follows. If X_i is an odd curve and X_j' is the reduced inverse image of X_i, then $a_j' = 2a_i$. If X_i is an even curve meeting some odd curves and X_j' is the inverse image of X_i, then $a_j' = a_i$. Finally, if X_i is an even curve meeting no odd curves and if the inverse image of X_i is $X_j' \cup X_l'$, then $a_j' = a_l' = a_i$. The proofs of these rules are straightforward.

On the other hand, there is another important divisor on V' called the fundamental divisor, which we denote by Z. As in Artin [1, page 132], Z is the unique positive divisor on V' such that:

(1) $Z \cdot X_i' \leqq 0$, for every i,

(2) if C is a divisor such that $C \cdot X_i' \leqq 0$ for every i, then $Z \leqq C$.

Let R be a normal two-dimensional local ring with maximal ideal q. For simplicity, assume that the residue field of R is algebraically closed. Let $\beta: Y \to \operatorname{Spec}(R)$ be a resolution of $\operatorname{Spec}(R)$. Let $\beta^{-1}(q) = Y_1 \cup \cdots \cup Y_d$, where the Y_i are distinct irreducible curves. Then in this general setting M and Z are defined as above and we have the following propositions.

PROPOSITION 2. *If Z, M, R, q, and $Y_1 \cup \cdots \cup Y_d$ are as above, then $Z \leqq M$.*

Proof. We show that $M \cdot Y_j \leqq 0$ for every j. Let w_j denote the valuation determined by $Y_j \subseteqq Y$. Clearly if $M = \sum_{i=1}^{d} m_i Y_i$, then $m_i = \min\{w_i(f_1), \cdots, w_i(f_r)\}$, where the minimum is taken over a basis $f_1, \cdots, f_r$ of q. Denote the divisor of f_i on Y by (f_i). Then $(f_i) = F_i + G_i$, where F_i is a linear combination of the Y_j and G_i involves no Y_j. We obtain

$$0 = (f_i) \cdot Y_j = F_i \cdot Y_j + G_i \cdot Y_j \, .$$

Now $G_i \cdot Y_j \geqq 0$, so $F_i \cdot Y_j \leqq 0$. Let $F_i = \sum_{l=1}^{s} b_{il} Y_l$. Then

$$M = \min(F_1, \cdots, F_r) = \sum_{l=1}^{s} \Bigl(\min_{i=1,\cdots,r} \{b_{il}\}\Bigr) Y_l$$

and so $M \cdot Y_j \leqq 0$ [1, page 131].

PROPOSITION 3 [6, *Lemma* 2.8]. *Let C_1 and C_2 be two divisors on Y with support in $\bigcup_{i=1}^{d} Y_i$. Assume that $C_1 \cdot Y_j \leqq 0$ for every j and that $C_1 \leqq C_2$. Then $(C_1^2) \geqq (C_2^2)$ and $C_1 = C_2$ if and only if $(C_1^2) = (C_2^2)$.*

Proof. Let $C_1 + B = C_2$. Then

$$(C_2^2) = (C_1^2) + 2C_1 \cdot B + B^2 \leqq (C_1^2)$$

since $C_1 \cdot B \leqq 0$ and $B^2 \leqq 0$. If $(C_1^2) = (C_2^2)$, then $C_1 \cdot B \leqq 0$ implies that $B^2 = 0$. Thus $B = 0$ since the intersection matrix for the Y_j's is negative definite.

Let us also prove a lemma which will be useful in §3.

