PACIFIC JOURNAL OF MATHEMATICS
Vol. 80, No. 2. 1979

BOUNDARY CONTINUITY OF SOME HOLOMORPHIC FUNCTIONS

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For certain bounded domains D in C^n any continuous function on $D \cup \partial_{A(D)}$, which is holomorphic on D automatically extends continuously to D^-.

For a bounded domain D in C^n let $A(D)$ be the sup normed algebra of functions continuous on D^- and holomorphic on D, and let $\partial = \partial_{A(D)}$ denote its Šilov boundary, so $\partial \subset \partial D$. Of course this inclusion can be proper, and in his thesis [0] A. Aytuna raised the question of whether every bounded continuous function on $\partial \cup D$ holomorphic on D necessarily extends continuously at all points of $\partial D \backslash \partial$. Aytuna showed this held when $n \geqq 2$ for the half-ball $D = \{z: |z| < 1, \operatorname{Re} z_1 > 0\}$, where $\partial D \backslash \partial = \{z: |z| < 1, \operatorname{Re} z_1 = 0\}$ is a union of analytic discs and normal family arguments apply. In fact there are simple domains for which continuous extension fails, as we shall see below (§ 3), while it holds rather trivially for starlike domains; our purpose here is to point out some classes of domains for which it holds, and indeed something stronger obtains, by virtue of some elementary function algebra facts combined with the Oka-Weil approximation theorem.

Recall that $K \subset D^-$ is a peak set for $A(D)$ if there is an f in $A(D)$ with $f(K) = 1$ and $|f| < 1$ on $D^- \backslash K$. $P(K)$ will denote the closure in $C(K)$ of the analytic polynomials and $H^\infty(D)$ the bounded holomorphic functions on D.

THEOREM 1. *Suppose $\partial D \backslash \partial$ is differentiable and covered by a union of peak sets K for $A(D)$, for each of which*

(1.1) *f holomorphic near K implies $f|K$ is uniformly approximable by polynomials, and*

(1.2) *$x \in K \backslash \partial$ implies $(0, \varepsilon_x)\nu_x + K \subset D$, where ν_x is the inward unit normal to ∂D at x, and $x \rightarrow \varepsilon_x$ is a positive continuous function on $\partial D \backslash \partial$.*

If h is bounded and holomorphic on D, and, for one of our peak sets K_0, has a continuous extension to $D \cup (\partial \cap K_0)$, then h extends continuously to $D \cup K_0$.

In particular if h extends continuously to $D \cup \partial$ it extends to an element of $A(D)$; in fact in this case we need not assume h bounded on D. Hypothesis (1.1) holds if each peak set K is polynomially convex by the Oka-Weil approximation theorem (cf. [3], which will also be our reference for facts on uniform algebras).

An alternative to (1.1) is the assumption that each K provides the spectrum of the (necessarily closed [3]) algebra $A(D)|K$: for then K is the joint spectrum of the coordinate functions and the functional calculus [3, p. 76] applies to assert any f holomorphic near K has $f|K \in A(D)|K$, which is just the property used. (Note that if D^- is the spectrum of $A(D)$, each K has this property.) Hypothesis (1.2), as we shall see, is one of many which might be used; the assumption that we can move all of K into D can be relaxed considerably in the presence of additional hypotheses, with some complication in the argument; in fact we need only know that (as more smoothness of $\partial D \backslash \partial$ guarantees) near each $x_0 \in \partial D \backslash \partial$ not too small chunks of each K can be moved along a normal into D, with rather more information on our K_0.

THEOREM 2. *Suppose $\partial D \backslash \partial$ is a C^2 manifold and is covered by a union of peak sets K for $A(D)$ which are polynomially convex. Suppose that for one of these, K_0, there are $v_j \to 0$ in C^n with $v_j + K_0 \subset D$. Then if h is bounded and holomorphic on D and extends continuously to $D \cup (\partial \cap K_0)$, h extends continuously to $D \cup K_0$.*

More generally, suppose K_0 only contains polynomially convex subsets K_j for which $K_j + v_j \subset D$ for $v_j \to 0$ in C^n, while for some $z_0 \in K_0 \backslash \partial$

(2.1) *each probability measure λ on $\partial \cap K_0$ representing z_0 on polynomials is a w^* cluster point of a bounded sequence $\{\lambda_j\}$, where λ_j is a complex measure on K_j multiplicative on polynomials.*

Then any bounded holomorphic h on D continuous on $D \cup (\partial \cap K_0)$ has a unique cluster value at z_0. (The first assertion follows from the second by taking $K_j = K_0$, $\lambda_j = \lambda$.)

