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ON COMMON FIXED POINT SETS OF COMMUTATIVE MAPPINGS

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Let C be a compact convex subset of a locally convex topological vector space X. Anzai and Ishikawa recently proved that if $T_1, \cdots, T_n$ is a finite commutative family of continuous affine self-mappings of C, then $F(\sum_{i=1}^n \lambda_i T_i) = \bigcap_{i=1}^n F(T_i)$ for every λ_i such that $0 < \lambda_i < 1$ and $\sum_{i=1}^n \lambda_i = 1$, where $F(T)$ denotes the fixed point set of T. It is natural to question whether the conclusion of their theorem is dependent on the topological properties of X, C and T_i—in this case, the linear topology, the compactness and the continuity. We shall see that this is not; the theorem can be formulated in an algebraic context.

Our theorem, when applied to Hausdorff topological vector spaces, yields a better version of Anzai-Ishikawa's theorem (see Corollary 2).

DEFINITION 1. A subset B of a real vector space is said to be (algebraically) bounded if $\bigcap_{\varepsilon>0} \varepsilon(C - C) = \{0\}$, where $C = C_0(B)$, the convex hull of B.

Every bounded convex subset of a Hausdorff topological vector space is algebraically bounded. Every bounded subset of a locally convex Hausdorff topological vector space is algebraically bounded.

THEOREM 1. *Let C be a convex subset of a real vector space X and $T_1, \cdots, T_n$ a finite commutative family of affine self-mappings of C. If the set $D = \{T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n} x : 0 \leqq m_i < \infty, i = 1, \cdots, n\}$ is bounded for each $x \in C$, then $F(\sum_{i=1}^n \lambda_i T_i) = \bigcap_{i=1}^n F(T_i)$ for every $0 < \lambda_i < 1$ with $\sum_{i=1}^n \lambda_i = 1$.*

LEMMA 1. *Let x_n be a sequence in a Banach space such that $x_n \to x$. Then the sequence y_n defined by*

$$y_n = (1/2^n)(x_0 + {}_nC_1 x_1 + \cdots + {}_nC_i x_i + \cdots + x_n)$$

converges to x.

Proof. For an arbitrary $\varepsilon > 0$, choose m such that $\|x_i - x\| < \varepsilon/2$ for $i \geqq m$. Choose $N \geqq m$ such that

$$1/2^n(1 + {}_nC_1 + \cdots + {}_nC_{m-1}) < \varepsilon/(2D)$$

for all $n \geqq N$, where D is a number such that $\|x_i - x\| \leqq D$ for all $i \geqq 0$. Then

$$
\begin{aligned}
\|y_n - x\| \\
&= \|(1/2^n)(x_0 - x + {}_nC_1(x_1 - x) + \cdots + {}_nC_{m-1}(x_{m-1} - x) + \cdots + x_n - x)\| \\
&\leqq (1/2^n)(1 + {}_nC_1 + \cdots + {}_nC_{m-1})D + (\varepsilon/2)(1/2^n)({}_nC_m + \cdots + 1) \\
&< \varepsilon
\end{aligned}
$$

for all $n \geqq N$.

REMARK 1. The above lemma is also a consequence on Silverman-Toeplitz's theorem on regular method of summability.

Proof of Theorem 1. We may assume that $n = 2$. The inclusion $\bigcap_1^n F(T_i) \subset F(\sum_1^n \lambda_i T_i)$ is obvious. Let $A = \lambda_1 I + \lambda_2 T_1$, $B = \lambda_2 I + \lambda_1 T_2$ and $T = (1/2)(A + B)$. Then $T = (1/2)(I + \lambda_1 T_1 + \lambda_2 T_2)$. Moreover, $F(T) = F(\lambda_1 T_1 + \lambda_2 T_2)$, $F(A) = F(T_1)$ and $F(B) = F(T_2)$. Let $x \in F(\lambda_1 T_1 + \lambda_2 T_2) = F(T)$. For every n, we have

