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ON THE MEIJER TRANSFORM OF GENERALIZED FUNCTIONS

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An extension of the Meijer transform to a certain space generalized functions (distributions) is provided. The validity of the inversion formula in the distributional sense is established. Characterization theorem for the distributional Meijer transform is proved and a structure formula for the Meijer transformable generalized functions is given. An operation-transform formula is obtained, which together with the inversion formula, is applied in solving certain integrodifferential equations.

1. Introduction. During the past decade a number of integral transforms have been extended to various classes of generalized functions. Some of these extensions have been incorporated by Zemanian in his monograph [14]. The Meijer transform of ordinary functions has been studied by many authors [2], [5], [8], and [9] but its distributional theory has not yet been explored. The aim of the present paper is to extend the Meijer transform to a certain space of generalized functions and to establish certain related results. The novelity of the extension lies in the construction of the testing function space where instead of taking a differential operator one has to think of an integrodifferential operator of a certain kind.

Let k, m, and z be complex variables, let t, σ, and ω be real variables in R^1, and set $s = \sigma + i\omega$. The Whittaker functions $W_{k,m}(z)$ and $M_{k,m}(z)$ are defined by the series [7, pp. 9-10]

$$(1)\qquad M_{k,m}(z) = z^{(1/2)+m}e^{-(1/2)z}{}_1F_1\left(\frac{1}{2} + m - k; 1 + 2m; z\right)$$

and

$$(2)\qquad W_{k,m}(z) = \frac{\pi}{2\sin m\pi}\left(\frac{-M_{k,m}(z)}{\Gamma\left(\frac{1}{2}-m-k\right)\Gamma(1+2m)} + \frac{M_{k,-m}(z)}{\Gamma\left(\frac{1}{2}+m-k\right)\Gamma(1-2m)}\right).$$

The function $M_{k,m}(z)$ is analytic everywhere except at the points $2m = -1, -3, -5, \cdots$, where it has simple poles. At these points, however, the function $M_{k,m}(z)/\Gamma(1 + 2m)$ is analytic. The function $W_{k,m}(z)$ is defined for all real and complex values of k, m, and z. It is a many valued function of z. We shall take as its principal branch that which lies in the z-plane cut along the negative real axis. It is a fact that $W_{k,m}(z) = W_{k,-m}(z)$ [7, p. 11], therefore, we lose no generality in restricting according to $0 \leqq \operatorname{Re} m < \infty$.

The asymptotic behaviors of Whittaker functions for large values of z are the following [2, pp. 734–735]. For any fixed $\varepsilon > 0$ and $|z| \to \infty$,

$$e^{-1/2z} W_{k,m}(z) = e^{-1/2z} z^k \{1 + 0(|z|^{-1})\} \left(-\frac{3}{2}\pi + \varepsilon < \arg z < \frac{3}{2}\pi - \varepsilon\right) \tag{3}$$

$$\begin{aligned} e^{1/2z} M_{k,m}(z) = {} & \frac{\Gamma(1+2m)}{\Gamma\left(\frac{1}{2} - k + m\right)} e^z z^{-k}\{1 + O(|z|^{-1})\} \\ & + \frac{\Gamma(1+2m)}{\Gamma\left(\frac{1}{2} + k + m\right)} e^{-(k-m-1/2)\pi i} z^k \{1 + O(|z|^{-1})\} \\ & \left(-\frac{1}{2}\pi + \varepsilon < \arg z < \frac{3}{2}\pi - \varepsilon\right) \end{aligned} \tag{4}$$

$$\begin{aligned} e^{1/2z} M_{k,m}(z) = {} & \frac{\Gamma(1+2m)}{\Gamma\left(\frac{1}{2} - k + m\right)} z^{-k}\{1 + O(|z|^{-1})\} \\ & + \frac{\Gamma(1+2m)}{\Gamma\left(\frac{1}{2} + k + m\right)} e^{(k-m-1/2)\pi i} z^k \{1 + O(|z|^{-1})\} \\ & \left(-\frac{3}{2}\pi + \varepsilon < \arg z < \frac{1}{2}\pi - \varepsilon\right). \end{aligned} \tag{5}$$

The other results that we shall need are the following differentiation formula [7, p. 25]

$$\frac{d}{dx}\{e^{-1/2x} x^{m-1/2} W_{k,m}(x)\} = -e^{-1/2x} x^{m-1} W_{k+1/2,m-1/2}(x) \tag{6}$$

and the indefinite integral [2, p. 733]

$$\begin{aligned} (x - t) \int & e^{1/2xs} M_{k-1/2,m}(xs) e^{-1/2ts} W_{k+1/2,m}(ts) s^{-1} ds \\ = {} & \frac{1}{m-k}[2mx^{1/2}s^{-1/2}e^{1/2xs}M_{k,m-1/2}(xs)e^{-1/2ts}W_{k+1/2,m}(ts) \\ & - (k+m)t^{1/2}s^{-1/2}e^{1/2xs}M_{k+1/2,m}(xs)e^{-1/2ts}W_{k,m-1/2}(ts)]. \end{aligned} \tag{7}$$

Now, we reproduce Meijer's inversion theorem in the original form.

