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ON CHARACTERIZATIONS OF EXPONENTIAL POLYNOMIALS

PHILIP G. LAIRD

This paper considers some characterizations of exponential polynomials in $C(G)$, the set of all continuous complex valued functions on a σ-compact locally compact Abelian group G. For $f \in C(G)$, U_f will denote the subspace of $C(G)$ obtained by taking finite linear combinations of translates of f. It is known that f is an exponential polynomial if and only if U_f is of finite dimension. Our main result is to show that f is an exponential polynomial when U_f is closed in $C(G)$ if $C(G)$ is given the topology of convergence uniform on all compact subsets of G.

Further characterizations of exponential polynomials are given when G is real Euclidean n-space, R^n.

A function $b \in C(G)$ is additive if $b(x+y) = b(x) + b(y)$ for all $x, y \in G$ and $g \in C(G)$ is an exponential if $g(x+y) = g(x)g(y)$ for all $x, y \in C(G)$. An exponential polynomial is a finite linear combination of terms $h = b_1^{q_1} b_2^{q_2} \cdots b_m^{q_m} g$ where $b_1, b_2, \cdots, b_m$ are additive, $q_1, q_2, \cdots, q_m$ are nonnegative integers and g is an exponential.

If f is an exponential polynomial, it is easy to see that U_f is finite dimensional. For if h is as above, then $T_\alpha h\colon x \to h(x-\alpha)$ is a finite linear combination of terms $b_1^{r_1} b_2^{r_2} \cdots b_m^{r_m} g$ for each $\alpha \in G$ where $r_j = 0, 1, \cdots, q_j$ for $j = 1, 2, \cdots, m$. A result of Engert [5] shows that if U_f is finite dimensional, then f is an exponential polynomial. The proof of this result when G is any σ-compact locally compact Abelian group is naturally more involved than when G is merely R or R^n. Proofs for the case of $C(R)$ may be found in Anselone and Korevaar [1] and Loewner [8] who also refers to $C(R^n)$.

Throughout this paper, the only topology considered on $C(G)$ is that of convergence uniform on all compact subsets of G. With G being σ-compact, let G be the countable union of compact sets K_p. Let $S_p(f) = \sup\{|f(x)|\colon x \in K_p\}$ and $d(f, g) = \sum_{p=1}^{\infty} 2^{-p} \min(1, S_p(f-g))$ for $f, g \in C(G)$. Then d is a metric for $C(G)$ and $C(G)$ is complete in this metric.

With such a topology for $C(G)$, if U_f is finite dimensional, it is closed. The converse to this is shown here (Theorem 3) so that in $C(G)$,

$$f \text{ is an exponential polynomial} \Longleftrightarrow U_f \text{ is finite dimensional}$$
$$\Longleftrightarrow U_f \text{ is closed in } C(G)\,.$$

In showing that when U_f is closed, it is then finite dimensional, the following notation shall be used throughout. As above, assume that $G = \bigcup_{p=1}^{\infty} K_p$ where each K_p is compact. For a given function f in $C(G)$, set

$$S_p = \Big\{ g \in C(G)\colon g = \sum_{k=1}^{p} a_k T_{\beta_k} f$$
$$\text{where } |a_k| \leqq p \text{ and } \beta_k \in K_p \text{ for } k = 1, 2, \cdots, p \Big\} .$$

It is clear that $U_f = \bigcup_{p=1}^{\infty} S_p$. The method of proof is one suggested by Edwards [4], pages 38–39 in establishing the result for functions on the circle group.

LEMMA 1. *S_p is pointwise equicontinuous in $C(G)$.*

Proof. Let $x \in G$ and $\varepsilon > 0$. Let B denote the set of all neighborhoods of 0 in G. It suffices to show that there is a $U \in B$ such that

$$|f(x - \alpha) - f(y - \alpha)| < \varepsilon/p^2 \text{ for all } \alpha \in K_p \text{ and all } y \text{ with } y - x \in B .$$

Then

$$|g(x) - g(y)| < \sum_{k=1}^{p} |a_k| \varepsilon/p^2 \leqq \varepsilon$$

whenever $y - x \in U$ and $g \in S_p$.

Set $F = x - K_p$ so if $\alpha \in K_p$, $\beta = x - \alpha \in F$. For each $\beta \in F$, there exists $V_\beta \subset B$ such that $|f(z) - f(\beta)| < \varepsilon/2p^2$ whenever $z - \beta \in V_\beta$. For this V_β, there is a $W_\beta \in B$ such that $W_\beta + W_\beta \subset V_\beta$. With $\{\beta + W_\beta\colon \beta \in F\}$ forming an open cover for the compact set F, select a finite subcover $\{\beta_j + W_{\beta_j}\}_{j=1}^{m}$. Let $W = \bigcap_{j=1}^{m} W_{\beta_j}$ and $U = W \cap (-W)$ so $U \in B$. If $\alpha \in K_p$ and $x - \alpha \in F$, $x - \alpha \in \beta_l + W_{\beta_l}$ say. Then

$$y - \alpha = y - x + x - \alpha \in U + x - \alpha \subset \beta_l + V_{\beta_l}$$

which also contains $x - \alpha$. Hence $f(x - \alpha)$ and $f(y - \alpha)$ differ from $f(\beta_l)$ by amounts in modulus less than $\varepsilon/2p^2$ and the result follows.

