RESTRICTIONS OF CERTAIN FUNCTION SPACES TO CLOSED SUBGROUPS OF LOCALLY COMPACT GROUPS

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Let G be a locally compact group, and $E(G)$ either the space $C_u(G)$ of bounded left and right uniformly continuous functions on G, the space $W(G)$ of weakly almost periodic functions on G, or the Fourier-Stieltjes algebra $B(G)$ of G. Let $E(G)|_H$ be the space of restrictions of $E(G)$-functions to the closed subgroup H of G. A necessary and sufficient condition is given for an $E(H)$-function to belong to $E(G)|_H$ when H is a normal subgroup of G. It is also shown that $E(G)|_H$ is all of $E(H)$ when H is any closed subgroup of a [SIN]-group. The techniques employed here can be used to deal with other function spaces.

Let $C(G)$ be the space of bounded continuous complex-valued functions on G with the uniform norm, $\|\ \|_\infty$, and $\beta(G)$ be the Stone-Cech compactification of G, i.e., the maximal ideal space of $C(G)$, to which $C(G)$-functions extend naturally, via the Gelfand transform. The left translation operator is denoted λ:

$$[\lambda(g)u](g') = u(g^{-1}g') \qquad g, g'G, u \in C(G) .$$

The reader will recall that u in $C(G)$ is called weakly almost periodic if the set $\lambda(G)u$ of left translates of u is relatively compact in the weak topology of $C(G)$. Equivalently, one may require that for any sequence $\{g_j\}$ of elements of G there is a subsequence $\{g_j'\}$ such that $\lambda(g_j')u$ converges weakly in $C(G)$, or such that $\lambda(g_j')u$ converges pointwise on $\beta(G)$. This and other results on weakly almost periodic functions are summarized in [1]. The space $W(G)$ of weakly almost periodic functions in $C(G)$ is given the uniform norm.

The Fourier-Stieltjes algebra $B(G)$ of G is the algebra of coordinate functions

$$u: g \longrightarrow \langle \pi(g)\xi, \eta \rangle \qquad \xi, \eta \in \mathscr{H}_\pi$$

of continuous unitary representations π of G on Hilbert spaces $\mathscr{H}_\pi$; $B(G)$ is normed thus:

$$\|u\|_B = \min \{\|\xi\|_{\mathscr{H}_\pi} \cdot \|\eta\|_{\mathscr{H}_\pi}: u = \langle \pi\xi, \eta \rangle\} .$$

The basic facts about $B(G)$ can be found in [4]. We recall from [1] that

$$B(G) \subsetneqq W(G) \subsetneqq C_u(G) .$$

The spaces $B(G)$, $W(G)$, and $C_u(G)$ have many similar properties. For example, both left and right translations act continuously and isometrically on each space, their functorial properties are similar, and they are all stable under continuous automorphisms of G ([1, 1.8], [4, 2.20], [8, 34.48]). We shall find more examples below.

We let $E(G)$ denote one of $C_u(G)$, $W(G)$ or $B(G)$. If H is a closed subgroup of G, then the space of restrictions $E(G)|_H$ is a subspace of $E(H)$. We are interested in knowing which $E(H)$-functions have extensions in $E(G)$. It is known that $E(G)|_H = E(H)$ when G is abelian ([1, 3.16], [8, 34.48]), but this is generally false for nonabelian G. A. Douady observed that if H is a closed normal abelian subgroup of a locally compact group G and χ is a continuous character of H such that, for any neighborhood $\mathscr{U}$ of the identity e in G, there are s in $\mathscr{U}$ and h in H such that $\chi(s^{-1}hs) \neq \chi(h)$, then χ cannot be extended to a function u in $B(G)$ (quoted in [5, p. 204]; see also [8, 34.28]). Applying this observation to the semidirect product group $G = T \times_s C$, where the torus T acts on the complex numbers C by multiplication, we conclude that $B(G)|_C \neq B(C)$. Ching Chou [2, p. 192] showed that $W(G)|_C \neq W(C)$; his example shows also that $C_u(G)|_C \neq C_u(C)$.

For an arbitrary locally compact group G and closed subgroup H it is known that $B(G)|_H = B(H)$ if H is open or compact or the connected component of the identity G_0 of G or the center of G ([10, Prop. 1.1], [11, §2]); $W(G)|_H = W(H)$ if H is open ([1, 3.14]—normality is not needed) or compact or G_0 (analogous to the $B(G)$-proof); $C_u(G)|_H = C_u(H)$ if H is open or compact or G_0.

