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A SELECTION THEOREM FOR GROUP ACTIONS

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Let a Polish group G act continuously on a Polish space X, inducing an equivalence relation E. Let E_Y be the restriction of E to an invariant Borel subset Y of X. Assume E_Y is countably separated. Then it has a Borel transversal.

Throughout, let Γ be a continuous action of a Polish group G on a Polish space X. Thus X is a separable space admitting a complete metric, while G is a topological group whose topology is separable and admits a complete metric, and Γ is a continuous function $G\times X \to X$ satisfying $\Gamma(g^{-1}, \Gamma(g, x)) = x$ and $\Gamma(g, \Gamma(h, x)) = \Gamma(gh, x)$ for all $x \in X$ and $g, h \in G$. We write gx for $\Gamma(g, x)$, and for subsets of X write gA for $\{gx: x \in A\}$. Γ induces an equivalence relation E on X: xEy iff $gx = y$ for some $g \in G$. $W \subset X$ is *invariant* if $gW = W$ for all $g \in G$. Let $Y \subset X$ be an invariant Borel set, E_Y the restriction of E to Y. A *transversal* or *selector-set* for an equivalence relation is a set composed of exactly one representative from each equivalence class. Let us assume E_Y is *countably separated*, i.e., that there exist invariant Borel $Z_0, Z_1, Z_2, \cdots \subset Y$ such that for all $x, y \in Y$:

$$xEy \longleftrightarrow \forall m(x \in Z_m \longleftrightarrow y \in Z_m) \tag{0}$$

our goal is to prove the following selection result:

THEOREM. Under the above hypotheses, E_Y has a Borel transversal. It should be mentioned that a number of special cases and overlapping results have been known to and applied by C^*-algebraists for some time now. The construction of the required transversal proceeds in four stages.

Stage A. It will prove convenient to reserve the letters m, n plain and with subscripts to range over the set I of natural numbers, and to reserve s, t plain and with subscripts to range over the set Q of finite sequences of natural numbers. We let s^*m denote the *concatenation* of s and m, i.e., s with m tacked on at the end. We wish to define Borel sets $A(s)$ for overy $s \in Q$ of even length.

Case 1. $s =$ the empty sequence $\varnothing$. Set $A(\varnothing) = Y$.

Case 2. $s = a$ sequence (m, n) of length two. Set $A((m, n)) = Z_m$

if $n = 0$, and $Y - Z_m$ if $n > 0$.

Case 3. $s =$ a sequence of form t^*m^*n, where t has length $\geqq 2$, and $A(t)$ is a closed set. For such t we wish to define $A(t^*m^*n)$ for all m and n at once. In order to do so, we first fix a complete metric ρ compatible with the topology of X. For each m we then let $\{A(t^*m^*n)\colon n \in I\}$ be a family of closed sets of ρ-diameter $< 1/m$ whose union is $A(t)$.

Note that in every case so far we have:

$$(1) \qquad A(t) = \bigcap_m \bigcup_n A(t^*m^*n) .$$

Case 4. $s =$ a sequence of form t^*m^*n, where t has length $\geqq 2$, and $A(t)$ is not closed. Again, for such t we define all $A(t^*m^*n)$ at once.

But first we introduce by induction on countable ordinals α a slight modification of the usual hierarchies of Borel sets. Let Θ_0 be the family of all closed subsets of X. For a countable ordinal $\alpha > 0$, let Θ_α be the family of all sets of form $\bigcap_m \bigcup_n W_{mn}$ with the $W_{mn} \in \bigcup_{\beta<\alpha} \Theta_\beta$. Thus $\Theta_1 = F_{\sigma\delta}$, $\Theta_2 = F_{\sigma\delta\sigma\delta}$. For present purposes the *rank* of a Borel set W will mean the least α with $W \in \Theta_\alpha$.

Now returning to our Borel set $A(t)$ of rank $\alpha > 0$, we let the $A(t^*m^*n)$ be sets of rank $< \alpha$ satisfying (1) above. This completes the opening stage of the construction.

Stage B. Let us fix an enumeration $s_0, s_1, s_2, \cdots$ of the nonempty members of Q, such that if s_m is an initial segment of s_n, then $m < n$. Let F_n denote the set of all functions $\{s_0, \cdots, s_{n-1}\} \to I$. (So F_0 contains only the empty function $\varnothing$.) Let $F = \bigcup_n F_n$, and let F_∞ be the set of all functions $\{s_i\colon i \in I\} \to I$. We reserve the letters σ, τ plain and with subscripts to range over F. We say τ is an *immediate proper extension* of σ, and write $\sigma \Subset \tau_\alpha$, if for some n, $\sigma \in F_n$, $\tau \in F_{n+1}$, and τ extends σ.

For $\psi \in F \cup F_\infty$ and $s = (m_0, m_1, \cdots, m_{k-1}) \in \operatorname{dom} \psi$ we define:
$\psi^+(s) = (m_0, n_0, m_1 n_1, \cdots, m_{k-1}, n_{k-1})$, where
$n_0 = \psi((m_0))$ and $n_1 = \psi((m_0, m_1)), \cdots, n_{k-1} = \psi(s)$.