LEMMA 2. *S_p is compact in $C(G)$.*

Proof. Use is made of the condition that in $C(G)$, a closed equicontinuous set S is compact if $S[x] = \{f(x)\colon f \in S\}$ is compact in C (see, for example, [3], page 34 or [6], page 234). With f being continuous and $x \in G$, $\{f(x - \beta)\colon \beta \in K_p\}$ is compact whence $S_p[x]$ is compact in C. To show that S_p is closed, let $\{g_q\}$ be any Cauchy sequence in S_p with $g_q=\sum_{k=1}^{p} a_{q,k}T_{\beta_{q,k}}f$. Since $|a_{q,1}|\leqq p$ for all positive integers q, a convergent subsequence $a_{q',1}$ may be found with limit, say a_1, and $|a_1|\leqq p$. Continue in this manner to find convergent subsequences $\{a_{r,k}\}_{r=1}^{\infty}$ for $k=1, 2, \cdots, p$ with respective limits a_k where $|a_k|\leqq p$. Now use $\{\beta_{r,k}\}_{r=1}^{\infty}\subset K_p$ for $k=1, 2, \cdots, p$ and K_p is compact to find convergent subsequences $\{\beta_{v,k}\}_{v=1}^{\infty}$. With $a_{v,k}\to a_k$, $|a_k|\leqq p$ and $\beta_{v,k}\to\beta_k\in K_p$ as $v\to\infty$ for $k=1, 2, \cdots, p$, it follows that $g_v\to g$ for some $g\in S_p$ So $g_q\to g$ as $q\to\infty$ showing that S_p is closed. Hence S_p is compact in $C(G)$.

THEOREM 3. *If U_f is closed in $C(G)$, then U_f is finite dimensional.*

Proof. Since $U_f=\bigcup_{p=1}^{\infty} S_p$ is closed in the metric space $C(G)$, it follows by Baire's category theorem applied to U_f that there must be as S_p that is not nowhere dense. As this S_p is closed, it must have a nonvoid interior. Hence U_f contains a compact neighbourhood of zero. So, by Riesz's theorem (see, for example [3], page 65) U_f is finite dimensional.

The remainder of this article, concerns exponential polynomials in $C(R^n)$. These functions in $C(R^n)$ are finite linear combinations of terms $x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n}\exp(a_1x_q+a_2x_2+\cdots+a_nx_n)$ where $x=(x_1, x_2, \cdots, x_n)\in R^n$, $p_1, p_2, \cdots, p_n$ are nonnegative integers and $a_1, a_2, \cdots, a_n$ are complex numbers. In restricting G to be R^n, little economy of the proof of Theorem 3 is gained except for Lemma 1. However, it is considerably easier to show for $C(R^n)$ compared with $C(G)$ that if U_f is finite dimensional, then f is an exponential polynomial. A new and simple proof is as follows.

Suppose that U_f has finite dimension m where $m>1$. (If $m=0$, $f=0$ and if $m=1$ a simpler version of the following suffices.) Let $g_1, g_2, \cdots, g_m$ be a basis of U_f and $g=(g_1, g_2, \cdots, g_m)$. Then $T_\alpha g=A(\alpha)g$ where $A(\alpha)$ is an $m\times m$ complex matrix. From $T_{\alpha+\beta}=T_\alpha T_\beta$, one finds that $A(\alpha+\beta)=A(\alpha)A(\beta)$ and $A(0)=I$, the unit matrix. Since $T_\alpha f\to T_\beta f$ as $\alpha\to\beta$, $A(\alpha)$ is continuous. So $z\in R^n$ near 0 may be chosen and fixed so that $A(z)$ is nonsingular. It is clear from

$$A(x)=\left(\int_{x_1}^{x_1+z_1}\cdots\int_{x_n}^{x_n+z_n}A(y)dy\right)(A(z))^{-1},$$

that each partial derivative of A exists. Letting $\{e_1, e_2, \cdots, e_n\}$ be the standard basis for R^n,

$$D_jg=\lim_{h\to 0}(A(-he_j)-A(0))g/h=C_jg,$$

where the matrix $C_j=D_jA(0)$. So $D_j(\exp(-C_jx_j)g)=0$ showing that $g = \exp(C_j x_j)\phi_j$ where ϕ_j is independent of x_j for $j = 1, 2, \cdots, n$ and ϕ_j takes value in R^m.