LEMMA 1. *Let $h\colon Y' \to Y$ be the blow up of $p \in Y$, with $\beta(p) = q$. Let M_Y and $M_{Y'}$ denote the divisors of the maximal ideal on Y and Y'. Then $h^{-1}(M_Y) \leqq M_{Y'}$.*

Proof. Let $D = h^{-1}(p)$ and $h^{-1}(Y_i) = Y_i' + n_i D$. Certainly the coefficients of Y_i' in $h^{-1}(M_Y)$ and $M_{Y'}$ are equal. Let $\mathcal{O}_p$ denote the local ring of p on Y. Then $q\mathcal{O}_p = t\mathfrak{a}_p$, where $\mathfrak{a}_p$ is an ideal primary for the maximal ideal of $\mathcal{O}_p$ and t is a local equation of M_Y at p. Let v_D deeote the valuation determined by D. Then

$$v_D(q) = v_D(t) + v_D(\mathfrak{a}_p) \, ,$$

and since, at D, $h^{-1}(M_Y)$ has coefficient $v_D(t)$ and $M_{Y'}$ has coefficient $v_D(q)$, we have proved the lemma. Note that $q\mathcal{O}_Y$ is invertible if and only if $h^{-1}(M_Y) = M_{Y'}$.

Let us now return to the case of surface singularities of multiplicity two. We wish to determine the possible values for the two integers Z^2 and M^2 on a resolution of Spec (A), where A is an in §1 and A has maximal ideal m'. Let $\beta: Y \to \mathrm{Spec}(A)$ and any resolution of Spec (A) and let $\beta^{-1}(m') = Y_1 \cup \cdots \cup Y_d$, where the Y_i are distinct irreducible curves. By [6, Theorem 2.7] if $m'\mathcal{O}_Y$ is locally principal, then $M^2 = -2$ on Y. If $m'\mathcal{O}_Y$ is not locally principal, then consider a resolution $\alpha: W \to \mathrm{Spec}(A)$ such that $m'\mathcal{O}_W$ is locally principal (V' for example), with $\lambda: W \dashrightarrow Y$. Denote the divisor of the maximal ideal on W by M'. Lemma 1 and the remark following it then imply that $\lambda^{-1}(M) < M'$. But then Proposition 3 implies that

$$0 > M^2 = (\lambda^{-1}(M))^2 > (M')^2 = -2$$

and thus $M^2 = -1$. Combining the two above cases we obtain that $-2 \leqq M^2 < 0$ for any resolution of Spec (A). Propositions 2 and 3 then imply that $-2 \leqq Z^2 < 0$. These bounds for Z^2 and M^2 give us the following corollary to Proposition 3.

COROLLARY. *With Z and M as above, if $M^2 = -1$, then $Z = M$.*

Proof. $Z^2 \geqq M^2 = -1$ implies that $Z^2 = -1$. Proposition 3 then implies that $Z = M$.

Note that $m'\mathcal{O}_Y$ is not invertible in the above corollary since $m'\mathcal{O}_Y$ is invertible if and only if $M^2 = -2$.

Let us make the following two remarks. If $Z^2 = -2$ on some resolution, then $Z^2 = -2$ on every resolution [6, Proposition 2.9] and hence $Z = M$ on every resolution by Proposition 3. Again using Proposition 3, if $Z < M$ on some resolution, then we must have that $M^2 = -2$ and $Z^2 = -1$.

We need the following general proposition.

PROPOSITION 4. *Let Z be the fundamental divisor for a resolution of Spec (R), where R is as in Proposition 2. Let $Y = Y_1 \cup \cdots \cup Y_d$ be the support of Z, with Y_i distinct irreducible curves. Let $Z = \sum_{i=1}^d r_i Y_i$ and let $B = \sum_{i=1}^d b_i Y_i$ be a divisor whose support is contained in Y, where $b_i \geqq 0$ for all i. Suppose that $Z^2 = -1$, $B^2 = -2$, and $B \cdot Y_i \leqq 0$ for every i. Then the following two conditions hold.*

(1) *There exists a unique integer i_0 such that $Z \cdot Y_{i_0} = -1$, $r_{i0} = 1$, and $Z \cdot Y_j = 0$ for $j \neq i_0$.*

(2) *There exists a unique integer k_0 such that $B \cdot Y_{k_0} = -1$, $b_{k0} = 2$, and $B \cdot Y_j = 0$ for $j \neq k_0$.*