THEOREM (Meijer). *Let $F(s)$ be an analytic function on the half plane $\operatorname{Re} s > a \geqq 0$. For some real constant $c > a$, let the integral*

$$\int_{-\infty}^{\infty} |F(c+iy)|\,dy$$

converge. Moreover, assume that $F(s)$ *is bounded according to* $|F(s)| < A,\ A > 0$ *for* $\operatorname{Re} s \geqq c$ *and that* $F(x+iy) \to 0$ *as* $x \to \infty$ *uniformly for* $-\infty < y < \infty$. *Finally, assume that* $\operatorname{Re} k \leqq -\operatorname{Re} < 1/2$. *Then, for* $\operatorname{Re} s > c$,

$$(8) \qquad F(s) = \int_0^\infty e^{-1/2st}\, W_{k+1/2,m}(st)(st)^{-k-1/2} f(t)\,dt$$

where

$$(9) \qquad f(t) = \frac{\Gamma(1-k+m)}{2\pi i \Gamma(1+2m)} \int_{c-i\infty}^{c+i\infty} e^{1/2tz} M_{k-1/2,m}(tz)(tz)^{k-1/2} F(z)\,dz\ .$$

2. An integrodifferential operator. From the differential equation satisfied by Whittaker functions [7] it is a simple exercise to show that the kernels

$$(10) \qquad K(x) \triangleq e^{-1/2x}\, W_{k+1/2,m}(x) x^{-k-1/2}$$

and

$$(11) \qquad H(x) \triangleq \frac{\Gamma(1-k+m)}{\Gamma(1+2m)} e^{1/2x} M_{k-1/2,m}(x) x^{k-1/2}$$

satisfy the integrodifferential equations

$$(12) \qquad \Delta_x K(\alpha x) = -\alpha K(\alpha x)$$

and

$$(13) \qquad \nabla_x H(\alpha x) = \alpha H(\alpha x)$$

respectively, where Δ_x and ∇_x are defined as below:

$$(14) \qquad \Delta_x \triangleq \Delta_x^{k,m} \triangleq x^{-1}(x^{-2k} D^{-1} x^{2k-1})(x^{1-k+m} D x^{k-m})(x^{1-k-m} D x^{k+m})$$

$$(15) \qquad \nabla_x \triangleq \nabla_x^{k,m} \triangleq x^{-1}(x^{2k} D^{-1} x^{-2k-1})(x^{1+k+m} D x^{-k-m})(x^{1+k-m} D x^{-k+m})$$

in Δ_x we interpret $D^{-1} = \int_\infty^x \cdots dt$ and in ∇_x, $D^{-1} = \int_0^x \cdots dt$.

REMARK. The operator Δ_x can be applied on any $C^\infty(R+)$ function ϕ any number of times which satisfies the asymptotic orders

$$(16) \qquad \phi^{(r)}(x) = O(x^{\alpha - r})\ , \quad x \longrightarrow \infty\ , \qquad r = 0, 1, 2, \cdots$$

where $\alpha + 2\operatorname{Re} k < 0$. If $\phi^{(r)}(x)$ possess exponentially small aymptotic orders as $x \to \infty$, then this condition does not apply. The operator ∇_x can be applied to any $C^\infty(R+)$ function ϕ any number of times which satisfies the asymptotic orders

(17) $$\phi^{(r)}(x) = O(x^{\alpha - r}) , \quad x \longrightarrow 0+ , \qquad r = 0, 1, 2, \cdots$$

where $\alpha > 2 \operatorname{Re} k$. Furthermore, if $\phi(x) \in C^\infty(R+)$ is of compact support in $(0, \infty)$ then the two interpretations of D^{-1} are identical and the aforesaid asymptotic order conditions are not required.

Some properties of these operators are described below.

LEMMA 1. *Let $\phi \in C^\infty(R+)$ with the asymptotic orders* (16) *(or* (17) *in case of ∇_x), then the integration operator $(x^{-2k}D^{-1}x^{2k-1})$ and the differentiation operator $(x^{1-k+m}Dx^{k-m})$ occuring in Δ_x (or in ∇_x) when acting on ϕ in succession are commutative.*

Proof. A simple computation shows that

$$\begin{aligned}
&(x^{1-k+m}Dx^{k-m})(x^{-2k}D^{-1}x^{2k-1})\phi(x) \\
&\quad = x^{1-k+m}Dx^{-m-k}\int_\infty^x y^{2k-1}\phi(y)dy \\
&\quad = \phi(x) - (m + k)x^{-2k}\int_\infty^x y^{2k-1}\phi(y)dy , \quad \alpha + \operatorname{Re} 2k < 0
\end{aligned}$$

and

$$\begin{aligned}
&(x^{-2k}D^{-1}x^{2k-1})(x^{1-k+m}Dx^{k-m})\phi(x) \\
&\quad = x^{-2k}D^{-1}x^{k+m}[x^{k-m}\phi'(x) + (k - m)x^{k-m-1}\phi(x)] \\
&\quad = \phi(x) - (m + k)x^{-2k}\int_\infty^x y^{2k-1}\phi(y)dy , \quad \alpha + \operatorname{Re} 2k < 0 .
\end{aligned}$$

This proves the lemma.

