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LOCALE GEOMETRY

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We commence with a locale $\mathscr{L}$ (that is, a complete Heyting algebra) and introduce the notion of an $\mathscr{L}$-valued betweenness relation on a set. The concept of an $\mathscr{L}$-valued geometry is then formulated and the relevant versions of the Radon, Helly and Carathéodory theorems are proved.

Introduction. The abstract theory of join systems was developed by W. Prenowitz [8] and [9] as an aid to studying descriptive and spherical geometries. This notion of join system has since been further developed by V. W. Bryant and R. J. Webster [1] to enable the corresponding axiomatic treatment of such results as the Radon, Helly and Carathéodory theorems. It is this aspect of the theory with which the present article is concerned.

We commence this article by extending the notion of a join system so that it is no longer necessarily two-valued. More precisely, given a locale lattice $\mathscr{L}$, we introduce the notion of an $\mathscr{L}$-valued betweenness relation $(-, -, -)\colon X \times X \times X \to \mathscr{L}$ on a set X; if $(x, y, z) = p \in \mathscr{L}$ we might say that the point z lies on the segment (x, y) with "probability p". This loose description is related to theories of multivalued logic which arise in topos theory. Indeed, one can develop join systems in a reasonably complete topos in terms of multivalued join systems over the category of sets; see §4. These notions, in turn, give rise to the forms of the Radon, Helly and Carathéodory theorems dicussed in §3.

We emphasize here that, in this preliminary article, we do not deal with multigroups (after W. Prenowitz) nor do we enter into all aspects of dimension theory (after V. W. Bryant and R. J. Webster). Also we leave the proof of the more basic elementary deductions as simple exercises for the reader; these results are used without reference.

1. $\mathscr{L}$-forms. Let $\mathscr{L}$ be a locale and let X be a set. A *symmetric $\mathscr{L}$-form* on X is a function $X(-, -)\colon X \times X \to \mathscr{L}$ such that $X(x, x) = 1$, $X(x, y) = X(y, x)$, $\sup_y X(x, y) \wedge X(y, z) = X(x, z)$. A *functional* on X is a set map $A\colon X \to \mathscr{L}$ such that $A = \sup_x A(x) \wedge X(x, -)$. A *singleton*, or *point* is a functional of the form $\bar{x} = X(x, -)\colon X \to \mathscr{L}$. Thus each functional is an "expansion of singletons" or an "internal colimit of points". For notational convenience we shall represent $\bar{x}$ simply by x unless we wish to emphasize the distinction.

The ordered set of functionals on X is denoted $\mathrm{Fnl}\,(X, \mathscr{L})$; it is a sublocale of $\mathscr{L}^X$. Note that if $A: X \to \mathscr{L}$ is any functional then $A \geqq x$ iff $A(x) = 1$.

A *map* of $\mathscr{L}$-forms $f: (X, X(-, -)) \to (X', X'(-, -))$ is a set map $f: X \to X'$ such that $X'(fx, fy) = X(x, y)$ for all $x, y \in X$.

2. **Convexity spaces.** An *$\mathscr{L}$-preconvexity space* is a set X equipped with a symmetric $\mathscr{L}$-form $X(-, -): X \times X \to \mathscr{L}$ and a map $(-, -, -): X \times X \times X \to \mathscr{L}$ which is functional in each variable separately. A *map* of preconvexity space is a map $f: X \to X'$ of $\mathscr{L}$-forms such that $(fx, fy, fz) = (x, y, z)$ for all $x, y, z \in X$. The resultant category is denoted $\mathscr{L}pc$.

Given $X \in \mathscr{L}pc$ we define the *convolutions*:

$$AB(x) = \sup_{y,z} A(y) \wedge B(z) \wedge (y, z, x)$$
$$A/B(x) = \sup_{y,z} A(y) \wedge B(z) \wedge (z, x, y) .$$

Then $\bar{x}\bar{y} = (x, y, -)$ is the *join* of x to y, while $\bar{x}/\bar{y} = (y, -, x)$ is the *extension* of x by y.

