PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 1, 1979

THE DIMENSION OF THE KERNEL OF A PLANAR SET

MARILYN BREEN

Let S be a compact subset of R^2. We establish the following: For $1 \leqq k \leqq 2$, the dimension of $\ker S$ is at least k if and only if for some $\varepsilon > 0$, every $f(k)$ points of S see via S a common k-dimensional neighborhood having radius ε, where $f(1) = 4$ and $f(2) = 3$. The number $f(k)$ in the theorem is best possible.

We begin with some definitions: Let S be a subset of R^d. For points x and y in S, we say *x sees y via S* if the segment $[x, y]$ lies in S. The set S is *starshaped* if there is some point p in S such that, for every x in S, p sees x via S. The set of all such points p is called the (convex) *kernel* of S, denoted by $\ker S$.

A well-known theorem of Krasnosel'skii [5] states that if S is a compact set in R^d, then S is starshaped if and only if every $d+1$ points of S see a common point via S.

Although various results have been obtained concerning the dimension of the set $\ker S$ (Hare and Kenelly [3], Toranzos [6], Foland and Marr [2], Breen [1]), it still remains to set forth an appropriate analogue of the Krasnosel'skii theorem for sets whose kernel is at least k-dimensional, $1 \leqq k \leqq d$. Hence the purpose of this work is to investigate such an analogue for subsets of the plane.

The following terminology will be used. Throughout the paper, $\operatorname{conv} S$, $\operatorname{cl} S$, $\operatorname{int} S$, $\operatorname{bdry} S$, and $\ker S$ denote the convex hull, closure, interior, boundary, and kernel, respectively, of the set S. If S is convex, $\dim S$ represents the dimension of S. Finally for $x \neq y$, $R(x, y)$ denotes the ray emanating from x through y and $L(x, y)$ is the line determined by x and y.

2. The results.

We begin with the following theorem for sets whose kernel is 1-dimensional.

THEOREM 1. *Let S be a compact set in R^2. The dimension of $\ker S$ is at least 1 if and only if for some $\varepsilon > 0$, every 4 points of S see via S a common segment of radius ε. The number 4 is best possible.*

Proof. The necessity of the condition is obvious. Hence we need only establish its sufficiency.

By Krasnosel'skii's theorem in R^2, S is starshaped, so we may select a point z in $\ker S$. Moreover, we assert that every 4 points of S see a common segment of length ε having z as endpoint (we refer to such a segment as an ε-interval at z): For x_1, x_2, x_3, x_4 in S, these points see a common 2ε-interval $[a, b]$ in S, and since $z \in \ker S$, $\operatorname{conv}\{z, x_i, a, b\} \subseteq S$ for each $1 \leqq i \leqq 4$. Hence x_i sees $\operatorname{conv}\{z, a, b\}$ for every i. Certainly one of the edges $[z, a]$, $[z, b]$ of the triangle (possibly degenerate) $\operatorname{conv}\{z, a, b\}$ has length at least ε, and this edge satisfies our assertion.

To complete the proof, we consider two cases.

Case 1. Assume that $z \in \operatorname{int} S$. Let N be a disk about z of radius $r \leqq \varepsilon$ contained in S. If $N = S$ the result is immediate, so assume that $S \sim N \neq \phi$. For $y \in S \sim N$, we define C_y to be the subset of N seen by y. Since S is starshaped, S is simply connected, so C_y is convex. Let $[a_y, b_y]$ be the intersection of C_y with the line perpendicular to $L(y, z)$ at z, and let δ_y be the smaller of the lengths of the segments $[a_y, z]$ and $[b_y, z]$, say the length of $[a_y, z]$.

If $\operatorname{glb} \delta_y > 0$, then $\cap C_y$ contains a neighborhood of z, contained in $\ker S$. Hence we may assume $\operatorname{glb} \delta_y = 0$.

Let $\{y_n\}$ be a sequence of points in S such that $\delta_{y_n} \to 0$ as $n \to \infty$. Let y_0 be a limit point of $\{y_n\}$ and assume y_n converges to y_0. Set $L = L(y_0, z)$ and call the open halfplanes into which L divides the plane L_1 and L_2. Without loss of generality, we assume that for each n, the corresponding a_n lies in the closed halfplane $\operatorname{cl} L_2$ determined by L.