1. One of the main function algebra facts we shall use is that if K is a peak set for $A(D)$ then $A(D)|K$ is closed in $C(K)$ [3]; another is that $\partial \cap K$ provides a boundary for this algebra. (Any representing measure for $x \in K$ on ∂ must be carried by K, hence by $\partial \cap K$, as one sees by applying it to f^n, where $f \in A(D)$ peaks on K, and letting $n \to \infty$.) Both enter our proof of Theorem 1 which has been greatly simplified by T. W. Gamelin, to whom I would like to express my thanks.

Let $E \subset D^-$ be closed and contain the Šilov boundary ∂. The basic step in our proof of Theorem 1 is the more general.

LEMMA 1. *Suppose $z_0 \in \partial D \backslash \partial$ lies in the peak set K for $A(D)$, K satisfies (1.1), and $v_j + K \subset D$ for a sequence $v_j \to 0$ in C^n.*

Then if

$$(*)\qquad \limsup_{z\in D\to z_0}|g(z)| \leqq \limsup_{z\in D\to E}|g(z)|,\ g\in H^\infty(D)\ ,$$

any $f\in H^\infty(D)$ *which extends continuously at all points of* $E\cap K$ *extends continuously at* z_0.

For the proof, define $f_j\in C(K)$ by $f_j(z) = f(z+v_j)$. Because of (1.1) and our hypothesis that $v_j + K\subset D$ we know f_j is a uniform limit of polynomials on K, hence of elements of $A(D)|K$; since this algebra is closed, $f_j\in A(D)|K$. But f extends continuously at all points of $E\cap K$ so the sequence $\{f_j|E\cap K\}$ converges uniformly on $E\cap K$, a set which includes the boundary $\partial\cap K$ for $A(D)|K$, so that in fact $\{f_j\}$ converges in $A(D)|K$. Its limit is the restriction to K of some $g\in A(D)$, and evidently for $w\in E\cap K$, $g(w)=\lim_{z\in D\to w} f(z)$. Thus $f-g$ tends to zero at each point of $E\cap K$, and given $\varepsilon>0$, by compactness we have a neighborhood U of $E\cap K$ in D^- for which $|f-g|<\varepsilon$ on $U\cap D$.

Now let $h\in A(D)$ peak on K, and let V be an open neighborhood in D^- of $E\backslash U$ at a positive distance from K. Since $h^m\to 0$ uniformly on V we have an m for which $|(f-g)h^m|<\varepsilon$ on V, and the same is true on $U\cap D$ because $|f-g|<\varepsilon$ and $|h|\leqq 1$ there. Thus

$$\limsup_{z\in D\to E}|(f-g)h^m(z)|\leqq\varepsilon$$

so that $\varepsilon\geqq\limsup_{z\in D\to z_0}|(f(z)-g(z))h^m(z)|=\limsup_{z\in D\to z_0}|f(z)-g(z)|$ by (*). Since ε is arbitrary $g(z_0)$ evidently provides the unique cluster value for f at z_0, yielding our conclusion.

A simple condition insuring (*) is provided by

LEMMA 2. *Suppose that for each sequence* $\{z_j\}$ *in* D *converging to* z_0 *there are closed sets* $\mathscr{K}_j$ *and* E_j *for which* $z_j\in\mathscr{K}_j$, $E_j\subset\mathscr{K}_j\subset D$, $\limsup E_j\subset E$, *while* $\sup|f(\mathscr{K}_j)| = \sup|f(E_j)|$, *all* $f\in H^\infty(D)$. *Then* (*) *holds.*

Let M denote the left side of (*) and choose z_j in D so that $|f(z_j)|\to M$, $z_j\to z_0$. With E_j and $\mathscr{K}_j$ as above we have $w_j\in E_j\subset D$ for which $|f(w_j)|\to M$, and by hypothesis the w_j accumulate only in E, so trivially (*) follows.