An interesting consequence of these definitions is the following *Kan-extension principle*: If f and g are polynomials of n-variables in the convolution operations AB and A/B, and $f(x_1, \cdots, x_n) \geqq g(x_1, \cdots, x_n)$ for all points $x_1, \cdots, x_n$ then $f(A_1, \cdots, A_n) \geqq g(A_1, \cdots, A_n)$ for all functionals $A_1, \cdots, A_n$.

An *$\mathscr{L}$-convexity space* is an $\mathscr{L}$-preconvexity space which satisfies the following axioms:

C1. (symmetry) $(x, y, z) = (y, x, z)$.

C2. (idempotence) $(a, a, x) = X(a, x)$, $(a, x, a) = X(a, x)$.

C3. (associativity) $\sup_w (y, v, w) \wedge (w, z, x) = \sup_w (v, z, w) \wedge (y, w, x)$.

C4. (transposition) $\sup_w (z, w, y) \wedge (x, w, v) \leqq \sup_w (x, y, w) \wedge (z, v, w)$.

C5. (cancellation) $\sup_w (x, y, w) \wedge (x, z, w) = X(y, z) \vee (x, y, z) \vee (x, z, y)$.

The full subcategory of $\mathscr{L}pc$ comprising the $\mathscr{L}$-convexity spaces is denoted $\mathscr{L}c$.

The following propositions are immediate from the axioms.

PROPOSITION. $xy/xz = y/z \vee xy/z \vee y/xz$.

PROPOSITION. *$AB = BA$, $(AB)C = A(BC)$, $A \leqq AA$ and $A \leqq A/A$, $(A/B)/C = A/BC$, $A(B/C) \leqq AB/C$, and $A/(B/C) \leqq AC/B$.*

PROPOSITION. (i) $xA/x = A \vee xA \vee A/x$,
(ii) $xA/xB = A/B \vee xA/B \vee A/xB$,
(iii) $x/xB = x/B$.

The following relations are easily deduced by iterated use of the preceding proposition:

LEMMA 2.1 (*Radon*).

$$\frac{x_0 \cdots x_n}{x_0 \cdots x_n} = \bigvee \left\{ x_{i_0} \cdots x_{i_r}, \frac{x_{i_0} \cdots x_{i_s}}{x_{i_{s+1}} \cdots x_{i_r}};\ i_0, \cdots, i_r \textit{ all different} \right\}.$$

LEMMA 2.2 (*Carathéodory*). *For* $n \geqq r$

$$\frac{x_0 \cdots x_n}{x_0 \cdots x_r} = \bigvee \left\{ x_{i_0} \cdots x_{i_p}, \frac{x_{i_0} \cdots x_{i_q}}{x_{i_{q+1}} \cdots x_{i_p}};\ i_0, \cdots, i_p \textit{ all different and } p - q \leqq r \right\}.$$

For the remainder of this section we shall suppose that X is a fixed $\mathscr{L}$-convexity space. A functional $A: X \to \mathscr{L}$ is said to be *convex* if $AA = A$; note that singletons are convex (C2). The *convex hull* of a functional A is defined to be $\bigvee_{n=1}^{\infty} A^n$.

PROPOSITION. (i) *If* $A_1, \cdots, A_n$ *are convex then so are* $A_1 \cdots A_n$ *and* A_1/A_2.
(ii) *The convex hull of a functional* A *is the intersection in* $\mathrm{Fnl}(X, \mathscr{L})$ *of all the convex functionals which contain* A.

A functional $A: X \to \mathscr{L}$ is said to be *linear* if it is convex and $A/A = A$. The *linear hull* of a functional A is defined to be $\bigvee_{m,n=1}^{\infty} A^m/A^n$ and is denoted by $[A]$.