We now show that every two points of S see a common ε-interval at z in $\operatorname{cl} L_1$: Otherwise, some members x_1 and x_2 of S would see no such interval, and there would exist points q_1 and q_2 in bdry $N \cap L_2$ such that every ε-interval at z seen by both x_1 and x_2 would lie in the convex region bounded by rays $R(z, q_1)$ and $R(z, q_2)$. However, for δ_n sufficiently small, y_n sees no ε-interval at z in this region, impossible since x_1, x_2, y_n see a common ε-interval at z. Thus the result is established.

Assume that the points of bdry $N \cap \operatorname{cl} L_1$ are ordered in a clockwise direction from s_0 to t_0, where s_0 and t_0 denote the endpoints of the interval $N \cap L$. For each y in S, there exist s_y and t_y on bdry $N \cap \operatorname{cl} L_1$ such that y sees $[s_y, z] \cup [t_y, z]$ via S and such that s_y and t_y are, respectively, the first and last points on bdry $N \cap \operatorname{cl} L$ having this property. Finally, let E_y denote the convex hull of all segments

$[z, a_y]$ seen by y, where $a_y \in \text{bdry}\, N \cap \text{cl}\, L_1$. Certainly y sees E_y via S.

We say $a < b$ on $\text{bdry}\, N \cap \text{cl}\, L_1$ if a precedes b in our clockwise order. Since every pair of points of S sees a common ε-interval at z in $\text{cl}\, L_1$, it follows that $\text{lub}\, s_y \leqq \text{glb}\, t_y$. Let $s_1 = \text{lub}\, s_y$ and $t_1 = \text{glb}\, t_y$. Then for each y we have $s_0 \leqq s_y \leqq s_1 \leqq t_1 \leqq t_y \leqq t_0$. If $s_0 = s_1$ or $t_1 = t_0$, the proof is complete. Hence we assume that $s_0 \neq s_1$ and $t_1 \neq t_0$, so that $\text{conv}\,\{s_1, z, t_1\} \cap L = \{z\}$. If for some positive number r', the set $\cap E_y$ contains an interval of length r' in $\text{conv}\,\{s_1, z, t_1\}$, the proof is finished. Otherwise, for every $1/n$ there is some w_n in S for which $E_{w_n} = E_n$ does not contain $M(z, 1/n) \cap \text{conv}\,\{s_1, z, t_1\}$, where $M(z, 1/n)$ denotes the $1/n$-disk centered at z. Hence the sequence of sets E_n converges to $[s_0, t_0]$.

In this case, every point of S sees some ε-interval at z on L: Suppose on the contrary that for some x in S, x sees neither $[s_0, z]$ nor $[z, t_0]$ via S. Then there exist points p_1 and p_2 in $\text{bdry}\, N \cap L_1$ and points p_1' and p_2' in $\text{bdry}\, N \cap L_2$ such that every ε-interval at z seen by x lies either in the convex region bounded by $R(z, p_1) \cup R(z, p_2)$ or in the convex region bounded by $R(z, p_1') \cup R(z, p_2')$. However, for n sufficiently large, the points y_n and w_n defined previously see no common ε-interval at z in either of these regions, impossible since every 4 points of S see a common ε-interval at z. Thus the assertion is proved.

Finally, we have to show that for at least one of the segments $[s_0, z]$ and $[z, t_0]$, every point of S sees this segment via S: Otherwise, there would exist points $u, v \in S$, $p_1, p_2 \in \text{bdry}\, N \cap L_1$ and $p_1', p_2' \in \text{bdry}\, N \cap L_2$ such that the ε-segments at z seen by both u and v would be either in the convex region bounded by $R(z, p_1) \cup R(z, p_2)$ or in the convex region bounded by $R(z, p_1') \cup R(z, p_2')$. This contradicts the fact that u, v, w_n, y_n see a common ε-segment at z for each value of n. We conclude that $\ker S$ is a full 1-dimensional, and the proof for Case 1 is complete.

Case 2. Assume that $z \in \text{bdry}\, S$. There are two possibilities to consider.