In this paper the space $E(G)|_H$ is characterized when H is a closed normal subgroup of G. Further, it is shown that $E(G)|_H = E(H)$ for any closed subgroup of a [SIN]-group G, but that this is false if G belongs to the larger class of [IN]-groups.

1. **The extension theorems.** Let N be a closed normal subgroup of G. We write h^g for the conjugate $g^{-1}hg (g, h \in G)$. Functions on G will be denoted by u, v, and w, and those on N by x, y, and z. Given such functions u and x, we define u^g and x^g by the formulae:

$$u^g(h) = u(h^g) \qquad\qquad g, h \in G$$
$$x^g(n) = x(n^g) \qquad\qquad g \in G, n \in N .$$

The maps $u \to u^g$ and $x \to x^g$ are isometries of $E(G)$ and $E(N)$.

THEOREM 1. *Let N be a closed normal subgroup of the locally compact group G. Then*

$$E(G)|_N = \{x \in E(N): \|x^g - x\|_E \longrightarrow 0 \quad as \quad g \longrightarrow e\} \, ,$$

and, if $x \in E(G)|_N$, *then*

$$\|x\|_E = \inf \{\|u\|_E: u \in E(G), u|_N = x\} \, .$$

Proof. We have already remarked that $E(G)|_N \subseteq E(N)$; it is further true that

$$\|u|_N\|_E \leqq \|u\|_E \, .$$

If u is in $E(G)$, so is u^g, and as g tends to e in G,

$$\|u^g - u\|_E \longrightarrow 0 \, .$$

Consequently

$$E(G)|_N \subseteq \{x \in E(N): \|x^g - x\|_E \longrightarrow 0 \quad as \quad g \longrightarrow e\} \, .$$

We remark at this point that if χ is a continuous character of N and belongs to $E(G)|_N$ there must be a neighborhood $\mathscr{U}$ of e in G such that, if g belongs to $\mathscr{U}$, then

$$\|\chi^g - \chi\|_E < 1 \, .$$

It then follows easily that $\chi^g = \chi$ for g in $\mathscr{U}$. This is a proof of Douady's observation quoted in the introduction.

To complete the proof of the theorem is less trivial. As in the proof of the open mapping theorem, it suffices to show that, for any x in $E(N)$ satisfying $\|x^g - x\|_E \to 0$ as $g \to e$, and for any small positive ε, there is some u in $E(G)$ for which

$$\|u\|_E = \|x\|_E$$

and

$$\|u|_N - x\|_E < \varepsilon \, .$$

Given such an x and ε, we can find neighborhoods $\mathscr{U}$ of e in G and $\mathscr{X}$ of e in N such that

(1) $$\|x^g - x\|_E < \varepsilon/2 \qquad\qquad g \in \mathscr{U}$$

(2) $$\|\lambda(n^{-1})x - x\|_E < \varepsilon/2 \qquad\qquad n \in \mathscr{X} \, .$$

Let $\mathscr{V}$ be a compact neighborhood of e in G such that

(3) $$\mathscr{V} \subseteq \mathscr{U}$$

(4) $$\mathscr{V}^{-1} \cdot \mathscr{V} \cap N \subseteq \mathscr{X} \, .$$

Take a nonnegative continuous function v on G such that

(5) $$\mathrm{supp}\,(v) \subseteqq \mathscr{Y}$$

(6)
$$
\begin{aligned}
1 &= \int_{G/N} d\dot{g} \Big[\int_N dn\, v(gn)\Big]^2 \\
&= \int_{G/N} d\dot{g} \int_N dn'\, v(gn') \int_N dn\, v(gn) \\
&= \int_{G/N} d\dot{g} \int_N dn'\, v(gn') \int_N dn\, v(gn'n) \\
&= \int_G dg \int_N dn\, v(g)v(gn) \, ;
\end{aligned}
$$

here we have assumed that the Haar measure dg, $d\dot{g}$, and dn of G, G/N and N are normalized so that

$$
\int_{G/N} d\dot{g} \int_N dn\, w(gn) = \int_G dg\, w(g) \, , \qquad\qquad w \in C_c(G)
$$

($C_c(G)$ the space of compactly supported functions) ([8, 28.54]). We now define the function u on G by the formula

$$
u(g') = \int_G dg \int_N dn\, v(g'g)v(gn)x(n) \, , \qquad\qquad g' \in G \, .
$$

This is a variation of a device of H. Reiter [13], who considered functions w of the form

$$
w(g') = \int_N dn\, v(g'n)x(n^{-1}) \, , \qquad\qquad g' \in G \, .
$$

We shall check later that u belongs to $E(G)$ and that $\|u\|_E \leqq \|x\|_E$. For the moment we obtain an expression for $u|_N$ and estimate $\|u|_N - x\|_E$.

