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AN ANTI-OPEN MAPPING THEOREM FOR FRÉCHET SPACES

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It is well-known that completeness is necessary for the usual open mapping theorem for Fréchet spaces. In contrast, it is shown that, with the obvious exception of ω, each infinite-dimensional Fréchet space has another distinct complete topology with the same continuous dual.

By a space or subspace, we mean an infinite-dimensional locally convex Hausdorff topological vector space over either the real or the complex scalars. Our notation generally follows Robertson and Robertson [7]. In particular, X' and $\sigma(X, X')$ denote the continuous dual and the weak topology on X, respectively. Denote by ω (respectively, ϕ) the space formed by the product (respectively, direct sum) of countably-many copies of the scalar field. We use c_0, l_1 and l_∞ to denote the Banach sequence spaces (with their usual norms) of, respectively, null sequences, absolutely summable sequences and bounded sequences.

Our main result can be stated as:

THEOREM. *Each Fréchet space $(X, \zeta) \neq \omega$ has a topology η, so that, $\sigma(X, X') < \eta < \zeta$ and the space (X, η) is complete.*

By the open mapping theorem, (X, η) is a complete space which is not barrelled. In Section one we prove the theorem for the special cases of $(X, \zeta) = c_0$ (Case I) and (X, ζ) a nuclear space with a continuous norm (Case II). Then in Section two we reduce the theorem to these special cases.

We will have occasion to use Grothendieck's characterization of the completion of the space (X, ζ) as the set of linear functionals on X' which are $\sigma(X', X)$-continuous on ζ-equicontinuous sets (see Robertson and Robertson [7], p. 103). Berezanskii's [4] (see also [2, pp. 61–62]) notion of inductive semi-reflexivity is used in Case II. In particular, complete nuclear spaces are inductive semi-reflexive, and the topology constructed from $\{\mu_n\}$ in Case II is complete in any inductive semi-reflexive space. The only other fact used about nuclear spaces is that their topology can be defined by means of (semi-) inner products (see Case II and Schaefer [7] p. 103).

Perhaps it is worth pointing out, that there are always lots of differently-defined complete topologies on each complete separable space (see Bellenot [1], [2] and with Ostling [3]): the difficulty is in showing that these topologies are really different.

1. Two special cases. First we prove the theorem for the following special cases:

Case I. The Banach space c_0: Let ξ be the norm topology on c_0 and let $\mathscr{U}$ be a free ultrafilter on the set of positive integers (i.e., $\cap \mathscr{U} = \emptyset$). For each $A \in \mathscr{U}$ and $K > 0$ let

$$E(A, K) = \{x = (x_n) \in l_1 \colon ||x||_1 \leqq K \text{ and } x_m = 0 \text{ for each } m \in A\} .$$

Let η be the topology of uniform convergence of the collection of sets

$$\{E(A, K) \cup \{y^n\} \colon A \in \mathscr{U}, K > 0, \{y^n\} \text{ a } l_1\text{-norm-null-sequence}\} .$$

Since finite sets are η-equicontinuous and each of the sets above are ξ-equicontinuous, we have $\sigma(c_0, l_1) \leqq \eta \leqq \xi$.

To see that $\eta < \xi$, note that if $\eta = \xi$ there would be a set $E(A, K) \cup \{y^n\}$ whose polar is contained in the unit ball of c_0. Since y^n is a l_1-norm-null-sequence, there is an M, so that $m \geqq M$ implies that $|y^n_m| < 2^{-1}$, for each n. (Where y^n is the sequence $\{y^n_m\}_m$.) Since $\mathscr{U}$ is free, A must be infinite and there is a $k \geqq M$ with $k \in A$. Consider $x \in c_0$, the vector which is the zero sequence, except that it is 2 in the kth position. Clearly x is not in the unit ball of c_0, but it is the polar of $E(A, K) \cup \{y^n\}$, a contradiction.

Consider X, the completion of (c_0, η), as a subspace of the algebraic dual of l_1. Since each l_1-norm-null-sequence is η-equicontinuous, $X \subset l_\infty$. Suppose D is a subset of the positive integers with $D \notin \mathscr{U}$. Then, since $\mathscr{U}$ is an ultrafilter, D^c, the complement of D, is an element of $\mathscr{U}$. Thus the η-topology restricted to the subspace $\{x \in c_0 \colon x_n = 0 \text{ if } n \notin D\}$ is the norm topology. It follows that for each $f = (f_n) \in X$, the subsequence $\{f_n \colon n \in D\}$ is a null-sequence, since f is $\sigma(l_1, c_0)$-continuous on $E(D^c, 1)$. Let $f = (f_n) \in l_\infty$ with $A = \{n \colon |f_n| \geqq \delta\}$ infinite, for some $\delta > 0$. If $A \notin \mathscr{U}$, then $f \notin X$ by the above, so assume $A \in \mathscr{U}$. Write $A = B \cup C$, a disjoint union of infinite sets, one of them is not in $\mathscr{U}$, and thus $f \notin X$. Therefore $X = c_0$ and (c_0, η) is complete.