PROPOSITION. (i) *The linear hull of a functional* A *is the intersection in* $\mathrm{Fnl}(X, \mathscr{L})$ *of all the linear functionals which contain* A.
(ii) *If* A *is convex then* A/A *is linear.*
(iii) *If* A *is convex then* $[A] = A/A$.
(iv) $[x_0 \cdots x_n] = x_0 \cdots x_n/x_0 \cdots x_n$.

3. The Radon, Helly and Carathéodory theorems. Henceforth in this section we suppose that X is a fixed $\mathscr{L}$-convexity space. We shall also suppose that whenever we consider a set $\{x_0, \cdots, x_n\}$ then the $\bar{x}_i$'s are distinct (recall that $\bar{x}_i$ is denoted simply by x_i). The product functional of $M = \{x_1, \cdots, x_n\}$ is denoted by $M^* = x_1 \cdots x_n$.

A set $\{x_0, \cdots, x_n\}$ of singletons is said to be *strongly dependent* if there exists an $i(0 \leqq i \leqq n)$ such that $[x_0 \cdots x_{i-1}x_{i+1} \cdots x_n](x_i) = 1$. If every set of $n+2$ singletons is strongly dependent then we say that X has *dimension* $\leqq n$.

THEOREM 3.1 (*Radon*). *If* $\{x_0, \cdots, x_{n+1}\}$ *is a set of* $n+2$ *singletons in a convexity space of dimension* $\leqq n$ *then there exist disjoint nonempty subsets* M *and* N *of* $\{x_0, \cdots, x_{n+1}\}$ *such that* $M^* \wedge N^* \neq 0$.

Proof. The $n+2$ points lie in a space of dimension $\leqq n$ so we may assume, without loss of generality, that $[x_1 \cdots x_{n+1}](x_0) = 1$. By Lemma 2.1 we have either $N^*(x_0) \neq 0$ where N is a subset of $\{x_1, \cdots, x_{n+1}\}$ or $N^*/P^*(x_0) \neq 0$ where N and P are nonempty disjoint subsets of $\{x_1, \cdots, x_{n+1}\}$. Thus the result follows from taking $M = x_0$ in the first case and $M = \{x_0, P\}$ in the second case. In the first case we have $N^*(x_0) \neq 0$ implies $x_0 \wedge N^* \neq 0$ since $x_0 \wedge N^* = 0$ implies $x_0(x_0) \wedge N^*(x_0) = 0$ implies $N^*(x_0) = 0$, and in the second case we have $N^*/P^*(x_0) \neq 0$ implies $\sup_{u,v} N^*(u) \wedge P^*(v) \wedge (v, x_0, u) \neq 0$ implies $\sup_u N^*(u) \wedge x_0P^*(u) \neq 0$ implies there exists a $u \in X$ such that $N^*(u) \wedge x_0P^*(u) \neq 0$.

THEOREM 3.2 (*Helly*). *If* $A_0, \cdots, A_{n+1}$ *is a family of* $n+2$ *convex functionals on a convexity space of dimension* $\leqq n$ *and any* $n+1$ *of these functionals intersect with certainty then all the functionals have a nonzero intersection.*

Proof. For each $i(0 \leqq i \leqq n+1)$ there exists, by hypothesis, a singleton x_i such that

$$x_i \leqq A_0 \wedge \cdots \wedge A_{i-1} \wedge A_{i+1} \wedge \cdots \wedge A_{n+1} .$$

If $x_i = x_j$ for some $i \neq j$ then $x_i \leqq A_0 \wedge \cdots \wedge A_{n+1}$ and the result follows. Otherwise the singletons x_i are distinct so that, by Theorem 3.1, there exist nonempty disjoint subsets M and N of $\{x_0, \cdots, x_{n+1}\}$ such that $M^* \wedge N^* \neq 0$. Because $M^* \wedge N^* \leqq A_0 \wedge \cdots \wedge A_{n+1}$ the result follows.