For any n' in N,

(7)
$$
\begin{aligned}
u(n') &= \int_G dg \int_N dn\, v(g)v(n'^{-1}gn)x(n) \\
&= \int_G dg \int_N dn\, v(g)v(g(g^{-1}n'^{-1}g)n)x(n) \\
&= \int_G dg \int_N dn\, v(g)v(gn)[\lambda(n^{-1})x]^{gn}(n') \, .
\end{aligned}
$$

The map from $G \times N$ to $E(N)$ defined by the rule

$$
(g, n) \longrightarrow v(g)v(gn)[\lambda(n^{-1})x]^{gn}
$$

is continuous and compactly supported, so the vector-valued Riemann integral

(8)
$$
\int_G dg \int_N dn\, v(g)v(gn)[\lambda(n^{-1})x]^{gn}
$$

exists, and, by the pointwise equality (7), is equal to $u|_N$. We note from (6) that

$$x = \int_G dg \int_N dn\, v(g)v(gn)x \ ,$$

so that

$$\|u|_N - x\|_E \leqq \int_G dg \int_N dn\, v(g)v(gn)\|[\lambda(n^{-1})x]^{gn} - x\|_E$$

$$\leqq \int_G dg \int_N dn\, v(g)v(gn)(\|[\lambda(n^{-1})x]^{gn} - x^{gn}\|_E + \|x^{gn} - x\|_E)$$

$$= \int_G dg \int_N dn\, v(g)v(gn)(\|\lambda(n^{-1})x - x\|_E + \|x^{gn} - x\|_E) \ .$$

If $v(g) \cdot v(gn)$ is nonzero, then from (5), g and gn are in $\mathscr{V}$; from (1) and (3), $\|x^{gn} - x\|_E < \varepsilon/2$ and from (2) and (4), $\|\lambda(n^{-1})x - x\|_E < \varepsilon/2$. By applying (6) again, we conclude that

$$\|u|_N - x\|_E \leqq \varepsilon \ .$$

It remains to show that u belongs to $E(G)$ and that $\|u\|_E \leqq \|v\|_E$. The easiest case is where $E = C_u$; we leave this verification to the reader. Let us consider the case where $E = B$. Then x can be written in the form

$$x(n) = \langle \sigma(n)\xi, \eta \rangle \ , \qquad\qquad n \in N \ ,$$

where σ is a unitary representation of N on a Hilbert space $\mathscr{H}_\sigma$, ξ and η belong to $\mathscr{H}_\sigma$, and $\|v\|_B = \|\xi\|_{\mathscr{H}_\sigma} \cdot \|\eta\|_{\mathscr{H}_\sigma}$. Now, for any g' in G,

$$u(g') = \int_G dg \int_N dn\, v(g)v(g'^{-1}gn)x(n)$$

$$= \int_{G/N} d\dot{g} \int_N dn' \int_N dn\, v(gn')v(g'^{-1}gn'n)\langle \sigma(n)\xi, \eta \rangle$$

$$= \int_{G/N} d\dot{g} \int_N dn' \int dn\, v(g'^{-1}gn)v(gn')\langle \sigma(n)\xi, \sigma(n')\eta \rangle$$

$$= \int_{G/N} d\dot{g} \Big\langle \int_N dn\, v(g'^{-1}gn)\sigma(n)\xi, \int_N dn'\, v(gn')\sigma(n')\eta \Big\rangle \ ,$$

which the reader will recognize as a coordinate function of the representation π of G unitarily induced from σ (see [5, VI. 2]). Further,

$$\|u\|_B \leqq \Big\| \int_N dn\, v(gn)\sigma(n)\xi \Big\|_{\mathscr{H}_\pi} \cdot \Big\| \int_N dn\, v(gn)\sigma(n)\eta \Big\|_{\mathscr{H}_\pi} \ ,$$

and

$$\Big\| \int_N dn\, v(gn)\sigma(n)\zeta \Big\|_{\mathscr{H}_\pi} = \Big(\int_{G/N} d\dot{g} \Big[\Big\| \int_N dn\, v(gn)\sigma(n)\zeta \Big\|_{\mathscr{H}_\sigma} \Big]^2 \Big)^{1/2} \leqq \|\zeta\|_{\mathscr{H}_\sigma}$$

from (6), whence $\|u\|_B \leqq \|x\|_B$.