Case II. (X, ξ) is a nuclear Fréchet space with a continuous norm: Let $||\cdot||_1 \leqq ||\cdot||_2 \leqq \cdots$ be a sequence of continuous norms which define the ξ-topology on X. Since X is nuclear, we assume that the unit ball of each $||\cdot||_{k+1}$ is precompact in the norm $||\cdot||_k$ and that each $||x||_k^2 = \langle x, x\rangle_k$, for some continuous inner product $\langle\cdot\,,\cdot\rangle_k$ on $X \oplus X$. Let $||\cdot||_k$ also represent (the possibly infinite-valued) dual norm of $||\cdot||_k$ on X'. A sequence $\{a'_n\} \subset X'$ is called *k-admissible* if

$\{||a'_n||_k\}$ is bounded and the semi-norm, $\rho_a(x) = \sup_n |a'_n(x)|$, defined for $x \in X$, is stronger than $||\cdot||_1$. That is, there is a constant K, with

$$(*) \qquad ||x||_1 \leqq K\rho_a(x) \ , \quad \text{for each} \quad x \in X \ .$$

A nonincreasing null-sequence of positive reals $\{\lambda_n\}$ is said to be *k-discriminating*, if for each k-admissible sequence $\{a'_n\}$,

$$\limsup_n ||a'_n||_k \lambda_n^{-1} = \infty \ .$$

Note that if $\{\lambda_n\}$ is k-discriminating and $\{\mu_n\}$ is another nonincreasing null-sequence of positive reals so that $\lim_n \mu_n/\lambda_n = 0$, then $\{\mu_n\}$ is also k-discriminating.

First, we prove that for each k, there is a k-discriminating sequence. To see this, let $\{e_n\} \subset X$ be a sequence orthogonal in $\langle \cdot\ ,\ \cdot \rangle_k$ and orthonormal in $\langle \cdot\ ,\ \cdot \rangle_1$. (The $\{e_n\}$ can be chosen inductively, by picking $e_{n+1} \in (\bigcap_1^n \ker f_i) \cap (\bigcap_1^n \ker g_i)$, where f_i and g_i are the continuous linear functionals given by $f_i(x) = \langle e_i, x \rangle_1$ and $g_i(x) = \langle e_i, x\rangle_k$, $i = 1, 2, \cdots n$.) Re-order $\{e_n\}$ so that the sequence $\{||e_n||_k\}_n$ is nondecreasing. We claim that the sequence $\lambda_n = 1/n\, ||e_{n^2}||_k$ is k-discriminating. Suppose not, then there is k-admissible $\{a'_n\}$ with

$$(**) \qquad ||a'_n||_k \leqq \lambda_n \ .$$

Let $\delta = 2^{-1}K^{-1}$, where K is the constant in (*).

Inductively choose $f_n \in X$ and an integer-valued function ϕ, so that

(1) $||f_n||_1 = 1$ and $f_n \in \text{span}\{e_j\colon (n-1)^2 < j \leqq n^2\}$;

(2) $a'_{\phi(j)}(f_n) = 0$ for $j < n$; and

(3) $|a'_{\phi(n)}(f_n)| \geqq \delta$.

If f_j and $\phi(j)$ have been chosen for $j < n$, it is possible to choose f_n satisfying (1) and (2) since condition (2) puts $n-1$ constraints on f_n and f_n is chosen from a $(2n-1)$-dimensional space. Thus by (*) we can find a $\phi(n)$ such that $2^{-1}||f_n||_1 = 2^{-1} \leqq K|a'_{\phi(n)}(f_n)|$, i.e., that (3) is satisfied.