$$
\begin{aligned}
x &= \left(\frac{A + B}{2}\right)^n x \\
&= \frac{1}{2^n}(A^n x + {}_nC_1 A^{n-1} Bx + \cdots + {}_nC_i A^{n-i} B^i x + \cdots + B^n x)
\end{aligned}
\tag{1}
$$

and

$$
T_1 x = \frac{1}{2^n}(T_1 A^n x + {}_nC_1 T_1 A^{n-1} Bx + \cdots + {}_nC_i T_1 A^{n-i} B^i x + \cdots + T_1 B^n x)\,, \tag{2}
$$

where we make use of the commutativity of A and B and the affine property of T_1.

Following Anzai-Ishikawa's computation [1], we have

$$
\begin{aligned}
A^m y - T_1 A^m y &= \sum_{i=0}^{m} {}_mC_i \lambda_1^{m-i} \lambda_2^i T_1^i y - \sum_{i=1}^{m+1} {}_mC_{i-1} \lambda_1^{m-i+1} \lambda_2^{i-1} T_1^i y \\
&= \sum_{i=0}^{m+1} ({}_mC_i \lambda_1^{m-i} \lambda_2^i - {}_mC_{i-1} \lambda_1^{m-i+1} \lambda_2^{i-1}) T_1^i y,\ {}_mC_{-1} = {}_mC_{m+1} = 0 \\
&= \sum_{i=0}^{m_0} \mu_i T_1^i y - \sum_{i=m_0+1}^{m+1} (-\mu_i) T_1^i y \\
&= a_m \left(\sum_{i=0}^{m_0} \alpha_i T_1^i y - \sum_{i=m_0+1}^{m+1} \beta_i T_1^i y \right).
\end{aligned}
$$

Here, m_0 is the largest integer less than or equal to $\lambda_2(m + 1)$;

$$
\mu_i = {}_mC_i \lambda_1^{m-i} \lambda_2^i - {}_mC_{i-1} \lambda_1^{m-i+1} \lambda_2^{i-1}\,,
$$

$\mu_i \geqq 0$ for $0 \leqq i \leqq m_0$ and < 0 for $m_0 + 1 \leqq i \leqq m + 1$;

$$
a_m = \sum_{i=0}^{m_0} \mu_i = \sum_{i=m_0+1}^{m+1} (-\mu_i) = {}_mC_{m_0} \lambda_1^{m-m_0} \lambda_2^{m_0} \longrightarrow 0
$$

as $m \to \infty$; $\alpha_i = \mu_i/a_m \geqq 0$ for $0 \leqq i \leqq m_0$, $\beta_i = -\mu_i/a_m \geqq 0$ for $m_0 + 1 \leqq i \leqq m+1$, $\sum_{i=0}^{m_0} \alpha_i = 1$ and $\sum_{i=m_0+1}^{m+1} \beta_i = 1$.

Let $E = C_0(D)$. By the convexity of E, $A^m y - T_1 A^m y \in a_m(E - E)$ provided $T_1^i y \in E$ for $i = 0, \cdots, m+1$.

Since T_1 and T_2 are affine, $T_1^i A^k B^j x \in E$ for $j, k = 0, \cdots, n$; $i = 0, 1, \cdots$. It follows from (1) and (2) that

$$\begin{aligned} x - T_1 x &= \frac{1}{2^n}((A^n x - T_1 A^n x) + nC_1(A^{n-1}Bx - T_1 A^{n-1}Bx) + \cdots + (B^n x - T_1 B^n x)) \\ &\in \frac{1}{2^n}(a_n(E - E) + {}_nC_1 a_{n-1}(E - E) + \cdots + {}_nC_{n-1} a_1(E - E) + a_0(E - E)) \\ &\subseteqq \frac{1}{2^n}(a_0 + nC_1 a_1 + \cdots + {}_nC_i a_i + \cdots + a_n)(E - E)\ , \end{aligned}$$

the last inclusion being a consequence of the convexity of $E - E$.