The other case to consider is where E is W. By repeating the argument used to deduce (8) from (7), we deduce that, for fixed g' in G,

$$\lambda(g')u\,|_N = \int_G dg \int_N dn\, v(g'^{-1}g)v(gn)\lambda(n^{-1})x^{gn}\,,$$

as a vector-valued integral, from which we see that $\lambda(g')u\,|_N$ lies in $W(N)$ for each g' in G. The function u is uniformly continuous and supported by $\mathscr{V}^{-1}\mathscr{V}N$, where $\mathscr{V}^{-1}\mathscr{V}$ is a compact subset of G. It is trivial to show that $\|u\|_\infty \leqq \|x\|_\infty$, so application of the following lemma completes the proof of the theorem.

LEMMA. *Suppose that u belongs to $C_u(G)$ and that*
(i) *there is a compact subset K of G such that* $\mathrm{supp}\,(u) \subseteqq KN$,
(ii) $\lambda(g)u\,|_N$ *belongs to $W(N)$ for all g in G. Then u is in $W(G)$.*

Proof. Let $\{g_j\}$ be a sequence of elements of G. We want to show that $\{\lambda(g_j)u\}$ has a weakly convergent subsequence in $C(G)$. We let $p\colon G \to G/N$ be the canonical projection.

Case (a). $\{p(g_j)\}$ is unbounded in G/N.

In this case there is a subsequence $\{g'_j\}$ such that $\{p(g'_j)\}$ eventually avoids any compact subset of G/N. Let s be a nonnegative compactly supported function on G/N such that

$$|u(g)| \leqq s\circ p(g)\,, \qquad\qquad g \in G\,.$$

Then

$$|\lambda(g'_j)u| \leqq [\lambda(p(g'_j))s]\circ p\,.$$

Now $\{\lambda(p(g'_j))s\}$ converges to zero pointwise on $\beta(G/N)$. The map $s \to s\circ p$ injects $C(G/N)$ into $C(G)$, and so dually projects $\beta(G)$ into $\beta(G/N)$. It follows that $\{\lambda(p(g'_j))s\circ p\}$ converges to zero pointwise on $\beta(G)$; *a fortiori* $\{\lambda(g'_j)u\}$ converges pointwise to zero on $\beta(G)$, so converges weakly to zero in $C(G)$.

Case (b). $\{p(g_j)\}$ is bounded in G/N.

There is a compact set K' in G such that each g_j can be expressed, not necessarily uniquely, as $n_j k_j$, with n_j in N and k_j in K'. Since K' is compact, there is a subsequence $\{g'_j\}$ of $\{g_j\}$ such that the corresponding sequence $\{k'_j\}$ has a limit point k_0, say. Note that

$$\|\lambda(n'_j k'_j)u - \lambda(n'_j k_0)u\|_\infty = \|\lambda(k'_j)u - \lambda(k_0)u\|_\infty \longrightarrow 0$$

as $j \to \infty$. Hence, to show that $\{\lambda(g'_j)u\}$ has a weakly convergent subsequence, it suffices to show that $\{\lambda(n'_jk_0)u\}$ has one; it certainly suffices to show that the set $\lambda(Nk_0)u$ is relatively weakly compact.

We write $C^0(G)$ for the subspace of $C_u(G)$ of functions vanishing off $K'KN$. The Hahn-Banach theorem implies that the weak topology of $C^0(G)$ coincides with its relative weak topology as a subspace of $C(G)$. The function u lies in $C^0(G)$, as do the translates $\lambda(n'_jk_0)u$.

Next we denote by $C(K'K; C_u(N))$ the space of $C_u(N)$-valued continuous functions on $K'K$, and observe that $\sigma: C^0(G) \to C(K'K; C_u(N))$, defined by setting

$$[\sigma w(k)](n) = w(nk) \qquad k \in K'K, n \in N ,$$

is an isometric embedding of $C^0(G)$ into $C(K'K; C_u(N))$; the weak topology on $C^0(G)$ and the relative weak topology on $\sigma(C^0(G))$ therefore coincide. Further, if w satisfies the hypotheses of the lemma, then w is in $C(K'K; W(N))$. We denote by Λ the strongly continuous isometric representation of N on $C(K'K; C_u(N))$ by the formula

$$[\Lambda(n)t](k)(n') = t(k)(n^{-1}n') \qquad k \in K'K, n, n' \in N .$$

Then

$$\begin{aligned}
[\Lambda(n)\sigma w](k)(n') &= \sigma w(k)(n^{-1}n') \\
&= w(n^{-1}n'k) \\
&= \lambda(n)w(n'k) \\
&= [\sigma\lambda(n)w](k)(n') ,
\end{aligned}$$

so that, to prove the lemma, it will suffice to establish the following claim.