Case 2a. Suppose that there exist points s, t, u in S such that $z \in \text{int conv}\,\{s, t, u\}$. Then for two of these points, say s and t, no point of $[s, z)$ sees any point of $[t, z)$ via S. Then s and t see a common ε-interval at z in the closed region R' bounded by rays $R(t, z) \sim [t, z)$ and $R(s, z) \sim [s, z)$. We define R to be that minimal sector of a circle containing all ε-intervals at z seen by both s and

t. Then R is bounded by segments $[z, s_0]$ and $[z, t_0]$ in S, and since s, t, s_0, t_0 see a common ε-interval at z in R, certainly conv $\{s_0, z, t_0\} \subseteq S$. As before, order the points of bdry $R \sim ([z, s_0) \cup [z, t_0))$ in a clockwise direction, and say $a < b$ on bdry $R \sim ([z, s_0) \cup [z, t_0))$ if a precedes b in our clockwise ordering.

Assume that s_0 and t_0 are first and last points in our ordering. For each y in S, define D_y to be the convex hull of all ε-intervals at z in R seen by y, and let s_y and t_y be the first and last points of D_y in bdry $R \sim ([z, s_0) \cup [z, t_0))$. Clearly $s_1 \equiv \text{lub}\, s_y \leqq \text{glb}\, t_y \equiv t_1$. Furthermore, a simple geometric argument reveals that every y in S sees the region conv $\{s_0, z, t_0\} \cap D_y$ via S. But $s_0 \leqq s_y \leqq s_1 \leqq t_1 \leqq t_y \leqq t_0$ on bdry R, so conv $\{s_0, z, t_0\} \cap$ conv $\{s_1, z, t_1\} \subseteq$ conv $\{s_0, z, t_0\} \cap D_y$, and y sees conv $\{s_0, z, t_0\} \cap$ conv $\{s_1, z, t_1\}$ via S. This set is at least 1-dimensional and so dim ker $S \geqq 1$, the required result.

Case 2b. Suppose that $z \in$ bdry conv S. Then there must exist a line H supporting S at z, with S in the closed halfplane cl H_1 determined by H. Order the points $\{x: x \in \text{cl}\, H_1 \text{ and } \text{dist}\,(z, x) = \varepsilon\}$ in a clockwise direction, and assume that s_0 and t_0 are the first and last points of S in our ordering. Then conv $\{s_0, z, t_0\} \subseteq S$, since s_0 and t_0 see a common ε-interval at z.

If points s_0, z, t_0 are not collinear, then the argument in Case 2a above may be used to complete the proof. Hence consider the case in which s_0, z, t_0 lie in H. If $s_0 = t_0$, the proof is trivial, so assume $s_0 < z < t_0$. If s_0 and t_0 see a common interval at z in $H_1 \cup \{z\}$, then for some neighborhood N of z, $N \cap S$ is convex, and the argument of Case 1 may be adapted to finish the proof. In case s_0 and t_0 see no such interval, then using the fact that every 4 points see a common ε-interval at z, it is easy to show that for at least one of the segments $[s_0, z]$ and $[t_0, z]$, every point of S sees this segment via S. Hence we conclude that dim ker $S \geqq 1$ in Case 2, and the proof of Theorem 1 is complete.

The following example illustrates that the number 4 in Theorem 1 is best possible.

EXAMPLE 1. Let S be the set in Figure 1. Then every 3 points of S see via S at least one of the segments $[z, x_i]$, $1 \leqq i \leqq 4$, yet ker $S = \{z\}$.

Example 2 shows that the uniform lower bound ε on the segments seen by 4 points is necessary.

EXAMPLE 2. Let S be the set in Figure 2. Then every 4 points see a common segment on the x-axis, but ker S is the origin.

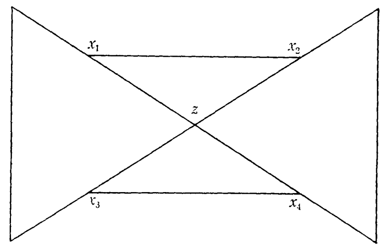

FIGURE 1

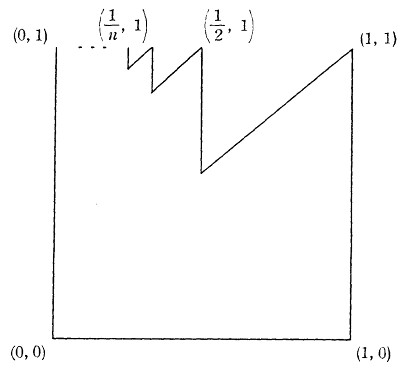

FIGURE 2

Our second theorem is not limited to the plane and is essentially a quantitative version of Krasnosel'skii's theorem.