COROLLARY. *The differentiation and integration operators as defined in Lemma 1 occuring in Δ_x and ∇_x when acting on $\phi \in C^\infty(R+)$ satisfying* (16) *in case of ∇_x and* (17) *in case of ∇_x can be switched in any order.*

Proof. Since two differentiation operators are commutative the result follows in view of Lemma 1.

3. The testing function space $\mathscr{T}_a^{k,m}(I)$. Let I denote the open interval $(0, \infty)$, $x \in I$ and let a be a real positive number and k and m be complex numbers. Assume that $\operatorname{Re} m \geqq 0$. Now, define $\mathscr{T}_a^{k,m}(I)$ to be the collection of all infinitely differentiable complex valued functions $\phi(x)$ on I with the properties (16) and

(18) $$\rho_n(\phi) \triangleq \rho_{a,n}^{k,m}(\phi) \triangleq \sup_{0<x<\infty} |e^{ax}x^{k+m}\Delta_x^n\phi(x)| < \infty , \quad n = 0, 1, 2, \cdots$$

where Δ_x is the integrodifferential operator defined by (14). The sequence $\{\rho_n\}_{n=0}^\infty$ is a separating collection of seminorms [14, p. 8] which generates the topology of $\mathscr{I}_a^{k,m}(I)$. It can be readily seen that $\mathscr{I}_a^{k,m}(I)$ is a locally convex, sequentially complete, Hausdorff topological vector space. The dual space of $\mathscr{I}_a^{k,m}(I)$ is denoted by $\mathscr{I}_a^{k,m\prime}(I)$.

Let $D(I)$ denote the space of infinitely differentiable complex valued functions with compact support on I, equipped with the usual topology. The dual space $D'(I)$ is the space of Schwartz distributions on I [14, pp. 33-34]. It is easily seen that $D(I) \subset \mathscr{I}_a^{k,m}(I)$ and that the topology of $D(I)$ is stronger than that induced on it by $\mathscr{I}_a^{k,m}(I)$. Hence the restriction of any $f \in \mathscr{I}_a^{k,m\prime}(I)$ to $D(I)$ is in $D'(I)$.

For $0 < a < b$ the space $\mathscr{I}_b^{k,m} \subset \mathscr{I}_a^{k,m}$, and the topology of $\mathscr{I}_b^{k,m}$ is stronger than the topology induced on it by $\mathscr{I}_a^{k,m}$. Consequently, the restriction of $f \in \mathscr{I}_a^{k,m\prime}$ to $\mathscr{I}_b^{k,m}$ is in $\mathscr{I}_b^{k,m\prime}$ and the convergence in $\mathscr{I}_a^{k,m\prime}$ implies convergence in $\mathscr{I}_b^{k,m\prime}$.

We notice that for every fixed s such that $\operatorname{Re} s > a > 0$ and $\operatorname{Re} m \geqq 0$, $(st)^{-k-1/2}e^{-1/2st}W_{k+1/2,m}(st)$ is a member of $\mathscr{I}_a^{k,m}(I)$.

4. The Meijer transform of generalized functions. Let f be a member of $\mathscr{I}_a^{k,m\prime}$ for some k, m, and a. Then, from the preceding argument it is clear that there exists some real number $\sigma_f \geqq 0$, depending upon f such that $f \in \mathscr{I}_a^{k,m\prime}$ for all $a > \sigma_f$ and $f \notin \mathscr{I}_a^{k,m\prime}$ for every $a < \sigma_f$.

Now recall the definition (10) of $K(z)$. Since $K(st) \in \mathscr{I}_a^{k,m}$ for every s such that $\operatorname{Re} s > a$ and $\operatorname{Re} m \geqq 0$, we may define the distributional Meijer transform of f by

$$F(s) \triangleq \mathscr{M}_{k,m}f(s) \triangleq \langle f(t), K(st)\rangle\ , \quad \operatorname{Re} s > \sigma_f \tag{19}$$

where σ_f is called the abscissa of definition.

LEMMA 2. *Let* $\operatorname{Re} m \geqq 0$, *and let* a *and* b $(>a)$ *be two real numbers. Then, for* $\operatorname{Re} \zeta \geqq b$, $\zeta \neq 0$, $-\pi < \arg \zeta \leqq \pi$ *and* $0 < t < \infty$,

$$|e^{at}(\zeta t)^{m-1/2}e^{-1/2\zeta t}W_{k+1/2,m}(\zeta t)| < A(1 + |\zeta|^{\lambda_r}) \tag{20}$$

where A *is a constant independent of* ζ *and* t, *and* $\lambda_r = \operatorname{Re}(m + k)$.