Proof. To prove part one we compute with Z as follows:

$-1 = Z \cdot Z = \sum_{j=1}^{s} r_j(Y_j \cdot Z)$. Noting that $Y_j \cdot Z \leqq 0$ for all i and that $r_j > 0$ for all j [1, page 132], we obtain part one. To prove part two we compute with B:

$$-2 = B \cdot B = \sum_{i=1}^{s} b_i(Y_i \cdot B) .$$

Since $Y_i \cdot B \leqq 0$ for all i and $b_i \geqq 0$ for all i, we have three cases.

Case 1. There exists an integer k_0 such that $B \cdot Y_{k_0} = -2$, $b_{k_0} = 1$, and $B \cdot Y_j = 0$ for $j \neq k_0$.

Case 2. There exist two distinct integers k_0 and l_0 such that $B \cdot Y_{k_0} = B \cdot Y_{l_0} = -1$, $b_{k_0} = b_{l_0} = 1$, and $B \cdot Y_j = 0$ for $j \neq k_0, l_0$.

Case 3 is part two of the present proposition.

We will show that Cases 1 and 2 cannot occur. First we need a computation. Since $Z < B$, let $Z' \neq 0$ be a divisor such that $B = Z + Z'$. Then

$$-2 = B^2 = Z^2 + 2Z \cdot Z' + (Z')^2 ,$$

and thus

$$-1 = 2Z \cdot Z' + (Z')^2 .$$

Since $(Z')^2 < 0$, and $Z \cdot Z' \leqq 0$, we must have that $Z \cdot Z' = 0$. But then

$$B \cdot Z = Z^2 + Z \cdot Z' = -1 .$$

Now it is easy to prove that Cases 1 and 2 are impossible. In fact, for Case 1 we obtain

$$-1 = B \cdot Z = \sum_{j=1}^{d} r_j(Y_j \cdot B) = -2r_{k_0} ,$$

and so $r_{k_0} = 1/2$ which is impossible. For Case 2 we compute similarly:

$$-1 = B \cdot Z = \sum_{j=1}^{d} r_j(Y_j \cdot B) = -r_{k_0} - r_{l_0} .$$

Thus $r_{k_0} + r_{l_0} = 1$ which is impossible since $r_j \geqq 1$ for all j [1, page 132]. This completes the proof of Proposition 4.

Under the assumptions of Proposition 4 we can also obtain the following information. The computation

$$-1 = B \cdot Z = \sum_{j=1}^{d} b_j(Y_j \cdot Z) = -b_{i_0}$$

yields $b_{i_0} = 1$. Also, since $b_{k_0} = 2$ we have that $i_0 \neq k_0$.

COROLLARY. *Suppose that the hypotheses of Proposition* 4 *are satisfied with* $B = M$ *(i.e., assume that* $Z < M$ *on the resolution). Assume that* Y_{k_0} *is rational and* $(Y_{k_0}^2) = -1$. *Let* $\alpha\colon Y \dashrightarrow V_0$ *be the map obtained by blowing down* Y_{k_0}. *Let* M_0 *be the divisor of the maximal ideal on* V_0 *and let* Z_0 *be the fundamental divisor or* V_0. *Then* $Z_0 = M_0$.

Proof. We have that $\alpha^{-1}(M_0) \cdot Y_{k_0} = 0$, and thus $\alpha^{-1}(M_0) < M$ by Lemma 1 and the remark following it. Then

$$M_0^2 = (\alpha^{-1}(M_0))^2 > M^2 = -2$$

by Proposition 3. Thus $M_0^2 = -1$ and we have that $Z_0 = M_0$ by the corollary to Proposition 3.