CLAIM. If t is in $C(K'K; W(N))$, then the set $\Lambda(N)t$ is relatively weakly compact.

To prove the claim, we consider first functions of the form $w \otimes y$:

$$w \otimes y: (k, n) \longrightarrow w(k)y(n) . \qquad k \in K'K, n \in N ,$$

where w and y are in $C(K'K)$ and $W(N)$ respectively. Clearly

$$\Lambda(n)w \otimes y = w \otimes \lambda(n)y .$$

Now suppose that Φ is a functional on $C(K'K; C_u(N))$. Then Φ gives rise to a functional Φ_w on $C_u(N)$ by the formula

$$\langle \Phi_w, y \rangle = \langle \Phi, w \otimes y \rangle .$$

Since y is in $W(N)$, any sequence $\{\lambda(n_j)y\}$ has a weakly convergent subsequence $\{\lambda(n'_j)y\}$; thus the sequence $\{\Lambda(n_j)w \otimes y\}$ has a weakly convergent subsequence $\{\Lambda(n'_j)w \otimes y\}$. We conclude that $\Lambda(N)w \otimes y$ is relatively weakly compact.

Now because $K'K$ is compact, any function in $C(K'K; W(N))$ can be approximated in norm by finite sums of functions of the form just considered. The set of functions t in $C(K'K; W(N))$ for which the set $\Lambda(N)t$ is relatively weakly compact is a closed linear subspace of $C(K'K; W(N))$ (c.f., the proof that $W(G)$ is a closed subspace of $C_u(G)[1, 1.6]$). This establishes the claim and completes the proof of the lemma.

A locally compact group G is said to have small invariant neighborhoods (written $G \in [\mathrm{SIN}]$) when there is a basis of neighborhoods of e in G-invariant under inner automorphisms. A function v on G is called central if $v(gg') = v(g'g)$ for all g and g' in G. A [SIN]-group has nonnegative continuous central functions v with arbitrarily small supports. Note that

$$\int v(g)dg = \int v(g'g)dg = \int v(gg')dg = \Delta(g') \int v(g)dg \,,$$

so [SIN]-groups are unimodular. A convenient reference for results on [SIN]-groups is the monograph of S. Grosser and M. Moskovitz [6].

THEOREM 2. *Suppose that H is a closed subgroup of the* [SIN]-*group G. Then*

$$E(G)|_H = E(H) \,,$$

and, if x is in $E(H)$,

$$\|x\|_E = \inf \{\|u\|_E \colon u \in E(G), \ u|_H = x\} \,.$$

Proof. As before, it suffice to show that, for any x in $E(H)$ and small positive ε, there exists u in $E(G)$ such that

$$\|u\|_E \leqq \|x\|_E$$
$$\|u|_H - x\|_E < \varepsilon \,.$$

Since G has small invariant neighborhoods, so does H. Then both G and H are unimodular, so there exists a G-invariant measure $d\dot{g}$ on the quotient space G/H (see [8, 15.24]); we assume that the Haar measures of G and H are adjusted so that

$$\int_G dgw(g) = \int_{G/H} d\dot{g} \int_H dhw(gh) \qquad w \in C_c(G) \,.$$

Let V be a compact neighborhood of e in G so that

$$(9) \qquad\qquad \|\lambda(h^{-1})x - x\|_E < \varepsilon \qquad\qquad h \in V^{-1}V \cap H \,,$$

and let v be a nonnegative continuous central function on G such that

(10)
$$\operatorname{supp}(v) \subseteqq V$$

(11)
$$\int_{G/H} d\dot{g} \int_{H} dh\, v(gh)^2 = 1 \; .$$

We define the function u on G by the formula

$$u(g') = \int_{G} dg \int_{H} dh\, v(g'g)v(gh)x(h) \qquad\qquad g' \in G \; .$$

For any h' in H,

$$u(h') = \int_{G} dg \int_{H} dh\, v(g)v(h'^{-1}gh)x(h)$$

$$= \int_{G} dg \int_{H} dh\, v(g)v(ghh'^{-1})x(h)$$

$$= \int_{G} dg \int_{H} dh\, v(g)v(gh)[\lambda(h^{-1})x](h') \; ,$$

since v is central and H is unimodular. This pointwise equality is very similar to the equality (7) of Theorem 1. By repeating the argument of Theorem 1, and using (9), (10), and (11), we deduce that

$$\| u |_{H} - x \|_{E} < \varepsilon \; .$$

The proof that u is in $E(G)$ if E is C_u is easy, and if E is B, is identical to the corresponding part of the proof of Theorem 1. We shall therefore examine the W-case only. We define $v(x)$ by the rule

$$v(x)(g) = \int_{H} dh\, v(g^{-1}h)x(h) \qquad\qquad g \in G \; ;$$

since $u = v * v(x)$ and $W(G)$ is closed under convolutions by $L^1(G)$-functions, it suffices to show that $v(x)$ is in $W(G)$.