Now in order to prove Theorem 1 it only remains to verify that we can apply Lemma 2 to any $z_0\in K_0\backslash\partial$ since in the presence of (*) Lemma 1 applies. So suppose $z_j\to z_0$, $z_j\in D$, and let $x_j\in\partial D$ be nearest z_j; taking j large we can assume x_j lies in $\partial D\backslash\partial$, and in fact in a compact neighborhood of z_0 in $\partial D\backslash\partial$ so that by (1.2) and the continuity of $x \to \varepsilon_x$ for some fixed $\varepsilon > 0$

$$(0, \varepsilon)\nu_{x_j} + K_j \subset D$$

for $j \geqq j_0$, where K_j is a peak set containing x_j. Now $z_j = x_j + t_j\nu_{x_j}$, where $t_j \to 0$ necessarily (indeed $t_j = \text{dist}(z_j, \partial D) \leqq \text{dist}(z_j, z_0)$), and we only have to take $\mathscr{K}_j = K_j + t_j\nu_{x_j} \subset D$ and $E_j = K_j \cap E + t_j\nu_{x_j}$: evidently $\limsup E_j \subset E$ while $\sup |f(E_j)| \geqq \sup |f(\mathscr{K}_j)|$ follows from the fact that $K_j \cap E \supset K_j \cap \partial$ is a boundary for $P(K_j)$, so that by translation E_j is a boundary for $P(\mathscr{K}_j)$, which contains $H^\infty(D)|\mathscr{K}_j$ because of (1.1). Our proof of Theorem 1 is now complete.

Note that our use of the differentiability of $\partial D \backslash \partial$ was needed only to allow us to satisfy the hypotheses of Lemma 2; this can be accomplished by various other hypotheses on D. For example

THEOREM 3. *Suppose D is a bounded domain in C^n and $K \subset \partial D$ is a peak set for $A(D)$ which is polynomially convex and for which $K + v_j \subset D$ for a sequence $v_j \to 0$ in C_n. Suppose a dense subset of D lies on positive dimensional subvarieties V of D all having $V^- \backslash V \subset \partial \cup \Delta$, where $\Delta \subset C^n$ is compact and disjoint from K.*

Then any $h \in H^\infty(D)$ which has a continuous extension to $D \cup (\partial \cap K)$ has a continuous extension to $D \cup K$.

Here we deduce (*) for $E = \partial \cup \Delta$ from the maximum principle for varieties, noting that we can restrict ourselves to a dense set of z in D on the left side of (*). So Lemma 1 implies Theorem 3 directly.

We should also note that our proof of Theorem 1 applies equally well to convex D, where the fact that $\partial D \backslash \partial$ is a union of polynomially convex peak sets follows from the fact that for each $z_0 \in \partial D$ one has a $w \in C^n$ for which

$$z \longrightarrow \text{Re}\,(z, w)$$

assumes its maximum over D^- at z_0, so a multiple of $z \to \exp(z, w)$ provides an element of $A(D)$ which peaks on a subset K of ∂D containing z_0; trivially K is convex and thus polynomially convex (via such functions of course). One has only to replace translation along normals by maps

$$\sigma_\varepsilon(z) = (1 - \varepsilon)z + \varepsilon z_0$$

where $z_0 \in D$ is fixed. (But in fact that $f \in H^\infty(D)$ continually extendable to all of $D \cup \partial$ has a continuous extension to D^- for D starlike is trivial: with z_0 now the star center $f_\varepsilon = f \circ \sigma_\varepsilon \in A(D)$, and $f_\varepsilon \to f$ uniformly on ∂, as $\varepsilon \to 0$, so that $f_\varepsilon \to g \in A(D)$; since $f_\varepsilon \to f$

pointwise on D, $f = g$ on D.) Of course convex domains are special cases of those for which, for our K_0, we have a sequence of holomorphic maps σ_j of D^- into D with $\sigma_j(k) \to k$ for each $k \in K_0 \cap \partial$, and such maps will serve in place of our translations $z \to z + v_j$ in Lemma 1; in particular, Theorem 3 holds if our assumption that $K + v_j \subset D$ for $v_j \to 0$ is replaced by the existence of such a sequence σ_j.

Part of our proof of Theorem 1 yields some information even when we have no continuity at the Šilov boundary.