THEOREM 2. *Let S be a compact set in R^2. The dimension of* $\ker S$ *is* 2 *if and only if for some* $\varepsilon > 0$, *every* 3 *points of S see via S a common neighborhood of radius ε. The number* 3 *is best possible.*

Proof. Again we need only establish the sufficiency of the condition. Clearly S is starshaped, so select z in $\ker S$. We observe that for every 3 points x_1, x_2, x_3 in S, there corresponds a connected subset T of S such that $\operatorname{dist}(z, t) = \varepsilon$ for each t in T and $\operatorname{conv}(T \cup \{z\})$ is a 2-dimensional subset of S. To verify this, let N be a neighborhood of radius ε seen by x_1, x_2, x_3. Then since $z \in \ker S$, $\operatorname{conv}(\{x_i, z\} \cup N) \subseteqq S$ for each i, so x_i sees $\operatorname{conv}(\{z\} \cup N)$ via S. Letting $T = \{y: y \in \operatorname{conv}(\{z\} \cup N), \operatorname{dist}(z, y) = \varepsilon\}$, T satisfies the requirements given above.

Furthermore, letting D denote the closed ε-disk about z, notice that $\operatorname{conv}(T \cup \{z\})$ is either D or a nondegenerate sector of D. If we associate with each set T the corresponding arc length $\delta(T)$ along bdry D, since S is compact, the numbers $\delta(T)$ are bounded below by some positive number δ. Therefore, for each $y \in S$, we may consider the collection G_y of all sectors of D seen by y for which the corresponding arc length on D is at least δ. Then using the sets G_y, the argument in Theorem 1 may be appropriately modified and in fact simplified to complete the proof. The details are straightforward and hence are omitted.

To see that the number 3 of Theorem 2 is best possible, consider the following easy example.

EXAMPLE 3. Let S be the set in Figure 3. Then every two points of S see one of the regions A_i via S, $1 \leqq i \leqq 3$, yet $\ker S = \phi$.

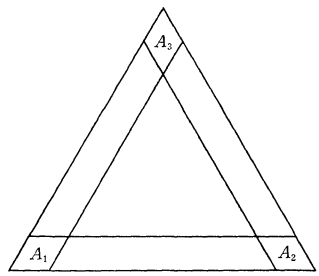

FIGURE 3

In conclusion, it is interesting to notice that both Theorems 1 and 2 fail completely and in fact no $f(k)$ is possible without the requirement that S be compact.

EXAMPLE 4. To see that our set must be closed, let S denote the unit disk with its center removed. Then every j-member subset of S sees via S an open sector having arc length $2\pi/2^j$, and every denumerable set of points sees a radius of S. Yet the set is not starshaped.

EXAMPLE 5. To show that S must be bounded, consider the following example by Hare and Kenelly [4]: Define $T_n = \{(x, y): n - 1 \leqq y \leqq n, n \leqq x + y\}$, and let $S = \bigcup T_n$. Then every finite subset of S sees via S a common disk of radius 1/2 in T_1, yet S is not starshaped.

REFERENCES

1. Marilyn Breen, *Sets in R^d having $(d-2)$-dimensional kernels*, Pacific J. Math., (to appear).
2. N. E. Foland and J. M. Marr, *Sets with zero dimensional kernels*, Pacific J. Math., **19** (1966), 429-432.
3. W. R. Hare, Jr. and J. W. Kenelly, *Concerning sets with one point kernel*, Nieuw Arch. Wisk., **14** (1966), 103-105.
4. ———, *Intersections of maximal starshaped sets*, Proc. Amer. Math. Soc., **19** (1968), 1299-1302.

5. M. A. Krasnosel'skii, *Sur un critère pour qu'un domaine soit étoile,* Math. Sb., (61) **19** (1946), 309-310.
6. F. A. Toranzos, *The dimension of the kernel of a starshaped set,* Notices Amer. Math. Soc., **14** (1967), 832.

Received January 15, 1978.

THE UNIVERSITY OF OKLAHOMA
NORMAN, OK 73019