Proof. The proof can be given by following the technique of Zemanian [14, p. 184] and using the estimates

$$|z^{m-1/2}e^{-1/2z}W_{k+1/2,m}(z)| < A \quad \text{for} \quad \operatorname{Re} m \geqq 0 \quad \text{and} \quad |z| \leqq 1$$

and

$$|z^{m-1/2}e^{-1/2z}W_{k+1/2,m}(z)| < B|z|^{\lambda_r}e^{-\operatorname{Re} z} \quad \text{for} \quad |z| > 1\ .$$

These estimates can easily be obtained from the series representation (2) and the asymptotic expansion (3).

THEOREM 1. (*Analiticity of* $F(s)$). *For* $\operatorname{Re} s > \sigma_f$, *let* $F(s)$ *be the Meijer transform of* $f \in \mathscr{J}_a^{k,m'}$ *defined by* (19). *Then,* $F(s)$ *in analytic and*

$$\frac{d}{ds}F(s) = \left\langle f(t), \frac{\partial}{\partial s}K(st)\right\rangle \tag{21}$$

where $\operatorname{Re} m \geqq 0$.

Proof. Using the differentiation formula (6), series representation (2) and the asymptotic expansion (3) we observe that $\partial/\partial s K(st) \in \mathscr{J}_a^{k,m}(I)$ and hence the right-hand side of (21) is meaningful. Using Lemma 2 and following the technique of Zemanian [14] used in proving Theorem 6.5-1, p. 185, the proof can be given.

5. **Inversion and uniqueness.** In this section we shall prove an inversion theorem for the distributional Meijer transform and then deduce an uniqueness theorem.

LEMMA 3. *For* $\operatorname{Re} s > \sigma_f$, *let* $F(s)$ *be defined by* (19). *Let* $\phi \in D(I)$, *and set*

$$\psi(s) = \int_0^\infty K(st)\phi(t)dt\,, \quad \operatorname{Re} s > 0\,.$$

Then, for any fixed real number r *in* $(0, \infty)$,

$$\int_{-r}^{r} \psi(s)\langle f(\tau), K(s\tau)\rangle d\omega = \left\langle f(\tau), \int_{-r}^{r} \psi(s)K(st)d\omega\right\rangle \tag{22}$$

where $s = \sigma + i\omega$ *and* σ *is fixed with* $\sigma > \max(0, \sigma_f)$.

Proof. Consider the integral

$$I(\tau) = \int_{-r}^{r} \psi(s)K(s\tau)dw \tag{23}$$

where $\max(0, \sigma_f) < a < \sigma$. For $\operatorname{Re} m \geqq 0$ we can apply the operator Δ_τ within the integral sign in (23) and write

$$\begin{aligned} |e^{a\tau}\tau^{k+m}\Delta_\tau^{(n)} I(\tau)| &= \left|\int_{-r}^{r} \psi(s)e^{a\tau}s^n K(st)d\omega\right| \\ &\leqq \int_{-r}^{r} |\psi(s)s^{n-k-m}|\, A(1 + |s|^{\lambda_r})d\omega < \infty \\ &\quad \text{(by Lemma 2)}\,. \end{aligned}$$

This proves that $I(\tau) \in \mathscr{J}_a^{k,m}$ and hence the right-hand side of (22) is meaningful. The equality (22) can be proved by following the technique of Riemann sums [14, pp. 187-188].

LEMMA 4. *Let $\phi(x) \in D(I)$ and let its support be contained in $[c, d]$, where $0 < c < d < \infty$. Let $\operatorname{Re} m \geqq 0$, $\operatorname{Re}(m-k) \geqq 0$ and $\operatorname{Re} k < 1/2$. Then for fixed $\sigma > a \geqq 0$,*

$$W_r(\tau) \triangleq \frac{1}{2\pi}\int_{-r}^{r} K(s\tau)\int_0^\infty \phi(t)H(st)dtd\omega\ , \quad s = \sigma + i\omega$$

converges in $\mathscr{J}_a^{k,m}$ to $\phi(\tau)$ as $r \to \infty$.

Proof. In view of the definitions of the operators Δ_x and ∇_x, we have

$$\begin{aligned}\Delta_\tau^{(n)} W_r(\tau) &= \frac{1}{2\pi}\int_{-r}^{r} \Delta_\tau^{(n)} K(s\tau)\int_0^\infty \phi(t)H(st)dtd\omega \\ &= \frac{1}{2\pi}\int_{-r}^{r} K(s\tau)\int_0^\infty \phi(t)(-1)^n \nabla_t^{(n)} H(st)dtd\omega \\ &= \frac{1}{2\pi}\int_{-r}^{r} K(s\tau)\int_0^\infty \phi_n(t)H(st)dtd\omega\end{aligned}$$

where $\phi_n(t) \triangleq \Delta_t^{(n)}\phi(t)$, on integrating by parts with respect to t n times. Changing the order of integration we can write

$$\Delta_\tau^{(n)} W_r(\tau) = \int_c^d U_r(t, \tau)\phi_n(t)dt\ , \tag{24}$$

where

$$\begin{aligned}U_r(t, \tau) &= \frac{1}{2\pi i}\frac{\Gamma(m-k)}{\Gamma(2m+1)}\frac{\tau^{-k-1/2}}{t-\tau}t^{-k-1/2} \\ &\quad \times [2mt^{1/2}s^{-1/2}e^{1/2st}M_{k,m-1/2}(st)e^{-1/2s\tau}W_{k+1/2,m}(s\tau) \\ &\quad - (k+m)\tau^{1/2}s^{-1/2}e^{1/2st}M_{k+1/2,m}(st)e^{-1/2s\tau}W_{k,m-1/2}(s\tau)]_{\sigma-ir}^{\sigma+ir}\ .\end{aligned} \tag{25}$$