COROLLARY 4. *Let D be as in Theorem 1, or convex, and let K be one of our peak sets. If $f \in H^\infty(D)$ then $\mathrm{cl}(f, K)$, the set of cluster values of f at points of K, lies in $\mathscr{C} = \mathscr{C}(\mathrm{cl}(f, \partial \cap K))$, the closed convex hull of the set of cluster values at points of $\partial \cap K$.*

This follows precisely because (*) holds: if our inclusion were to fail, so some $w \in \mathrm{cl}(f, K) \backslash \mathscr{C}$, (necessarily a cluster value at some $z_0 \in K_0 \backslash \partial$), then so would (*) for $\exp(e^{i\theta} f)$, where θ is chosen so that $\mathrm{Re}(e^{i\theta} w) > \sup \mathrm{Re}(e^{i\theta} \mathscr{C})$.

2. The proof of Theorem 2 is more involved than that of Theorem 1 because we cannot make as great use of the closure of the algebra $A(D)|K$. The tubular neighborhood theorem [2, p. 9] allows us to deduce it from the more general result below, in which $\Pi(x, \delta)$ denotes the square polycylinder of radius δ about x.

THEOREM 2′. *Suppose $\partial D \backslash \partial$ is differentiable, and covered by a union of polynomially convex peak sets K for $A(D)$, and for each $x_0 \in \partial D \backslash \partial$ there are $\varepsilon, \delta, \eta > 0$ with $2\delta > \varepsilon$ for which $\Pi(x_0, \varepsilon + \delta) \cap \partial = \varnothing$, while $x \in \Pi(x_0, \varepsilon) \cap (\partial D \backslash \partial)$ implies*

$$(o, \eta)\nu_x + (K_x \cap \Pi(x, \delta)) \subset D \tag{2.0}$$

where K_x is one of our peak sets containing x and ν_x is again the inward unit normal. Finally suppose that for one of our peak sets K_0 we have a sequence of polynomially convex subsets K_j and $v_j \to 0$ in $\boldsymbol{C}^n$ with $v_j + K_j \subset D$, and for some $z_0 \in K_0 \backslash \partial$

(2.1) *each probability measure λ on $K_0 \cap \partial$ representing z_0 on polynomials is a w^* cluster point of a bounded sequence $\{\lambda_j\}$, where λ_j is a complex measure on K_j multiplicative on polynomials.*

Then if $h \in H^\infty(D)$ extends continuously to $D \cup (\partial \cap K_0)$ it has a unique cluster value at z_0.

To begin our proof of Theorem 2′, let B be the uniformly closed algebra of functions continuous $D \cup K_0$ and holomorphic on D, and let X be the closure in its spectrum of $D \cup K_0$ (hence of D itself), with ρ the restriction to X of the map dual to $A(D) \to B$. Since the elements of B are all continuous on $D \cup K_0$, the natural injection of that set into X is continuous, $1-1$, while ρ provides a continuous inverse, so $D \cup K_0$ is imbedded homeomorphically in X. In fact, D forms an open subset of X while ρ is $1-1$ over D, as is easily seen [4, p. 421]; moreover ρ is also $1-1$ over K since[1] no $\hat{b} \in \hat{B}$ (and hence no element of $C(X)$) can separate $\rho^{-1}(z)$ for $z \in K_0$: for if $x \in \rho^{-1}(z)$ and $\{z_\delta\}$ is a net in the dense subset D of X with $z_\delta \to x$ in X then $z_\delta = \rho(z_\delta) \to \rho(x) = z$ in D^-, so

$$\hat{b}(x) = \lim \hat{b}(z_\delta) = \lim b(z_\delta) = b(z) = \hat{b}(z)$$

since $b \in B$ is continuous on $D \cup K$.

Of course ρ maps X into, hence onto, D^-; moreover local maximum modulus and the fact that D is open in X shows $\partial_B \cap D = \varnothing$, so $\partial_B \subset \rho^{-1}(\partial D)$ since X forms a boundary for $\hat{B}$. In fact I claim $\partial_B \subset \rho^{-1}(\partial)$. If not some element $\hat{b}$ of $\hat{B}$ peaks at $x \in \rho^{-1}(\partial D \backslash \partial)$, and letting $x_0 = \rho(x) \in \partial D \backslash \partial$ we have by hypothesis $\varepsilon, \delta, \eta > 0$, with $\delta > 2\varepsilon$, $\Pi(x_0, \varepsilon + \delta) \cap \partial = \varnothing$, and