3. Statements and proofs of the theorems. The purpose of this section is to prove that Z equals M in the minimal resolution of certain double points of surfaces, among which are those in whose defining equation $z^2 = f(x, y)$, $f(x, y)$ is irreducible. We will show, for these double points, that Z equals M either in the resolution V' described in §1 or in the resolution obtained by blowing down a certain curve on V'. Note that M is locally principal on V', so that $Z = M$ on V' if and only if $Z^2 = -2$, and in that case $Z = M$ on every resolution. Now the minimal resolution can be obtained from V' by a succession of blowing downs [2, 7]. Hence the following proposition will imply that if Z equals M on some resolution then $Z = M$ on the minimal one.

PROPOSITION 5. *Let* R *be a normal two-dimensional local ring with algebraically closed residue field and maximal ideal* q. *Suppose* $\lambda\colon Y \to \mathrm{Spec}\,(R)$ *is a resolution of the singularity of* Spec R. *Let* $h\colon Y' \to Y$ *be the blow up of* $p \in Y$, *with* $\lambda(p) = q$. *Let* M_Y *and* $M_{Y'}$ *denote the divisors of the maximal ideal on* Y *and* Y', *and let* Z_Y *and* $Z_{Y'}$ *denote the fundamental divisors on* Y *and* Y'. *If* $M_{Y'} = Z_{Y'}$, *then* $M_Y = Z_Y$.

Proof. Let $Y_1, \cdots, Y_d$ be the irreducible components of $\lambda^{-1}(q)$. Let $D = h^{-1}(p)$ and $h^{-1}(Y_i) = Y'_i + n_i D$. Then $h^{-1}(M_Y) \cdot Y'_i = M_Y \cdot Y_i \leqq 0$ for all i [6, page 421]. Therefore $Z_{Y'} \leqq h^{-1}(M_Y)$ by the definition of $Z_{Y'}$.

Lemma 1 of §2 implies that $h^{-1}(M_Y) \leqq M_{Y'}$. Combining the above two inequalities we obtain

$$Z_{Y'} \leqq h^{-1}(M_Y) \leqq M_{Y'} .$$

But by assumption $Z_{Y'} = M_{Y'}$, and thus $h^{-1}(M_Y) = Z_{Y'}$. Now [6, Proposition 2.9] shows that $Z_{Y'} = h^{-1}(Z_Y)$, and thus $h^{-1}(M_Y) = h^{-1}(Z_Y)$, which implies that $M_Y = Z_Y$.

We now commence to prove that Z equals M on V' for certain double points.

THEOREM 1. *Let $f(x, y) \in k[[x, y]]$ be as in Proposition 1. Suppose that $f(x, y)$ has even order. Then on V' we have that Z equals M (and hence Z equals M on every resolution of $z^2 = f(x, y)$).*

Proof. Recall that $\phi: V \to \operatorname{Spec}(k[[x, y]])$ is obtained by successively blowing up closed points. In the first blowing up (the blowing up of m, the maximal ideal of $k[[x, y]]$) we obtain a curve which is the inverse image of m. This curve also has an inverse image in V, and we call it X_1. Let M and M_1 denote the divisors of the maximal ideals m' and m on V' and V. Recall that $M_1 = \sum_{i=1}^{n} a_i X_i$ and $M = \sum_{i=1}^{s} a_i' X_i'$, where

$$a_i = \min_{t \in m} \{v_i(t)\}$$

and

$$a_i' = \min_{u \in m'} \{v_i'(u)\} ,$$

with v_i and v_i' denoting the valuations determined by $X_i \subseteqq V$ and $X_i' \subseteqq V'$. Then X_1 is an even curve and $M_1 \cdot X_1 = -1$. If X_1 meets no odd curves in X, then $g^{-1}(X_1)$ is a disjoint union of two curves isomorphic to X_1 and the intersection number of M with each of these curves is -1. But this condition is incompatible with $Z < M$ by Proposition 4. If X_1 meets some odd curves, then we have that $M_1 \cdot X_1 = -1$ and $a_1 = 1$. Let $X_1' = g^{-1}(X_1)$. Then $M \cdot X_1' = -2$ and $a_1' = 1$, which, again, is incompatible with $Z < M$ by Proposition 4.