Suppose then that x is in $W(H)$ and that $\{g_j\}$ is a sequence of points of G. As before, we let $p: G \to G/H$ be the canonical projections and examine two cases.

  Case (a). $\{p(g_j)\}$ is unbounded in G/H.
  We claim that if $\{p(g'_j)\}$ is a subsequence tending to infinity, then $\{\lambda(g'_j)v(x)\}$ converges weakly to zero. First, let $\underline{1}$ be the function on H taking the value one on H. Then

$$|\lambda(g'_j)v(x)| \leqq \|x\|_\infty \lambda(g'_j)v(\underline{1}) \; ,$$

so it suffice to show that $\{\lambda(g'_j)v(1)\}$ converges to zero. Next, since v is central,

$$\lambda(g_j')v(\underline{1})(g) = \int_H dh\, v(g^{-1}g_j h)$$

$$= \int_H dh\, v(hg^{-1}g_j) .$$

From the second integral expression, we see that $\lambda(g_j')v(\underline{1})(gh) = \lambda(g_j')v(\underline{1})(g)(g$ in G, h in $H)$, so that $\lambda(g_j')v(\underline{1})$ is of the form $[\Lambda(g_j')s]\circ p$, for some continuous function $\Lambda(g_j')s$ on G/H. Again from the second integral expression, we see that $\lambda(g_j')v(\underline{1})(g)$ is nonzero only if $hg^{-1}g_j$ is in the support of v for some h in H, i.e., only if g is in $g_j^{-1}\,\text{supp}\,(v)^{-1}H$. Thus $\{\Lambda(g_j')s\}$ is a sequence of compactly supported functions on G/H. Finally, from the first integral expression, we note that $\lambda(g_j')v(\underline{1})(g)$ is nonzero only if $g^{-1}g_j h$ lies in the support of v for some h in H, i.e., only if g_j lies in the set $g\,\text{supp}\,(v)H$. Since $\{p(g_j)\}$ tends to infinity, $\{\lambda(g_j')v(\underline{1})(g)\}$ converges to zero.

The sequence $\{\Lambda(g_j')s\}$ is therefore a sequence of compactly supported function on G/H which converges pointwise to zero. Then $\{\Lambda(g_j')s\}$ converges pointwise to zero on $\beta(G/H)$, so that $\{[\Lambda(g_j')s]\circ p\}$ converges pointwise to zero on $\beta(G)$. In other words, $\{\lambda(g_j')v(\underline{1})\}$ converges weakly to zero.

Case (b). $\{p(g_j)\}$ is bounded in G/H.

In this case there is a compact set K in G such that each g_j can be written in the form $k_j h_j$, with k_j in K and h_j in H. Note that, for g in G,

$$\lambda(g_j)v(x) = \int_H dh\, v(g^{-1}k_j h_j h)x(h)$$

$$= \int_H dh\, v(g^{-1}k_j h)x(h_j^{-1}h)$$

$$= \int_H dh\, v(hg^{-1}k_j)\lambda(h_j)x(h) .$$

There is a subsequence g_j' of g_j such that $\{v(hg^{-1}k_j)\}$ converges to $v(hg^{-1}k_0)$, say, uniformly in g and h, and such that $\{\lambda(h_j)x\}$ converges weakly to y, say. It then follows that $\{\lambda(g_j)v(x)\}$ converges to $\{\lambda(k_0)v(y)\}$: first,

$$|[\lambda(g_j)v(x) - \lambda(k_0 h_j)v(x)](g)|$$

$$\leqq \int_H dh\, |v(hg^{-1}k_j) - v(hg^{-1}k_0)|\,\|\lambda(h_j)x\|_\infty$$

which converges to zero uniformly on G. Second,

$$\lambda(k_0 h_j)v(x) - \lambda(k_0)v(y) = \lambda(k_0)v(\lambda(h_j)x - y) .$$

Given a linear functional Φ on $C(G)$, then Φ', defined

$$\langle \Phi', z \rangle = \langle \Phi, \lambda(k_0)v(z) \rangle \qquad\qquad z \in C(H)$$

is a linear functional on $C(H)$. Since $\{\langle \Phi', \lambda(h_j)x - y \rangle\}$ converges to zero, $\{\langle \Phi, \lambda(k_0)v(\lambda(h_j)x-y) \rangle\}$ does also, and $\{\lambda(k_0)v(\lambda(h_j)x-y)\}$ converges weakly to zero.