Now, break up the integration (24) into integrations on $c < t < \tau - \delta$, $\tau - \delta < t < \tau + \delta$ and $\tau + \delta < t < d$ where $0 < \delta < c$ and denote the corresponding integrals by I_1, I_2, and I_3 respectively. We shall show first that

$$V_r(\tau) \triangleq e^{a\tau}\tau^{k+m}[I_2(\tau) - \phi_n(\tau)]\ , \qquad (n = 1, 2, \cdots)$$

converges uniformly to zero on $0 < \tau < \infty$ as $r \to \infty$. If either $\tau + \delta \leqq c$ or $\tau - \delta \geqq d$, then $I_2 \equiv 0$ and $\phi_n(\tau) \equiv 0$. Therefore, we consider the case $c - \delta < \tau < d + \delta$.

Now, for $s = \sigma \pm ir$ where $\sigma > 0$ is fixed, using the asymptotic orders (3), (4), and (5) we can write

$$
\begin{aligned}
V_r(\tau) = e^{a\tau}\frac{\tau^{m+k}}{\pi}\Big[\int_{\tau-\delta}^{\tau+\delta}&\frac{\sin r(t-\tau)}{t-\tau}e^{\sigma(t-\tau)}\phi_n(t)dt\\
&+\int_{\tau-\delta}^{\tau+\delta}\frac{\sin r(t-\tau)}{t-\tau}e^{\sigma(t-\tau)}\phi_n(t)\Big\{O\Big(\frac{1}{|st|}\Big)+O\Big(\frac{1}{|s\tau|}\Big)\\
&+O\Big(\frac{1}{|st|}\Big)O\Big(\frac{1}{|s\tau|}\Big)\Big\}dt\\
&-(k+m)\Big\{1+O\Big(\frac{1}{|s\tau|}\Big)\Big\}\int_{\tau-\delta}^{\tau+\delta}e^{\sigma(s-\tau)}t^{-1}\Big(\frac{e^{ir}(t-\tau)}{\sigma+ir}-\frac{e^{-ir}(t-\tau)}{\sigma-ir}\Big)\\
&\times\phi_n(t)\Big\{1+O\Big(\frac{1}{|st|}\Big)\Big\}dt\Big]\\
&-e^{a\tau}\tau^{k+m}\phi_n(\tau)\,. \qquad (26)
\end{aligned}
$$

It is a simple exercise to show that the second and third terms on the right-hand side of (26) are uniformly bounded on the domain

$$\Omega_1 \triangleq \{(t,\tau): c < t < d,\ c-\delta < \tau < d+\delta\}$$

by $\varepsilon/3$ for all $r > 1$ and δ sufficiently small, say $\delta = \delta_1$.

Next, the difference of the first and last term in (26) can be written as

$$\frac{1}{\pi}\int_{-\delta}^{\delta}G(x,\tau)\sin(rx)dx + e^{a\tau}\tau^{k+m}\phi_n(\tau)\Big[\frac{1}{\pi}\int_{-\delta r}^{\delta r}\frac{\sin y}{y}dy - 1\Big] \qquad (27)$$

where $G(x,\tau)$ is defined by

$$
\begin{aligned}
G(x,\tau) &= e^{a\tau}\tau^{m+k}\frac{1}{x}[e^{\sigma x}\phi_n(\tau+x)-\phi_n(\tau)] \quad && x \neq 0\\
&= e^{a\tau}\tau^{m+k}\phi_n'(\tau) && x = 0\,.
\end{aligned}
$$

Then $G(x,\tau)$ is a continuous function of (x,τ) for $x+\tau > 0$ and $\tau > 0$. Consequently, the first term in (27) can be made less than $\varepsilon/3$ for all $r > 1$ by choosing δ small enough, say $\delta = \delta_2$. Now, fix $\delta = \min(\delta_1, \delta_2)$. Since the second term in (27) converges uniformly to zero on $0 < \tau < \infty$ as $r \to \infty$, we conclude that

$$\overline{\lim_{r\to\infty}}\,|V_r(\tau)| \leqq \varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, $V_r(\tau)$ converges uniformly to zero on $0 < \tau < \infty$ as $r \to \infty$.

Following the technique of Zemanian [14, pp. 191-194] it can be shown that

$$e^{a\tau}\tau^{k+m}I_1(\tau) \quad \text{and} \quad e^{a\tau}\tau^{k+m}I_3(\tau)$$

converge uniformly to zero on $0 < \tau < \infty$ as $r \to \infty$. This proves the lemma.

Now, we are able to establish the following inversion theorem.