To complete stage B of the construction, we define $B(\sigma)$ to be the intersection of all $A(\sigma^+(s))$ for $s \in \operatorname{dom} \sigma$. Unpacking all these definitions, one readily verifies that:

$$(2) \qquad B(\sigma) = \bigcup_{\sigma \Subset \tau} B(\tau) .$$

Another glance at the definitions (especially stage A, case 2) discloses:

$$(3) \qquad x \in B(\sigma) \;\&\; (m) \in \operatorname{dom} \sigma \longrightarrow (x \in Z_m \longleftrightarrow \sigma((m)) = 0) .$$

Stage C. Before launching into the next stage of the construction, we define for any $W \subset X$ the *Vaught transform* $W^{\#}$ of W to be $\{x \in X\colon \{g \in G\colon gx \in W\}$ is nonmeanger (2nd category) in $G\}$. One readily verifies that:

$W^{\#}$ is invariant.

W is invariant $\rightarrow W = W^{\#}$.

$(\bigcup_n W_n)^{\#} = \bigcup_n (W_n^{\#})$.

It is shown in [1] that

$$W \text{ is Borel} \longrightarrow W^{\#} \text{ is Borel}$$

which will be all-important for us.

Now let us define $C(\sigma) = B(\sigma)^{\#}$. The above facts from Vaught's theory of group actions imply that each $C(\sigma)$ is an invariant Borel set, that $C(\varnothing) = Y$, and that:

$$C(\sigma) = \bigcup_{\sigma \Subset \tau} C(\tau) . \tag{4}$$

Now if $x \in C(\sigma)$, then some $gx \in B(\sigma)$, so applying (3) above, and recalling that the Z_m are invariant, we conclude:

$$x \in C(\sigma)\ \&\ (m) \in \operatorname{dom} \sigma \longrightarrow (x \in Z_m \longleftrightarrow \sigma((m)) = 0) . \tag{5}$$

Stage D. We say σ *lexicographically precedes* τ, and write $\sigma \lhd \tau$, if for some n and $i < n$ we have $\sigma \in F_n$, $\tau \in F_n$, $\sigma(s_j) = \tau(s_j)$ for all $j < i$, and $\sigma(s_i) < \tau(s_i)$. The relation $\lhd$ well orders each F_n.

Let $D(\sigma)$ be the invariant Borel set $C(\sigma) \cdot \bigcup \{C(\tau)\colon \tau \lhd \sigma\}$. Thus $D(\varnothing) = Y$ and by (4) and (5) we have:

$$D(\sigma) = \textstyle\sum_{\sigma \Subset \tau} D(\tau) \tag{6}$$

$$x \in D(\sigma) \text{ and } (m) \in \operatorname{dom} \sigma \longrightarrow (x \in Z_m \longleftrightarrow \sigma((m)) = 0) . \tag{7}$$

In (6), Σ denotes *disjoint* union.

Finally we are in a position to introduce the Borel set:

$$T = \bigcap_n \bigcup_{\sigma \in F_n} (B(\sigma) \cap D(\sigma)) .$$

We aim to show that T is the required transversal for E_Y. To this end we consider an arbitrary E-equivalence class $K \subset Y$ and verify that $T \cap K$ is a singleton.

To begin with, from (6) it is evident that there exists a sequence $\varnothing = \sigma_0 \Subset \sigma_1 \Subset \sigma_2 \Subset \cdots$ of elements of F such that $K \in D(\sigma_n)$ for each n, but $K \cap D(\sigma) = \varnothing$ for any other $\sigma \in F$. Let $\psi \in F_\infty$ be the union of these σ_n.

Recall that:

$$B(\sigma_n) = \cap \{A(\sigma_n^+(s_i))\colon i < n\} = \bigcap \{A(\psi^+(s_i))\colon i < n\} .$$

Let us consider the closely related sets:

$$L_n = \bigcap\{A(\psi^+(s_i)) \colon i < n \text{ and } A(\psi^+(s_i)) \text{ is a closed set}\}\,.$$

Manifestly the L_n are closed and nested, $L_{n+1} \subset L_n$. They are also nonempty. (To see this, note that $K \subset D(\sigma_n) \subset C(\sigma_n)$ implies $K \cap B(\sigma_n) \neq \emptyset$, and that $L_n \supset B(\sigma_n)$.) Finally, the ρ-diameters of the L_n converge to zero. (To see this, consider for any given m the sets $A(\psi^+((m)))$, $A(\psi^+((m, m)))$, $A(\psi^+((m, m, m)))$, $\cdots$. By stage A, case 4 of our construction, the ranks of these sets decrease until at some step we reach a closed set; then by stage A, case 3, at the very next step we get a closed set of ρ-diameter $< 1/m$.) Since ρ is complete, $\bigcap_n L_n$ is a singleton $\{y\}$.

Claim. $y \in A(\psi^+(s))$ for all s.

This is established by induction on the rank of the set involved: we know it already for rank 0, i.e., closed, sets. Suppose then $A(\psi^+(s))$ has rank $\alpha > 0$, and assume as induction hypothesis that the claim holds for sets of rank $< \alpha$, e.g., for the various $A(\psi^+(s)^*m^*n)$. Then for any m, letting $n = \psi(s^*m)$, we have $\psi^+(s^*m) = \psi^+(s)^*m^*n$, and so by induction hypothesis, $y \in A(\psi^+(s)^*m^*n)$. This shows $y \in \bigcap_m \bigcup_n A(\psi^+(s)^*m^*n) = A(\psi^+(s))$ as required to prove the claim.

From the claim it is immediate that $y \in \bigcap_n B(\sigma_n)$, and also that for any m, $y \in Z_m$ iff $\psi(m) = 0$. On the other hand, by (7) above, for any m, $K \subset Z_m$ iff $\psi(m) = 0$. But then by (0), $y \in K$. And this implies $y \in \bigcap_n D(\sigma_n)$. Putting everything together, $T \cap K = \{y\}$ as required.

References

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