Since $E - E$ is convex and $0 \in E - E$, we have $\varepsilon_1(E - E) \subseteqq \varepsilon_2(E - E)$ if $\varepsilon_1 < \varepsilon_2$. Hence by Lemma 1 and the boundedness of E, $\bigcap_1^n A(n)(E - E) = \{0\}$ where

$$A(n) = \frac{1}{2^n}(a_0 + {}_nC_1 a_1 + \cdots + {}_nC_i a_i + \cdots + a_n)\ .$$

It follows that $x = T_1 x$. Similarly $x = T_2 x$. This completes the proof.

COROLLARY 1. *Let C be a bounded convex subset of a vector space and $T_1, \cdots, T_n$ a finite commutative family of affine mappings of C. Then $F(\sum_{i=1}^n \lambda_i T_i) = \bigcap_{i=1}^n F(T_i)$ for all positive numbers λ_i, $i = 1, \cdots, n$ such that $\sum_{i=1}^n \lambda_i = 1$.*

COROLLARY 2. *Let C be a convex bounded (in the usual sense) subset of a Hausdorff topological vector space and $T_1, \cdots, T_n$ a finite commutative family of affine mappings of C. Then $F(\sum_i^n \lambda_i T_i) = \bigcap_1^n F(T_i)$ for all positive numbers λ_i, $i = 1, \cdots, n$ such that $\sum_1^n \lambda_i = 1$.*

REMARK 2. We note that the boundedness condition cannot be removed. The mappings $T_1 x = x + a$, $T_2 x = x - a$, $a \neq 0$ defined on R^1 are commutative and affine, with $F(T_1) = F(T_2) = \phi$ and $F((1/2)T_1 + (1/2)T_2) = R^1$.

COROLLARY 3. *Let C, T_i, $i=1, \cdots, n$ be defined as in Corollary 2. Assume that $T_1^p \cdots T_n^p = T_1 \cdots T_n$ for some $p \geqq 2$ and that for each $x \in C$ and each $i = 1, \cdots, n$, the set*

$$A_i x = \{T_{T_1}^{m_1} \cdots \hat{T}_i^{m_i} \cdots T_n^{m_n} X\colon 0 \leqq m_j < \infty,\ j = 1, \cdots, n\}$$

is bounded, where $\wedge$ *indicates that* $T_i^{m_i}$ *is missing. Then* $F(\sum_1^n \lambda_i T_i) = \bigcap_1^n F(T_i)$ *for all positive numbers* λ_i, $i = 1, \cdots, n$ *such that* $\sum_1^n \lambda_i = 1$.

Proof. If m_i, $i = 1, \cdots, n$ are n natural numbers and $m_j = \min\{m_i, i = 1, \cdots, n\}$, then

$$\begin{aligned} T_1^{m_1} \cdots T_n^{m_n} X &= T_1^{m_1-m_j} \cdots \hat{T}_j^{m_j-m_j} \cdots T_n^{m_n-m_j} T_1^{m_j} \cdots T_n^{m_j} x \\ &= T_1^{m_1-m_j} \cdots T_n^{m_n-m_j} T_1^k \cdots T_n^k x \in A_j(T_1^k \cdots T_n^k x) \end{aligned}$$

where k is an integer satisfying $0 \leqq k < p$. It follows that

$$\begin{aligned} Ax &= \{T_1^{m_1} \cdots T_n^{m_n} x : 0 \leqq m_i < \infty, i = 1, \cdots, n\} \\ &= \bigcup \{A_i(T_1^k \cdots T_n^k x) : i = 1, \cdots, n, k = 0, \cdots, p-1\} . \end{aligned}$$

Hence Ax, being a finite union of bounded sets is bounded.