THEOREM 2 (*Inversion*). *Let $F(s)$ be the distributional Meijer transform of $f \in \mathscr{I}_a^{k,m\prime}(I)$ for* $\operatorname{Re} s > \sigma_f$ *defined by*

$$F(s) \triangleq \langle f(t), (st)^{-k-1/2} e^{-1/2st} W_{k+1/2,m}(st) \rangle , \tag{28}$$

where $\operatorname{Re} m \geqq 0$, $\operatorname{Re}(m-k) \geqq 0$ *and* $\operatorname{Re} k < 1/2$. *Then for each* $\phi(x) \in D(I)$,

$$\begin{aligned} &\lim_{r\to\infty} \left\langle \frac{1}{2\pi i} \frac{\Gamma(1+m-k)}{\Gamma(1+2m)} \int_{\sigma-ir}^{\sigma+ir} F(s)(st)^{-k-1/2} e^{1/2st} M_{k-1/2,m}(st)ds, \phi(t) \right\rangle \\ &\quad = \langle f(t), \phi(t) \rangle \end{aligned} \tag{29}$$

where σ is any fixed number greater than a.

Proof. Recall the definitions (10) and (11) of $K(x)$ and $H(x)$ respectively. The theorem will be proved by establishing the following string of equalities.

$$\left\langle \frac{1}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} F(s)H(st)ds, \phi(t) \right\rangle \tag{30}$$

$$= \int_0^\infty \phi(t)dt \frac{1}{2\pi} \int_{-r}^{r} F(s)H(st)d\omega \quad (s = \sigma + i\omega) \tag{31}$$

$$= \frac{1}{2\pi} \int_{-r}^{r} \langle f(\tau), K(s\tau) \rangle \int_0^\infty \phi(t)H(st)dtd\omega \tag{32}$$

$$= \left\langle f(\tau), \frac{1}{2\pi} \int_{-r}^{r} K(s\tau) \int_0^\infty \phi(t)H(st)dtd\omega \right\rangle \tag{33}$$

$$\to \langle f(\tau), \phi(\tau) \rangle . \tag{34}$$

Since $\phi(t)$ is of compact support (30) is a repeated integral on (t, ω) and consequently (30) equals (31). Since by Theorem 1 $F(s)$ is analytic, for fixed r we can change the order of integration and arrive at (32). To which an application of Lemma 3 yields (33). Now, (33) goes into (34) by Lemma 4.

From the above inversion theorem the following uniqueness theorem can be deduced as a corollary.

COROLLARY. *Let $F(s) = \mathscr{M}_{k,m} f$ for* $\operatorname{Re} s > \sigma_f$, *let $G(s) = \mathscr{M}_{k,m} g$ for* $\operatorname{Re} s > \sigma_g$, *and let $F(s) = G(s)$ for* $\operatorname{Re} s > \max(\sigma_f, \sigma_g)$. *Then in the sense of equality in $D'(I)$, $f = g$.*

6. An operation-transform formula. Now, we shall obtain an operation-transform formula which may be used in solving certain integrodiffierential equations.

We define an operator $\Delta_x^*\colon \mathscr{J}_a^{k,m\prime}(I) \to \mathscr{J}_a^{k,m\prime}(I)$ by the relation

$$\langle \Delta_x^* f(x), \phi(x)\rangle \triangleq \langle f(x), \Delta_x \phi(x)\rangle$$

for all $f \in \mathscr{J}_a^{k,m\prime}(I)$ and $\phi \in \mathscr{J}_a^{k,m}(I)$. Let us call Δ_x^* as the adjoint of the operator Δ_x defined by (14). It can also be shown that for all $r = 1, 2, 3, \cdots$ and $\phi(x) \in \mathscr{J}_a^{k,m}(I)$,

$$\langle (\Delta_x^*)^r f(x), \phi(x)\rangle = \langle f(x), (\Delta_x)^r \phi(x)\rangle .$$

It can be readily seen from the definitions of the operators Δ_x and ∇_x given in §2 that if f is a regular generalized function in $\mathscr{J}_a^{k,m\prime}(I)$ generated by a member of $D(I)$, then

$$\Delta_x^* f \equiv \nabla_x f .$$

THEOREM 3. *Let $F(s)$ be the distributional Meijer transform of f for* $\operatorname{Re} s > \sigma_f$, *then for any positive integer r,*

$$\mathscr{M}_{k,m}[(\Delta_x^*)^r f] = (-s)^r F(s) . \tag{35}$$

The proof of trivial.

7. Characterization of Meijer transforms.

The following theorem gives a characterization of distributional Meijer transforms.

THEOREM 4 (*Characterization*). *Let* $\operatorname{Re} m \geqq 0$ *and* $\operatorname{Re} k \leqq -\operatorname{Re} m < 1/2$. *Then a necessary and sufficient condition for a function $F(s)$ to be the Meijer transform of some generalized function according to our definition given in §4 is that there be a half-plane* $\{s \mid \operatorname{Re} s \geqq b > 0\}$ *on which $F(s)$ is analytic and bounded according to*

$$|F(s)| \leqq P_b(|s|) \tag{36}$$

where $P_b(|s|)$ is a polynomial in $|s|$ depending in general on the choice of b.