$$(0, \eta)\nu_x + K_x \cap \Pi(x, \delta) \subset D \tag{1}$$

for any x in $\Pi(x_0, \varepsilon) \cap (\partial D \backslash \partial)$, where K_x is one of our peak sets containing x. Replacing b by b^k for k large we can suppose $|b| < 1/4$ on $D \backslash \Pi(x_0, 1/2n \min(\varepsilon, \eta))$, while $|b(z)| > 3/4$ for some $z \in \Pi(x_0, 1/2n \min(\varepsilon, \eta))$. Now let $x \in \partial D$ be nearest z, so $z = x + t\nu_x$, $0 < t < 1/2 \min(\varepsilon, \eta)$, and, since $x \in \Pi(x_0, \varepsilon)$, (1) applies. In particular this says $b(\cdot + t\nu_x)$ is analytic near the polynomially convex set $K_x \cap \Pi(x, \delta)$, and so lies in $P(K_x \cap \Pi(x, \delta))$ by Oka-Weil. But this algebra coincides with the uniform closure of $P(K_x)|(K_x \cap \Pi(x, \delta))$ clearly, whose Šilov boundary, by Rossi's local maximum modulus theorem [3], lies in the topological boundary $\partial_0 = \partial(K_x \cap \Pi(x, \delta)) = K_x \cap \partial\Pi(x, \delta)$ of $K_x \cap \Pi(x, \delta)$ in K_x (since $\Pi(x, \delta) \subset \Pi(x_0, \delta + \varepsilon)$ misses ∂). Because $t < \varepsilon$ and $\delta > 2\varepsilon$, $\partial_0 + t\nu_x$ lies in the closure of $D \backslash \Pi(x_0, 1/2n \min(\varepsilon, \eta))$, where $|b| < 1/4$, and so we obtain a contradiction $3/4 < |b(z)| = |b(x + t\nu_x)| \leqq \sup|b(\partial_0 + t\nu_x)| \leqq 1/4$, establishing the claim that $\partial_B \subset \rho^{-1}(\partial)$.

Moreover, if $g \in A(D) \subset B$ peaks on K_0 then $\hat{g} = g \circ \rho \in \hat{B}$ peaks on $\rho^{-1}K_0$, while the closed algebra $\hat{B}|\rho^{-1}K_0$ has, as a boundary, $\partial_B \cap \rho^{-1}K_0 \subset \rho^{-1}\partial \cap \rho^{-1}K_0 = \rho^{-1}(\partial \cap K_0)$ which is precisely $\partial \cap K_0$ since ρ is $1-1$ over K_0 as we saw earlier.

Now consider our function h holomorphic on D and continuous

[1] Here ^ is the Gelfand representation of B.

on $D \cup (\partial \cap K_0)$. We shall show

(i) h has a continuous extension h_0 to $X \backslash \rho^{-1}(K_0 \backslash \partial)$;

(ii) all probability measures λ on ∂_B representing our fixed $z_0 \in K_0 \backslash \partial$ are multiplicative on the closed subalgebra B_0 of $C(X \backslash \rho^{-1}(K_0 \backslash \partial))$ generated by B and h_0;

(iii) the subset ∂_B of $X \backslash \rho^{-1}(K_0 \backslash \partial)$ forms a boundary for B_0.

Once these facts are in hand our conclusion follows by noting that if X_0 is the closure of $X \backslash \rho^{-1}(K_0 \backslash \partial)$ in the spectrum M_{B_0} (hence that of D as well) and ρ_0 is the restriction to X_0 of the map dual to $A(D) \to B_0$, then all points of $\rho_0^{-1}(z_0)$, for our fixed $z_0 \in K_0 \backslash \partial$, are represented by measures λ on the boundary ∂_B for B_0(iii) which lie in the set of measures on ∂_B representing z_0 on B; since (ii) says these are all multiplicative on B_0 they all represent the same functional: if λ and λ' represented distinct functionals we'd have $b \in B_0$ with $\lambda(b) = 0$, $\lambda'(b) = 1$, whence multiplicativity of $(\lambda + \lambda')/2$ yields $1/2 = 1/2(\lambda + \lambda')(b^2) = (1/2(\lambda + \lambda')(b))^2 = 1/4$. Thus $\rho_0^{-1}(z_0)$ is a singleton, and of course this implies h has a unique cluster value at z_0 since if $z_j \to z_0$, $z_j \in D$, any cluster value of $\{h(z_j)\}$ is $\hat{h}(x)$ for some $x \in X_0$ with $\rho_0(x) = z_0$.

So it remains to prove (i)-(iii).