From $\exp(C_1 x_1)\phi_1 = \exp(C_2 x_2)\phi_2$ with $x_1 = 0$ $\phi_1(x_2, x_3, \cdots, x_n) = \exp(C_2 x_2)\phi_2(0, x_3, x_4, \cdots, x_n)$. Successively equating $\exp(C_j x_j)\phi_j = \exp(C_{j+1} x_{j+1})\phi_{j+1}$ with $x_j = 0$ for $j = 1, 2, \cdots, n-1$, we find

$$g = \exp(C_1 x_1) \exp(C_2 x_2) \cdots \exp(C_n x_n) d$$

where $d \in R^n$ is constant. As it is well known that the elements of $\exp(Cx)$ are exponential polynomials in x ([2], page 46), it follows that the components of g are exponential polynomials. Hence f is an exponential polynomial in $C(R^n)$ when U_f is finite dimensional.

Other characterizations of exponential polynomials in $C(R^n)$ are now given. For $C(R)$, one such is that of the set of all solutions to all nontrivial linear ordinary differential equations with constant coefficients. For $C(R^n)$ with $n > 1$, one cannot identify the set of all exponential polynomials with the set of all solutions to all nontrivial linear partial differential equations with constant coefficients. However, a necessary and sufficient condition that $f \in C(R^n)$ be an exponential polynomial is that there exists n nonzero linear differential operators $L_j = L_j(D_j)$ with constant coefficients where each L_j only involves the jth partial derivative D_j and $L_j f = 0$ for $j = 1, 2, \cdots, n$. A proof of this given by Laird [7], page 816, is reproduced here for completeness. The necessity of the condition is obvious. Conversely, if $f \in C(R^n)$ and if $L_1 f = 0$, then f is a finite sum of terms $A(x_2, x_3, \cdots, x_n)x_1^{q_1} \exp ax_1$. With $L_2 f = 0$, $L_2 A = 0$ and so each A is a finite sum of terms $B(x_3, x_4, \cdots, x_n)x_2^{q_2} \exp bx_2$. Continuing in this manner, one finds that f is an exponential polynomial.

The following is an extension of the above result.

THEOREM 4. *Let $f \in C(R^n)$ and let $A = (a_{jk})$ be a real nonsingular $n \times n$ real matrix. Then a necessary and sufficient condition that f be an exponential polynomial is that there exist n nonzero polynomials $P_1, P_2, \cdots, P_n$, each of one variable, such that*

$$P_j(a_{j1}D_1 + a_{j2}D_2 + \cdots + a_{jn}D_n)f = 0$$

for $j = 1, 2, \cdots, n$.

Proof. Let $u_k = \sum_{m=1}^{n} b_{km}x_m$ for $k = 1, 2, \cdots, n$ and $f(x) = g(u)$. Then

$$D_m f(x) = \sum_{k=1}^{n} \frac{\partial g}{\partial u_k} \frac{\partial u_k}{\partial x_m}$$

so that

$$\sum_{m=1}^{n} a_{jm} D_m f = \frac{\partial g}{\partial u_j}$$

when $B = (b_{km})$ is chosen so that $B^T = A^{-1}$. The given condition is then $P_j(D_j)g = 0$ for $j = 1, 2, \cdots, n$ which is equivalent to g and so to f being an exponential polynomial.

THEOREM 5. *Let $a \in R^n$, $f \in C(R^n)$ and $U_f(a)$ denote the subspace in $C(R^n)$ obtained from finite linear combinations of terms $f(x - ta)$ for $t \in R$. A necessary and sufficient condition that f be an exponential is that $U_f(a_j)$ be finite dimensional for n linearly independent vectors $a_1, a_2, \cdots, a_n$ in R^n.*

Proof. The necessity is easily seen from $U_f(a) \subset U_f$ for all $a \in R^n$, and if f is an exponential polynomial, then U_f is finite dimensional.

The converse, which has been recognized by Loewner [8] when $\{a_1, a_2, \cdots, a_n\}$ is the standard basis, may be shown directly, or as follows. Let $f_j(t) = f(ta_j)$ for all $t \in R$ and $j = 1, 2, \cdots, n$. If each $U_f(a_j)$ is finite dimensional in $C(R^n)$, then U_{f_j} is finite dimensional in $C(R^n)$. So each f_j is an exponential polynomial and there is a nonzero polynomial P_j so that $P_j(D)f_j = 0$. With $Df_j = a.\operatorname{grad} f$, the conditions of the sufficiency part of Theorem 4 are satisfied. Hence f is an exponential polynomial in $C(R^n)$.

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UNIVERSITY OF WOLLONGONG
WOLLONGONG, N. S. W. 2500
AUSTRALIA