LEMMA 3.3. *If* $x \leqq x_0 \cdots x_n$ *and* $M^*/N^*(x) \neq 0$ *where* M *and* N *are nonempty disjoint subsets of* $\{x_0, \cdots, x_n\}$ *then there exists a proper subset* P *of* $\{x_0, \cdots, x_n\}$ *such that* $P^*(x) \neq 0$.

Proof. The proof is by induction on the cardinal of N. Firstly, if $|N| = 1$, assume $N = x_0$ without loss of generality. Let $S = \{x_1, \cdots, x_n\}$. Now $x \leqq x_0 \cdots x_n$ implies $x_0 \leqq x/S^*$. Moreover, if

$M^*/x_0(x) \neq 0$ where M is a nonempty subset of S then $S^*/x_0(x) \neq 0$. Thus $0 \neq S^*/x_0(x) \leqq S^*/(x/S^*) \leqq S^*/x(x)$ since S^* is convex. But $S^*/x(x) \neq 0$ implies $\sup_u S^*(u) \wedge (x, x, u) \neq 0$ implies $S^*(x) \neq 0$ so $x_1 \cdots x_n(x) \neq 0$. Now suppose $|N| = r + 1$ and $x \leqq x_0 \cdots x_n$ and $M^*/N^*(x) \neq 0$. Without loss of generality let $N = \{x_0, \cdots, x_r\}$. The conditions $x \leqq x_0 \cdots x_n$ and $M^*/x_0 \cdots x_r(x) \neq 0$ imply that $x_1 \cdots x_n /x_1 \cdots x_r(x) \neq 0$ since $x \leqq x_0 \cdots x_n$ implies $x_0 \leqq x/x_1 \cdots x_n$. Thus $0 \neq M^*/x_0 \cdots x_r(x) \leqq (x_1 \cdots x_n)/(x/x_1 \cdots x_n)x_1 \cdots x_r(x)$ implies $x_1 \cdots x_n /xx_1 \cdots x_r(x) \neq 0$. But $x_1 \cdots x_n/xx_1 \cdots x_r(x) = ((x_1 \cdots x_n/x_1 \cdots x_r)/x)(x)$ so $x_1 \cdots x_n/x_1 \cdots x_r(x) \neq 0$. Thus, by Lemma 2.2, either $P^*(x) \neq 0$ where P is a nonempty subset of $\{x_1, \cdots, x_n\}$ or $Q^*/R^*(x) \neq 0$ where Q and R are nonempty disjoint subsets of $\{x_1, \cdots, x_n\}$ and $|R| \leqq r$.

THEOREM 3.4 (*Carathéodory*). *If $x \leqq x_0 \cdots x_{n+1}$ for singletons in a convexity space of dimension $\leqq n$ then there exists a proper subset P of $\{x_0, \cdots, x_{n+1}\}$ such that $P^*(x) \neq 0$.*

Proof. Without loss of generality let us assume $x_0 \leqq [x_1 \cdots x_{n+1}]$. Thus, by Lemma 2.2, either $M^*(x_0) \neq 0$ where M is a subset of $\{x_1, \cdots, x_{n+1}\}$ or $M^*/N^*(x_0) \neq 0$ where M and N are nonempty disjoint subsets of $\{x_1, \cdots, x_{n+1}\}$. In the first case $x_1 \cdots x_{n+1}(x) \neq 0$ and in the second case $x_1 \cdots x_{n+1}/N^*(x) \neq 0$. In order to establish these assertions let $S = \{x_1, \cdots, x_{n+1}\}$. In the first case note that $M^*(x_0) \neq 0$ implies $S^*(x_0) \neq 0$. But $x \leqq x_0S^*$ implies $x_0 \leqq x/S^*$ thus

$$\begin{aligned}0 \neq S^*(x_0) &= \sup_u S^*(u) \wedge X(x_0, u) = \sup_u S^*(u) \wedge x_0(u)\\ &\leqq \sup_u S^*(u) \wedge x/S^*(u) = \sup_{u\ v\ w} S^*(u) \wedge x(v) \wedge S^*(w) \wedge (w, u, v)\\ &= \sup_{u\ w} S^*(u) \wedge S^*(w) \wedge (w, u, x) = S^*(x)\end{aligned}$$