2. Corollaries. Hereafter, K denotes a compact group. We recall that the radical of a connected Lie group is the maximal connected solvable normal subgroup.

COROLLARY 1. *If G is a connected Lie group, the following properties are equivalent*:
   (i) *for every closed normal subgroup N of G, $E(G)|_N = E(N)$.*
   (ii) *the radical R of G is central.*

Proof. (i) $\Rightarrow$ (ii). The following claim is the key to this step: if A is a connected closed normal abelian subgroup of a connected Lie group H and $E(H)|_A = E(A)$, then A is central in H.

To establish the claim, we note that A can be written in the form $\mathbf{R}^b \times T^c$, T being the torus, that there exist $2b + c$ characters χ_j of $\mathbf{R}^b \times T^c$ such that

$$\cap \ker \chi_j = \{e\}$$

and that there is an open set U in H such that

$$\chi_j(h^{-1}ah) = \chi_j(a) \qquad\qquad j = 1, \cdots, 2b + c$$

by Douady's observation, cited in the introduction. Since H is connected we conclude that A is central, as claimed.

We now define inductively the solvable series of the radical R of G:

$$R_0 = R, \ R_{j+1} = [R_j, R_j] \qquad\qquad j = 0, 1, \cdots.$$

This series terminates:

$$R = R_0 \rhd R_1 \rhd \cdots \rhd R_{k-1} \rhd R_k = \{e\} \ ;$$

R_j/R_{j+1} is a nontrivial connected abelian Lie group and the R_j are characteristic in G.

We suppose temporarily that $k \geqq 2$, i.e., that R is not abelian, and derive a contradiction. The hypothesis (i) implies that $E(G)|_{R_{k-1}} = E(R_{k-1})$, so that R_{k-1} is central in G. Further,

$$
\begin{aligned}
E(G/R_{k-1})|_{R_{k-2}/R_{k-1}} &\cong E_{R_{k-1}}(G)|_{R_{k-2}} \\
&= E_{R_{k-1}}(R_{k-2}) \\
&\cong E(R_{k-2}/R_{k-1}) \ ,
\end{aligned}
$$

where, for example, $E_{R_{k-1}}(G)$ is the space of $E(G)$-functions constant on cosets of R_{k-1} in G. So R_{k-2}/R_{k-1} is central in G/R_{k-1}. However, R_{k-2} is not abelian, so contains a noncentral one-parameter subgroup A. Since R_{k-1} is central, $A \nleq R_{k-1}$, and AR_{k-1}/R_{k-1} is a one-parameter subgroup of G/R_{k-1}. Since R_{k-2}/R_{k-1} is central in G/R_{k-1}, AR_{k-1}/R_{k-1} is certainly normal in G/R_{k-1}, whence AR_{k-1} is normal in G. Finally, R_{k-1} is central in G and A is abelian but not central, so that AR_{k-1} is abelian but not central. We have just produced a noncentral normal abelian subgroup of G, which, by the claim, contradicts hypothesis (i). We conclude that $k \leq 1$, i.e., that R is abelian. Further, $E(G)|_R = E(R)$, whence R is central by the claim.

(ii) $\Rightarrow$ (i). Let $\widetilde{G}$ be the universal covering group of G, and p be the canonical projection of $\widetilde{G}$ onto G. Standard results of Lie theory imply that $\widetilde{G}$ can be written as a direct product

$$S_1 \times \cdots \times S_n \times A ,$$

where the S_j are simple (in the Lie algebra sense) and A is abelian (see, for example, [7], [9]).

If N is a closed normal subgroup of G, then $p^{-1}(N)$ is a closed normal subgroup of $\widetilde{G}$. Suppose that $(s_1, \cdots, s_n, a)$ is an element of $p^{-1}(N)$. Then $(e, \cdots, e, t_j, e, \cdots, e, e)$ normalizes $p^{-1}(N)$, so that

$$(s_1, \cdots, s_{j-1}, t_j^{-1} s_j t_j, s_{j+1}, \cdots, s_n, a) \in p^{-1}(N) \qquad \bullet$$

whence $(e, \cdots, e, s_j^{-1} t_j^{-1} s_j t_j, e, \cdots, e, e) \in p^{-1}(N)$. It follows that s_j lies in the discrete center of S_j or S_j is contained in $p^{-1}(N)$.