The special case when $n = 2$ and $p = 2$ can be given a simple direct proof. We shall illustrate it for the case $\lambda_1 = \lambda_2 = 1/2$. First we prove:

LEMMA 2. *For each* $n \geqq 1$, *there exists rational numbers (depending on* n *and not necessarily nonnegative)* $\lambda_1, \cdots, \lambda_{[n/2]}$ *such that*

$$\lambda_1 + \cdots + \lambda_{[n/2]} = 1 - (1/2^{n-1})$$

and such that the equation

$$\begin{aligned} D^n = (1/2^{n-1})\left(\frac{A^n + B^n}{2}\right) + \lambda_1 ABD^{n-2} + \cdots + \lambda_i A^i B^i D^{n-2i} + \cdots \\ + \lambda_{[n/2]} A^{[n/2]} B^{[n/2]} D^{n-2[n/2]} \end{aligned} \tag{3}$$

is valid for any two commutative affine mappings A, B *defined on a convex set, where* $D = 1/2(A+B)$. *(*$[m]$ *denotes the largest integer* $\leqq m$.*)*

Proof. If such rational numbers λ_i exist for a fixed n, then by putting $A = B = I$, we see that

$$\lambda_1 + \cdots + \lambda_{[n/2]} = 1 - (1/2^{n-1}) .$$

We shall prove by induction on n. For $n = 2$, $\lambda_1 = 1/2$. Assume that the lemma is true for $m \leqq n$. Then

$$\begin{aligned} D^{n+1} = D^n D = (1/2^n)\left(\frac{A^{n+1} + B^{n+1}}{2}\right) + \frac{1}{2}(1/2^{n-2})\left(\frac{A^{n-1} + B^{n-1}}{2}\right)AB \\ + \lambda_1 ABD^{n-1} + \cdots + \lambda_{[n/2]} A^{[n/2]} B^{[n/2]} D^{n-2[n/2]+1} . \end{aligned} \tag{4}$$

Making use of the induction hypothesis for $m = n - 1$, substitue

$$(1/2^{n-2})\left(\frac{A^{n+1} + B^{n+1}}{2}\right) = D^{n-1} - \mu_1 ABD^{(n+1)-4} - \cdots$$
$$- \mu_{[(n-1)/2]} A^{[(n-1/2]} B^{[(n-1/2]} D^{n-1-2[(n-1)/2]}$$

into (4). The proof will be then complete by collecting similar terms and making use of $[(n-1)/2] + 1 = [(n+1)/2]$ and $[(n+1)/2] - [n/2] = 0$ or 1.

COROLLARY 4. *Let C be defined as in Theorem 1 and A, B be two commutative affine self-mappings of C such that $A^2B^2 = AB$ and such that the sets $\{A^n x: n = 0, 1, 2, \cdots\}$ and $\{B^n x: n = 0, 1, 2, \cdots\}$ are bounded for each $x \in C$. Then $F((1/2)A + (1/2)B) = F(A) \cap F(B)$.*

Proof. Let $x \in F((1/2)A + (1/2)B)$. Using Lemma 2 and the condition $A^2B^2 = AB$, we have

$$x = \left(\frac{A + B}{2}\right)^n x = (1/2^{n-1})\left(\frac{A^n x + B^n x}{2}\right) + \left(1 - \frac{1}{2^{n-1}}\right)ABx . \tag{5}$$

Thus,

$$x - ABx = \frac{1}{2^{n-1}}\left(\frac{A^n x + B^n x}{2} - ABx\right) .$$

By the boundedness condition, we see that $ABx = x$. By (3) for $n = 2$, we have $x = (A^2x + B^2x)/2$. By applying A to $x = (1/2)Ax + (1/2)Bx$ we have $Ax = (1/2)A^2x + (1/2)x$ and hence $A^2x - x = 2(Ax - x)$. Thus by repeatedly replacing A, B by A^2 and B^2 in the above argument, we obtain $A^{2n}x - x = 2^n(Ax - x)$. This contradicts the boundedness of $\{A^n x: n = 0, 1, 2, \cdots\}$ unless $Ax = x$. Similarly $Bx = x$, completing the proof.

REFERENCES

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