since S^* is convex. Thus $x_1 \cdots x_{n+1}(x) \neq 0$. In the second case we have to show that $\sup_{u,v} S^*(u) \wedge N^*(v) \wedge (v, x, u) = \sup_u S^*(u) \wedge xN^*(u) \neq 0$. But we have

$$\begin{aligned}0 &\neq \sup_u S^*(u) \wedge x_0N^*(u) \leqq \sup_u S^*(u) \wedge (x/S^*)N^*(u)\\ &\leqq \sup_u S^*(u) \wedge xN^*/S^*(u) = \sup_{u\ v\ w} S^*(u) \wedge xN^*(v) \wedge S^*(w) \wedge (w, u, v)\\ &= \sup_v S^*(v) \wedge xN^*(v)\end{aligned}$$

since S^* is convex, as required. Thus either $P^*(x) \neq 0$ where P is a nonempty subset of $\{x_1, \cdots, x_{n+1}\}$ or $M^*/N^*(x) \neq 0$ where M and N are nonempty disjoint subsets of $\{x_1, \cdots, x_{n+1}\}$. The first case is as required while in the second case the result follows from Lemma 3.3.

REMARK. In the case $\mathscr{L} = 2$ these results reduce to the generalizations of Radon, Helly and Carathéodory theorems discussed by Bryant and Webster [1].

4. Examples. Examples of $\mathscr{L}$-convexity spaces can be generated by various different processes. Perhaps the most basic of these arises from the fact that $\mathscr{L}c$ is closed under colimits in $\mathscr{L}pc$ and $\mathscr{L}c$ has a generator (namely the one-point space). Thus, by the special adjoint-functor theorem (Mac Lane [7]), the inclusion $\mathscr{L}c \subset \mathscr{L}pc$ has a right adjoint, so every $\mathscr{L}$-preconvexity space has a canonical associated convexity space.

If X is an $\mathscr{L}$-convexity space then X^Z is an $\mathscr{L}^Z$-convexity space for all sets Z. Thus it is consistent to define, in a topos $\mathscr{E}$ (see Johnstone [6]) for which each $\mathscr{E}(Z, \Omega)$ is complete as a Heyting algebra, an Ω-convexity space as a map $(-, -, -): X \times X \times X \to \Omega$ in $\mathscr{E}$ such that $\mathscr{E}(Z, X)$ is an $\mathscr{E}(Z, \Omega)$-convexity space for all $Z \in \mathscr{E}$.

Another example arises as follows. Call a functional $A: X \to \mathscr{L}$ *left exact* if $A(x) \wedge A(y) = \sup_a A(a) \wedge X(a, x) \wedge X(a, y)$ and $\sup_a A(a) = 1$; a left-exact functional is always linear. Given $X \in \mathscr{L}c$ define $\hat{X}$ to be the set of all left-exact functionals from X to $\mathscr{L}$. On $\hat{X}$ define $\hat{X}(A, B) = \sup_x A(x) \wedge B(x)$ and $(A, B, C) = \sup_{x,y,z} A(x) \wedge B(y) \wedge C(z) \wedge (x, y, z)$. Then $\hat{X}$ is an $\mathscr{L}$-convexity space and $X \to \hat{X}$ is a map of $\mathscr{L}$-convexity spaces.

Finally, if $X \times X \times X \to \mathscr{L}_\lambda$, $\lambda \in \Lambda$, represents a set of convexity space structures on a set X, one for each $\lambda \in \Lambda$, the induced map $X \times X \times X \to \prod_\Lambda \mathscr{L}_\lambda$ is a convexity-space structure. This fact allows the construction of $\mathscr{L}$-valued convexity spaces from families of classical convexity spaces on X (see, for example, quasiconvexities [5]).

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