To see (i), note that

$$X = \rho^{-1}(D^-) = \rho^{-1}(D) \cup \rho^{-1}(\partial D \backslash K_0) \cup \rho^{-1}(\partial \cap K_0) \cup \rho^{-1}(K_0 \backslash \partial)$$

so $X \backslash \rho^{-1}(K_0 \backslash \partial) = \rho^{-1}(D) \cup \rho^{-1}(\partial D \backslash K_0) \cup \rho^{-1}(\partial \cap K_1)$, and we only have to see h, as a function on $D \subset X$, has a unique cluster value at each $x \in \rho^{-1}(\partial D \backslash K_0) \cup \rho^{-1}(\partial \cap K_0)$. For the second set this is our hypothesis on h, and for the first, if $g \in A(D)$ again peaks on K_0 then $b = (1 - g)h \in B$, so $\hat{b}(x)(1 - \hat{g}(x))^{-1}$ provides our unique cluster value at $x \in \rho^{-1}(\partial D \backslash K_0)$.

For (ii) recall that by hypothesis we have $v_j \to 0$ in $\boldsymbol{C}^n$ and polynomially convex sets $K_j \subset K_0$ with $v_j + K_j \subset D$, while each probability measure λ on $\partial \cap K_0$ representing z_0 on $P(K)$ is a w^* cluster point of a sequence $\{\lambda_j\}$, where λ_j is a complex measure on K_j multiplicative on polynomials.

As we know any probability measure λ' representing z_0 on B and carried by ∂_B is carried by $\partial_B \cap \rho^{-1}(K_0) \subset \rho^{-1}(\partial) \cap \rho^{-1}(K_0) = \rho^{-1}(\rho \cap K_0) = \partial \cap K_0$, and so $\lambda' = \lambda$ as above, as a measure representing z_0 on $A(D)$, hence on $A(D) | K_0 \supset P(K_0)$. If $\sigma_j(z) = z + v_j$ then trivially the translated measures $\sigma_j^* \lambda_j$ (defined by $\sigma_j^* \lambda_j(f) = \lambda_j((f \circ \sigma_j))$ still have λ as a w^* cluster point (now in the space of measures on X) since $b \circ \sigma_j \to b$ uniformly on $K_0 \cap \partial$ and $\{\lambda_j\}$ is bounded. Since $h \circ \sigma_j$ is analytic near K_j, and so in $P(K_j)$, as is $b \circ \sigma_j$ for $b \in B$,

$$\sigma_j^*\lambda_j(h^n b) = \lambda_j((h\circ\sigma_j)^n b\circ\sigma_j) = (\lambda_j(h\circ\sigma_j))^n\lambda_j(b\circ\sigma_j)$$
$$= \sigma_j^*\lambda_j(h)^n\sigma_j^*\lambda_j(b)$$

whence $\lambda(h^n b) = \lambda(h)^n\lambda(b)$, so λ is multiplicative on B_0, and (ii) holds.

Now only (iii), that $\partial_B \subset X\backslash\rho^{-1}(K_0/\partial)$ forms a boundary for B_0, remains to be seen. If $b\in B_0$ peaks at $x \in W = X_0\backslash(\partial_B \cup \rho_0^{-1}(K_0))$ and U is a compact neighborhood of x lying in W then for n sufficiently large $b_1 = b^n(1-g)(\equiv 0$ on K_0, g again our element of $A(D)$ peaking on $K_0)$ is an element of B which assumes its maximum modulus only within U, hence not on ∂_B, so ∂_B can not be a boundary for B. Thus the peak points for B_0 in X_0 lie in $\partial_B \cup \rho_0^{-1}(K_0)$, so $\partial_{B_0} \subset \partial_B \cup \rho_0^{-1}(K_0)$. Since the dense open subset D of X lies in $X_0\backslash\rho_0^{-1}(K_0)$, the peak set $\rho_0^{-1}(K_0)$ is nowhere dense in X_0. But now if $V = \partial_{B_0}\backslash\partial_B \subset \rho_0^{-1}(K_0)$ is nonvoid it must contain a peak point x_0 for B_0 (as a relatively open subset of ∂_{B_0} must), and since the element $1-\hat{g}$ of $\hat{B}_0$ vanishes on $\rho_0^{-1}(K_0) \supset V$ it must vanish on a neighborhood of x_0 in X_0 by [5, 2.1], despite the fact that we have seen $(1-\hat{g})^{-1}(0) = \rho_0^{-1}(K_0)$ is nowhere dense in X_0. We conclude $\partial_{B_0} \subset \partial_B$ so our proofs of (iii) and Theorem 2′ are complete.