Proof. Necessity. By Theorem 1 $F(s)$ is analytic function of s for $\operatorname{Re} s > \sigma_f$. Choose two real numbers a and b such that $\sigma_f < a < b$. Then, $K(st) \in \mathscr{J}_a^{k,m}$ for $\operatorname{Re} s > b$. Now, by the boundedness property of generalized functions [14, pp. 18-19], there exist a constant C and a nonnegative integer r such that

$$\begin{aligned}|F(s)| &\leqq C \max_{0\leqq n\leqq r} \rho_n(K(st)) \\ &= C \max_{0\leqq n\leqq r} \sup_{0<t<\infty} |e^{at} t^{k+m} \Delta_t^{(n)} \{e^{-1/2st}(st)^{-k-1/2} W_{k+1/2,m}(st)\}|\end{aligned}$$

$$= C \max_{0 \leqq n \leqq r} \sup_{0<t<\infty} |e^{at}t^{k+m}s^n e^{-1/2st}(st)^{-k-1/2} W_{k+1/2,m}(st)|$$
$$\leqq C|s|^{n-\lambda_r} e^{|\lambda_i|\pi} \sup_{0<t<\infty} |e^{at}(st)^{m-1/2} e^{-1/2st} W_{k+1/2,m}(st)|$$

where $\lambda_r = \mathrm{Re}\,(m+k)$ and $\lambda_i = \mathrm{Im}\,(m+k)$. The inequality (36) now follows from Lemma 2.

Sufficiency. Let q be a real number greater than 1 and let n be a positive integer such that $n-q$ is greater than or equal to the degree of $P_b(|s|)$. Then, $s^{-n}F(s)$ satisfies the assumptions of Meijer's theorem stated in §1 and therefore, for $\mathrm{Re}\, s > c > b$,

$$s^{-n}F(s) = \int_0^\infty g(t)e^{-1/2st} W_{k+1/2,m}(st)(st)^{-k-1/2}dt \tag{37}$$

where

$$g(t) = \frac{\Gamma(1-k+m)}{2\pi i \Gamma(1+2m)} \int_{c-i\infty}^{c+i\infty} s^{-n}F(s)e^{1/2st}M_{k-1/2,m}(st)(st)^{k-1/2}ds\ . \tag{38}$$

Now, consider the expression

$$\begin{aligned} \frac{g(t)e^{-ct}}{t^{k+m}(1+t^{-\lambda_r})} = \frac{1}{2\pi i}\frac{\Gamma(1-k+m)}{\Gamma(1+2m)} \int_{c-i\infty}^{c+i\infty} & s^{-n+k+m}(1+|s|^{-\lambda_r})F(s) \\ & \times \left[\frac{e^{-ct}e^{1/2st}M_{k-1/2,m}(st)(st)^{-m-1/2}}{(1+t^{-\lambda_r})(1+|s|^{-\lambda_r})}\right]ds\ . \end{aligned} \tag{39}$$

Using the series representation (1) and the asymptotic expansions (4) and (5) and following the technique of the proof of Lemma 2 it can be shown that

$$|e^{-ct}(st)^{-m-1/2}e^{1/2st}M_{k-1/2,m}(st)| \leqq D(1+|s|^{-\lambda_r})(1+t^{-\lambda_r})$$

on the line $s = c + i\omega$, $-\infty < \omega < \infty$ uniformly for all $t \in (0, \infty)$, where D is a constant independent of s and t. Furthermore,

$$\begin{aligned} |s^{-n-k+m}(1+|s|^{-\lambda_r})F(s)| &\leqq (|s|^{-n}P_b(|s|) + |s|^{\lambda_r - n}P_b(|s|))e^{|\lambda_i|\pi/2} \\ &\leqq E(|s|^{-q} + |s|^{\lambda_r - q})\ , \end{aligned}$$

where E is another constant. Since $q > 1$ and $\lambda_r \leqq 0$, it follows that for any $d>c$, $e^{-dt}g(t)(1+t^{\lambda_r})^{-1}$ is absolutely integrable on $0<t<\infty$, and consequently $e^{-dt}t^{-\lambda_s}g(t)$ is also absolutely integrable on the same interval. Hence $g(t)$ generates a regular distribution of $\mathscr{F}_d^{k,m\prime}(I)$. Therefore, (37) represents a distributional Meijer transform for $\mathrm{Re}\, s > d$.

Now, let $f = (-\Delta_x^*)^n g$. Then, by Theorem 3,

$$\mathscr{M}_{k,m}[f] = s^n \mathscr{M}_{k,m}[g] = F(s)$$

for at least $\mathrm{Re}\, s > d$. This completes the proof.

We conclude this section with the following structure theorem.

