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CONVOLUTION CUT-DOWN IN SOME RADICAL CONVOLUTION ALGEBRAS

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Let $\mathscr{A} = L^1_{\text{loc}}(\boldsymbol{R}^+)$ be the algebra of locally integrable functions on the positive real axis, with convolution as multiplication, given by

$$(f*g)(x) = \int_0^x f(x-t)g(t)dt\ .$$

Sometimes it is convenient to think of our functions as being defined on all of $\boldsymbol{R}$, but vanishing for negative x. We are interested in subalgebras of $\mathscr{A}$ that are Banach algebras in some norm, and that are radical in the sense that there exist no (nontrivial) complex homomorphisms. We call these algebras *radical convolution algebras*. Such algebras A present a challenge because there is no Fourier transform for them.

We are concerned with the problem of "convolution cut-down"; namely whether given a radical convolution algebra A and an $f \in A$, there must exist an $h \in A (h \neq 0$ such that $f*h \in L^1(\boldsymbol{R}^+)$. We show, at least, that one cannot always choose $h \in L^1$. As a corollary, we show that *simultaneous* convolution cut-down is not always possible.

A class of examples is formed by certain Beurling algebras $A_w = L^1_w(\boldsymbol{R}^+)$, where w is a positive weight function that satisfies

$$\text{(i)} \qquad w(x+y) \leqq w(x)w(y)$$

and where

$$\text{(ii)} \qquad \|f\| = \int_0^\infty |f(t)|\, w(t)dt\ .$$

It is not hard to show that for A_w to be a radical algebra, it is necessary and sufficient that

$$\text{(iii)} \qquad [w(x)]^{1/x} \longrightarrow 0 \quad \text{as} \quad x \to +\infty\ .$$

For example, we could choose $w(x) = \exp(-x^\alpha)$ for any $\alpha > 1$. (Note that the restriction, assumed by many authors, that $w(x) \geqq 1$, certainly does not apply here.)

Problem. Given a radical convolution algebra A and an f in A, does there exist an h in A ($h \neq 0$) such that $f*h \in L^1$?

This problem arose in discussion with Jamil A. Siddiqi, whom the author thanks for his help. An affirmative answer to the problem would be useful in solving certain natural problems about the ideal structure of A. To tell the truth, an affirmative answer is hardly to be expected, since convolution is known as a smoothing operator, and not as a reducing operator. We cannot at present prove a general negative answer. At least for some nonradical algebras, a negative answer is furnished by studying the distribution of the zeros of the Fourier transforms. We provide a partial negative solution in the radical case.

From now on, we assume that A is a radical convolution algebra, and we discard the function identically 0.

THEOREM. *It is not the case that for every $f \in A$ there is a $g \in L^1$ such that $f * g \in L^1$.*

COROLLARY. *There exists a pair $f, g \in A$ such that for no $h \in L^1$ are both $f*h$ and $g*h$ in L^1.*

We summarize this by saying that simultaneous convolution cut-down is impossible.

Proof of corollary. We proceed by contradiction. Given $f \in A$, choose $g \in A$ so that $f*g \in L^1$. Now choose $h \in A$ so that $g*h \in L^1$ and $(f*g)*h \in L^1$. Then with $k = g*h$, we would have $f*k \in L^1$ where $k \in L^1$, which contradicts the theorem.

Proof of theorem. We observe that for a function $h \in L^1(\boldsymbol{R}^+)$, its Fourier transform

$$h^\wedge(x) = \int_0^\infty e^{-itx} f(t)dt$$

has a natural extension as a function holomorphic in the lower half-plane $L = \{z: \operatorname{Im} z < 0\}$. If now $f*g = h$ with $g, h \in L^1(\boldsymbol{R}^+)$, we define for every $z \in L$:

$$V_z f = \operatorname{ord}_z h^\wedge - \operatorname{ord}_z g^\wedge$$

where $\operatorname{ord}_z H$ is the order of the zero of H at the point z. It is easy to see that V_z is well defined, for if $f*g = h$ and $f*G = H$ then $f*(g*H - G*h) = 0$ and by the famous Titchmarsh theorem that there are no divisors of zero in $\mathscr{A}$, we have $g*H - G*h = 0$, etc. Roughly, V_z plays the role of "the order of vanishing at z of the Fourier transform of f". Now it is easy to see that V_z extends to a valuation on the field A/A of formal quotients of functions in A. (That A/A makes sense as a field follows from the Titchmarsh theorem.)

By "valuation" we mean that

(i) $$V_z fg = V_z f + V_z g$$

(ii) $$V_z(f + g) \geqq \min(V_z f, V_z g) .$$

We appeal now to a theorem of Royden ([2], Proposition 5, p. 274) that any valuation V on A/A must be nonnegative on A. When applied to V_z, this implies that $h^\wedge/g^\wedge$ has no pole at z. If we now let

$$\phi_z f = \frac{h^\wedge(z)}{g^\wedge(z)}$$

then it follows that ϕ_z is a complex homomorphism of A. Since A is assumed to be radical, $\phi_z f = 0$, and thus $h^\wedge(z) = 0$ for all $z \in L$.

Therefore, by a simple argument, $h^\wedge(x)$ vanishes for all real x, and consequently $h = 0$, which is not allowed. This contradiction proves the theorem.

REFERENCES

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