A variant of our argument yields another version of Theorem 2.

THEOREM 2″. *Suppose $\partial D\backslash\partial$ is a C^2 manifold and is covered by a union of peak sets K for $A(D)$ each of which forms the spectrum of $A(D)|K$ (which is automatic if D^- is the spectrum of $A(D)$). Suppose that for one of these, K_0, $v_j + K_0 \subset D$ for $v_j \to 0$ in $\boldsymbol{C}^n$. Then any bounded holomorphic h on D extending continuously to $D \cup (\partial \cap K_0)$ extends continuously to $D \cup K_0$.*

This replacement of polynomial convexity for our K's is possible since here $K \cap \Pi(x,\delta)$ is the joint spectrum of the coordinate functions for the algebra $(A(D)|K\cap\Pi(x,\delta))^-$, so our argument that $\partial_B \subset \rho^{-1}(\partial)$ proceeds as before using the holomorphic calculus in place of Oka-Weil; a similar replacement occurs when we consider the functions $h\circ\sigma_j$ of course.

Note that when (1.1) in Theorem 1 is replaced by polynomial convexity of the peak sets K that result is contained in the assertion of Theorem 2′ (with all $K_j \equiv K_0$). We should also note a property of such domains: *for D as in Theorem* 1 (*or starlike*) *any $h \in H^\infty(D)$ bounded near ∂, say by $M > 0$, is bounded on D, and by the same constant.* (If h were not bounded by M then we have z_j in D with $|h(z_j)| \to \sup|h(D)| > M$, and we can assume $z_j \to z_0$, necessarily in $\partial D\backslash\partial$; if $x_j \in \partial D$ is nearest z_j and we take $j \geqq j_0$ all the x_j will lie in a compact neighborhood of z_0 in $\partial D\backslash\partial$ on which

$\varepsilon_x \geqq \varepsilon > 0$, so $z_j = x_j + t_j\nu_{x_j}$, with $t_j < \varepsilon$ for $j \geqq j_1$. But now since $K_{x_j} + t_j\nu_{x_j} \subset D$ while $h(\cdot + t_j\nu_{x_j})$ is in $P(K_j)$ as before,

$$|h(z_j)| \leqq \sup |h(K_{x_j} + t_j\nu_{x_j})| \leqq \sup |h(\partial \cap K_{x_j} + t_j\nu_{x_j})|$$

so $M < \sup|h(D)| = \lim |h(z_j)| \leqq \limsup |h(\partial \cap K_{x_j} + t_j\nu_{x_j})| \leqq M$, our contradiction.) As a consequence *if D is as in Theorem* 1 (*or starlike*) *and $h \in A(D)$ then $h(\partial \cup D)$ provides the entire range of h*, $h(D^-)$. To see this we need only show $0 \notin h(\partial \cup D)$ implies $0 \notin h(D^-)$. But the hypothesis implies $1/h$ is holomorphic on D and bounded on (and so near) ∂, so $1/h$ is bounded on D^- by our remark, and $0 \notin h(D^-)$. (Even when D^- is the spectrum of $A(D)$, so $h(D^-)$ is the spectrum of h, the familiar Banach algebra fact that $\partial h(D^-) \subset h(\partial)$, which thus implies $h(D) \cup \partial h(D^-) \subset h(\partial \cup D)$, does not quite yield this since $\partial h(D)$ may properly contain $\partial h(D^-)$.) More generally, for *$h \in H^\infty(D)$ the set of cluster values of h at all points of ∂D*, $\mathrm{cl}\,(h, \partial D) \subset h(D) \cup \mathrm{cl}\,(h, \partial)$ *for D as in Theorem* 1, by the same argument.