THEOREM 5. *Let f be an arbitrary element of $\mathscr{I}_a^{k,m\prime}(I)$. Then there exist bounded measurable functions $g_i(x)$ defined for $x > 0$ and for $i = 0, 1, 2, \cdots, r$ where r is some nonnegative integer depending upon f such that for arbitrary $\phi \in D(I)$, we have*

$$\langle f, \phi\rangle = \left\langle -\sum_{i=0}^{r} \nabla_x^i\left[e^{ax}x^{k+m}D_x^2 \int_0^x g_i(t)dt\right], \phi(x)\right\rangle , \tag{40}$$

where ∇_x is the integrodifferential operator defined by (15).

Proof. The proof is analogous to a number of proofs available in the literature [10, pp. 272-274; 6, pp. 14-15] and therefore is omitted.

8. **Applications.** Now we will apply our inversion theory to the solution of certain integrodifferential equations.

(a) *Solution of $P(\Delta_x^*)u = g$.* Let P be any polynomial. For $\operatorname{Re} m \geqq 0$ and $\operatorname{Re} k \leqq -\operatorname{Re} m < 1/2$, consider the operational equation

$$P(\Delta_x^*)u = g \quad 0 < x < \infty \tag{41}$$

where g is a given Meijer transformable generalized function and u is unknown generalized function.

Now to determine u, using (35) we apply the distributional Meijer transformation to (41) and get

$$P(-s)U(s) = G(s)$$

where $G(s) = \mathscr{M}_{k,m}g$ for $\operatorname{Re} s > \sigma_g$. Let σ_p be the largest of the real parts of the roots of $P(-s) = 0$. Then $G(s)/P(-s)$ satisfies hypotheses of Theorem 4 on some half-plane $\{s \mid \operatorname{Re} s \geqq b > \max(0, \sigma_g, \sigma_p)\}$ and hence it is a distributional Meijer transform of some $u \in \mathscr{I}_b^{k,m\prime}$. We may apply the inversion formula (29) to get u. Thus

$$u(x) = \lim_{r\to\infty} \frac{\Gamma(1 - k + m)}{2\pi i\Gamma(1 + 2m)} \times \int_{\sigma-ir}^{\sigma+ir} [G(s)/P(-s)](st)^{-k-1/2}e^{1/2st}M_{k-1/2,m}(st)ds \tag{42}$$

in the sense of equality in $D'(I)$, which is a solution of (41). This solution is in fact a restriction of $u \in \mathscr{I}_b^{k,m\prime}(I)$ to $D(I)$, and is unique in view of the corollary following Theorem 2.

By arguments preceding Theorem 3 one can easily verify that u as determined by (42) is also a solution to the distributional integrodifferential equation

$$P(\nabla_x^{k,m})u = g \;. \tag{43}$$

(b) *Solution of* $P(\nabla_x^{-k,m})\phi = \psi$. Suppose that ψ is a given Meijer transformable conventional function possessing the asymptotic properties:

$$\begin{aligned}\psi(x) &= O(e^{ax}) \quad x \longrightarrow \infty\\ &= O(x^{\rho}) \quad x \longrightarrow 0+\end{aligned}$$

where $a > 0$ and $\operatorname{Re}(\pm m - k) + \rho + 1 > 0$. We wish to find ϕ such that

$$P(\Delta_x^{-k,m})\phi = \psi \;. \tag{44}$$

If we assume that

$$\begin{aligned}\phi^{(r)}(x) &= O(e^{bx}) \;, \quad x \longrightarrow \infty\\ &= O(x^{\beta - r}) \;, \quad x \longrightarrow 0+\end{aligned}$$

for each $r = 0, 1, 2, \cdots$, we can apply Meijer transform (8) to (44) and get

$$\int_0^\infty P(\Delta_x^{-k,m})\phi(x)K(sx)dx = \Psi(s) \tag{45}$$

where $\operatorname{Re} s > \max(a, b)$ and $\Psi(s)$ is a Meijer transform of $\psi(x)$. Now, using the formula [2, p. 733]

$$\frac{d}{dz}\{z^k e^{-1/2z} W_{k,m}(z)\} = -z^{k-1}e^{-1/2z}W_{k+1,m}(z)$$

and integrating by parts the left-hand side of (45), we get

$$P(-s)\Phi(s) = \Psi(s)$$

where $\Phi(s)$ is the Meijer transform of $\phi(x)$. If we further assume that $\operatorname{Re} s \geqq c > \max(a, b, \sigma_q)$, where σ_q is the largest of the real parts of roots of $P(-s) = 0$, we find that $\Psi(s)/P(-s)$ satisfies conditions of Meijer's theorem (given in §1), and hence is the Meijer transform of some function $\phi(x)$ defined by

$$\begin{aligned}\phi(x) = \frac{\Gamma(1 - k + m)}{2\pi i \Gamma(1 + 2m)} &\int_{c-i\infty}^{c+i\infty} e^{1/2xs} M_{k-1/2,m}(xs)(xs)^{-k-1/2}\\ &\times [\Psi(s)/P(-s)]ds \;.\end{aligned} \tag{46}$$

Following the technique of proof of sufficiency part of Theorem 4 it can be shown that $\phi(x)$, as a regular distribution, is a member of $\mathscr{J}_d^{k,m\prime}(I)$, where $d > c$.

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