Now let U be a small neighborhood of e in $\widetilde{G}$ of the form $U_1 \times \cdots \times U_n \times U_0$ such that p, restricted to U, is a local isomorphism. We define a continuous projection q of U into $U \cap p^{-1}(N)$ by the rule

$$q(t_1, \cdots, t_n, b) = (t_1', \cdots, t_n', e) ,$$

where $t_j' = t_j$ if S_j is contained in $p^{-1}(N)$ and $t_j' = e$ otherwise. We observe that, for $(t_1, \cdots, t_n, b)$ in U and $(s_1, \cdots, s_n, a)$ in $p^{-1}(N)$,

$$(t_1, \cdots, t_n, b)^{-1}(s_1, \cdots, s_n, a)(t_1, \cdots, t_n, b)$$
$$= q(t_1, \cdots, t_n, b)^{-1}(s_1, \cdots, s_n, a)q(t_1, \cdots, t_n, b) ,$$

since t_j' is exactly t_j unless s_j is central.

We now pass to the quotient group G, where we have a continuous projection, again called q, of $p(U)$ into $p(U) \cap N$ such that

$$g^{-1}ng = qg^{-1}nqg \qquad\qquad g \in p(U), n \in N .$$

If x is in $E(N)$ and g is in $p(U)$, $x^g = x^{qg}$. Thus

$$\|x^g - x\|_E = \|x^{qg} - x\|_E .$$

As $g \to e$ in $p(U)$, $qg \to e$ in $p(U) \cap N$, so $\|x^g - x\|_E \to 0$; by Theorem 1, $E(N) = E(G)|_N$, as required.

COROLLARY 2. *If G is a connected Lie group, the following properties are equivalent:*
  (i) *for every closed subgroup H of G, $E(G)|_H = E(H)$.*
  (ii) *G is of the form $R^b \times K$.*

Proof. (ii) $\Rightarrow$ (i). A group of the form $R^b \times K$ has small invariant neighborhoods, so Theorem 2 applies.

(i) $\Rightarrow$ (ii). Assuming that (i) holds, the first condition of Corollary 1 is satisfied, so the radical R of G is central, and G/R is semisimple. If G/R is noncompact, it has an Iwasawa decomposition KAN, where K is compact, A is abelian, and N is nilpotent. The group A normalizes but does not centralize N. For details, see [7, Chap. VI].

Let G' be the solvable subgroup of G such that $G'/R = AN$. If H is a closed normal subgroup of G', then

$$E(G')|_H = E(G)|_{G'}|_H = E(G)|_H = E(H) ,$$

so that G' is a solvable group satisfying the condition of Corollary 1, and therefore G' is abelian. But then the quotient group G'/R is abelian, a contradiction. Thus G/R must be compact. Structure theory of such Lie groups now implies the required result ([6, Th. 4.3]).

3. **Examples.** The following are examples of groups G with normal abelian subgroups N for which $E(N) \neq E(G)|_N$. All are semidirect products of the form $G = H \times_s N$. To find an example of an $E(N)$-function with no extension in $E(G)$, is suffices to find a character χ of N such that $\chi^h \neq \chi$ for small nonidentical h in H. Alternatively, Chou's construction can be used to provide examples of $W(N)$-functions with no uniformly continuous extensions.

EXAMPLE 1. The "$ax + b$" group. Take G to be the semidirect product $R_+^* \times_s R$, where the multiplicative group of positive integers acts on R by multiplication, and N to be R.

EXAMPLE 2. The Heisenberg group. Let G be $R \times_s R^2$, where R acts on R^2 by the rule $a(b, c) = (b, c + ab)$, and let N be R^2.

EXAMPLE 3. Let G be $T \times_s C$, where the torus acts on C by multiplication, and let N be C.

EXAMPLE 4. If G is the Heisenberg group and D the central discrete subgroup of elements $(0, (0, n))(n \in \mathbf{Z})$, then G/D is a connected Lie group with compact neighborhoods of the identity invariant under inner automorphisms, but is not a [SIN]-group. Extension from the normal abelian subgroup $\mathbf{R}^2/D$ is generally not possible.

We conclude with a rather different example.

EXAMPLE 5. SL(2, $\mathbf{R}$). For the group SL(2, $\mathbf{R}$), it can be shown that any B-function can be written as the sum of a constant function and a function which vanishes at infinity (see [3]). The group SL(2, $\mathbf{R}$) contains many one-parameter subgroups H which are non-compact; for such subgroups, the restriction space $B(G)|_H$ is much smaller than $B(H)$.

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