An improvement of Theorem 2 can be obtained via the basic lemma of [1], viz: for *$a \in A(D)$ and $E \subset \boldsymbol{C}$ a closed set of* (inner logaritmic) *capacity zero, if h has a single cluster value at each point of $\partial \cap (K_0\backslash a^{-1}(E))$ the same is true at z_0 if $z_0 \in K_0\backslash a^{-1}(E)$*. Here one argues exclusively with Jensen measures. h_0 is now only continuous on $Y = (X\backslash\rho^{-1}(K_0\backslash\partial))\backslash(a^{-1}(E) \cap \partial)$, but since $\lambda(a^{-1}(E)) = 0$ for each Jensen measure λ representing $z \in K_0\backslash(\partial \cup a^{-1}(E))$ on B (by [1, Lemma 1]), $h_0 \circ \sigma_j \to h_0$ a.e. λ, so as before one concludes all such Jensen measures for z coincide on the subalgebra B_0 of $C(Y)$ generated by B and h_0, and represent the same functional ϕ_0 on B_0. On the other hand by the proof of (iii) ∂_{B_0} is the closure in M_{B_0} of $\partial_B\backslash a^{-1}(E)$, and it is easy to see the new points lie in $\rho_0^{-1}a^{-1}(E) = \hat{a}^{-1}(E)$; since each element of $\rho_0^{-1}(z)$ is represented by a Jensen measure λ on ∂_{B_0}, which necessarily vanishes on $\hat{a}^{-1}(E)$, λ is in fact carried by ∂_B, and represents z on B. But now λ represents ϕ_0, and $\rho_0^{-1}(z)$ is a singleton as desired.

3. There are simple domains for which continuous extension fails. Here is one which amounts to a union of two convex domains, deformed so that two discs in the boundary meet in precisely their common center, and thus obstruct continuous extension; as will be noted there is a vast gap between the example and the domains previously considered.

In $\boldsymbol{C}^3$ let

$$D_1 = \{t(z_1, 0, 2i) + (1-t)w\colon 0 < t < 1,\ |z_1| < 1,\ |w-(0, 0, 2)| < 1\}\ ,$$
$$D_2 = \{t(0, z_2, -2i) + (1-t)w\colon 0 < t < 1,\ |z_2| < 1,\ |w-(0, 0, 2)| < 1\}\ ,$$

and $D_0 = D_1 \cup D_2$. Note that $\pi_3 D_0$ lies in the open right half plane, while $\pi_3 D_0^-$ meets the imaginary axis in $\pm 2i$; evidently $\pi_3^{-1}(i\boldsymbol{R}) \cap D_0^-$ is the union of the two closed discs

$$\Delta_1 = \{(z_1, 0, 2i): |z_1| \leqq 1\}, \quad \Delta_2 = \{(0, z_2, -2i): |z_2| \leqq 1\}.$$

Thus if we set $\rho(z_1, z_2, z_3) = (z_1, z_2, z_3^2)$ and $D = \rho D_0$ then ρ maps D_0^- onto D^-, and D_0 biholomorphically onto D. In fact ρ is $1-1$ on D^- except at the centers $(0, 0, \pm 2i)$ of Δ_1, Δ_2 both of which map to $(0, 0, -4)$, which is precisely $\rho\Delta_1 \cap \rho\Delta_2$. Thus the function $h = \pi_3 \circ \rho^{-1}$ on $D^- \backslash \{(0, 0, -4)\}$, which provides the square root of the third coordinate, is continuous, and holomorphic on D. Since it yields values near both $\pm 2i$ in each neighborhood of $(0, 0, -4)$ in D^-, it has no continuous extension to D^-. Finally $(0, 0, -4) \notin \partial_{A(D)}$, (essentially since each $\rho\Delta_i$ is an analytic disc), and we are done.

(One can easily modify D_0 so that $\partial D \backslash \rho\Delta_1 \cup \rho\Delta_2$ lies in $\partial_{A(D)}$; moreover $\rho\Delta_1 \cup \rho\Delta_2$ can be made a peak set (as in the example, where $(4 - z_3)/8$ is the peaking function), and polynomially convex.)

Added in proof (April 1, 1979). I am indebted to H. Alexander for the following simpler (and basically different) example. In $\boldsymbol{C}^2$ let $D = \{(z, w): |z| < |w| < 1\}$. Then $\partial = \{(z, w): |z| = |w| = 1\}$, essentially since any point with $|z| = |w| < 1$ lies in a disc where $f \in A(D)$ must be analytic (as $\lim_{r \nearrow 1} f(r\cdot, \cdot)$); but $f = z/w$ is a bounded continuous function on $D^- \backslash \{(0, 0)\}$ analytic on D which has no continuous extension to D^- since $(0, 0)$ lies on too many analytic discs. (For the same reason $(0, 0)$ lies in no proper peak set.)

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Received January 26, 1977 and in revised form July 14, 1978. Work supported in part by the NSF.

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