If $f(x, y)$ has odd order, then Theorem 1 does not hold in general. In fact, if $f(x, y) = y(x^4 + y^6)$, then in the minimal resolution of $z^2 = f(x, y)$ we have that $Z < M$. This example was given by Henry B. Laufer. Notice however that $f(x, y) = y(x^4 + y^6)$ is reducible. If we assume that $f(x, y)$ is irreducible at $(0, 0)$, then we can prove that $Z = M$ in the minimal resolution.

THEOREM 2. *Let $f(x, y) \in k[[x, y]]$ be as in Proposition 1. Suppose that $f(x, y)$ has odd order and is irreducible at $(0, 0)$. Then Z equals M on the minimal resolution of $z^2 = f(x, y)$.*

Proof. Let X_1 be as in the proof of Theorem 1 and let X_c be defined similarly as curves and on V for $c = 2, \cdots, n$. Then X_1 is an odd curve and we set $X'_1 = (g^{-1}(X_1))_{red}$. We have two cases to consider.

(1) Suppose that the first quadratic transform of $f(x, y)$ has the same multiplicity as $f(x, y)$. Then on V we have that $X_1 \cdot X_2 = 1$ and $X_1 \cdot X_j = 0$ for $j > 2$. Thus $(X_1^2) = -2$ and so $(X'_1)^2 = -1$ since X_1 is an odd curve. Note also that X'_1 is rational since X_1 is odd. Thus we can apply the corollary to Proposition 4 ($k_0 = 1$).

Let us make two remarks here before continuing with the proof. Since $f(x, y)$ is irreducible at (0, 0) it is easy to see that X_i is rational for all i. This follows because it can be shown that each X_i meets at most 3 other curves in X and thus the genus of an even curve meeting some odd curves is $(N - 2)/2$, where N must be 2. Also note that the proof of Case 1 above still holds if we assume instead that some quadratic transform of $f(x, y)$ has the same multiplicity as $f(x, y)$, where $f(x, y)$ is not necessarily irreducible at (0, 0).

(2) Suppose the first quadratic transform of $f(x, y)$ does not have the same multiplicity as $f(x, y)$. Assume that $Z < M$ on V'. Then Proposition 4 shows that there exists an integer i_0 such that $Z \cdot X'_{i_0} = -1$, $Z \cdot X'_j = 0$ for $j \neq i_0$, and $a'_{i_0} = 1$. It is clear from the definition of the integers a_i that $a_1 = a_2 = 1$ and $a_i > 1$ for $i > 2$. We have two possibilities to check. Suppose that X_2 is an odd curve. Let $X'_2 = (g^{-1}(X_2))_{red}$. Then since X_1 and X_2 are odd curves we have that $a'_1 = a'_2 = 2$ and $a'_2 \geqq 2$ for $i > 2$. This contradicts Proposition 4 since a'_{i_0} must be 1. Now suppose that X_2 is an even curve. Since $f(x, y)$ is irreducible it can easily be checked that X_2 meets only one other curve in X. In fact, if $(X_2^2) = -c$, then X_2 meets only X_{c+1}. This curve cannot be odd since each even curve meets an even number of odd curves, as stated in §1. Thus X_2 meets no odd curves and so $g^{-1}(X_2)$ consists of two disjoint isomorphic copies of X_2, say X'_2 and X'_3. Now $a'_1 = 2$ and $a'_i \geqq 2$ for $i > 3$. Thus, since $a'_{i_0} = 1$, i_0 must be either 2 or 3. But if Z has nonzero intersection number with one of X'_2 and X'_3, then it must have it with the other. In fact, the automorphism of $L = K(z)$ given by $z \mapsto -z$ leaves Z fixed and interchanges X'_2 and X'_3. Thus we have a contradiction since Proposition 4 insists that i_0 must be unique.

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