Let $A(n) = \{j\colon (n-1)^2 < j \leqq n^2\}$ and suppose $f_n = \sum_{i \in A(n)} a_i e_i$. Since $\{e_i\}$ is orthonormal in $\langle \cdot\ ,\ \cdot \rangle_1$, condition (1) implies $\sum_{i \in A(n)} |\alpha_i|^2 = 1$. But $\{e_i\}$ is orthogonal in $\langle \cdot\ ,\ \cdot \rangle_k$ hence

$$||f_n||_k = \left[\sum_{i \in A(n)} |\alpha_i|^2 ||e_i||_k^2\right]^{1/2} \leqq ||e_{n^2}||_k = n^{-1}\lambda_n^{-1} \ .$$

Thus by condition (3), we have

$$(***) \qquad \delta \leqq |a'_{\varphi(n)}(f_n)| \leqq ||a'_{\phi(n)}||_k \cdot ||f_n||_k \leqq n^{-1}\lambda_n^{-1} ||a'_{\phi(n)}||_k \ .$$

On the other hand, condition (2) implies that ϕ is $1-1$ and hence $\phi(n) \geqq n$, infinitely often. Thus (**) implies $||a'_{\phi(n)}||_k \leqq \lambda_{\phi(n)} \leqq \lambda_n$, infinitely often. Combining with (***) yields

$$0 < \delta \leqq n^{-1}\lambda_n^{-1}\lambda_n = \frac{1}{n}\ , \quad \text{for infinitely many} \quad n\ ,$$

a contradiction.

For $k \geqq 1$, let $\{\lambda_n^k\}$ be a k-discriminating. The sequence $\lambda_n = n^{-1}\min\{\lambda_n^j : j \leqq n\}$ is thus k-discriminating for each $k \geqq 1$. Let $\mu_n = \lambda_{n(n+1)}$, and let η be the topology of uniform convergence on sequences $\{a_n'\} \subset X'$ with the property that there is an integer k and constant K with $\|a_n'\|_k \leqq K\mu_n$. It is easy to check that $\sigma(X, X') < \eta \leqq \xi$. Note that if U is any ξ-neighborhood of the origin and if ρ_{U^0} is the gauge functional of U-polar in X', then there is an integer k and a constant K so that $a' \in X'$ implies $\|a'\|_k \leqq K\rho_{U^0}(a')$. Thus by Bellenot [2, p. 62 and Th. 4.1, p. 64], η is a ξ-rotor topology and (X, η) is complete.

To show that $\eta < \xi$, we will prove that $\|\cdot\|_1$ is not η-continuous on X. By Robertson and Robertson [7, p. 46], the η-neighborhoods of the origin are polars of finite unions of the above sequences (as sets of values in X'). (Note that it is possible for $\lim(\mu_n/\mu_{2n}) = \infty$, and so we must consider finite unions.) Suppose $\|\cdot\|_1$ is η-continuous, then there is a finite number of sequences $\{b_{n,i}'\}_n$, $1 \leqq i \leqq j$, used to define η, so that $\|x\|_1 \leqq \sup\{|b_{n,i}'(x)| : 1 \leqq i \leqq j,\ n = 1, 2, \cdots\}$ for each $x \in X$. Let k and K be so that $\|b_{n,i}'\|_k \leqq K\mu_n$, for $1 \leqq i \leqq j$ and each n. Let $\{a_n'\}$ be a listing of values in X' contained in the sequences $\{b_{n,i}'\}_n$, $1 \leqq i \leqq j$, so that $\{\|a_n'\|_k\}$ is nonincreasing. It follows that $\{a_n'\}$ is k-admissible. Since $n > j \geqq i \geqq 1$ implies $(n+1)(n+2) \geqq nj + i$, and $\{\lambda_n\}$ is nonincreasing, $\lambda_{(n+1)(n+2)} \leqq \lambda_{nj+i}$. Thus if $m \geqq j^2 + j + 1$, then $m = nj + i$ with $n > j \geqq i \geqq 1$, and

$$\|a_m'\|_k \leqq K\mu_{n+1} = K\lambda_{(n+1)(n+2)} \leqq K\lambda_m\ .$$

Hence $\limsup_m(\|a_m'\|_k/\lambda_m) < \infty$ and since $\{\lambda_m\}$ is k-discriminating, $\{a_n'\}$ is not k-admissible. This contradiction completes the proof of the theorem for this case.

2. The general case. The following two lemmas are of a general nature. The first lemma shows that completeness is a "three space property" while the second is used often in the proof of the theorem. The referee has pointed out that Lemma 1 is known, we include a proof for completeness.

LEMMA 1. *Let X be a space, Y a closed subspace of X and $Z = X/Y$, the quotient. If Y and Z are complete, then X is complete.*

Proof. Let $\phi: X \to Z$ be the quotient map and let $j: X \to \hat{X}$ be the injection of X into its completion $\hat{X}$. Since Z is complete, ϕ

extends to a map $\hat{\phi}: \hat{X} \to Z$ so that $\hat{\phi} \circ j = \phi$. Furthermore, since Y is complete, $j(Y)$ is closed in $\hat{X}$, and thus we can construct the quotient $W = \hat{X}/j(Y)$ with quotient map $\psi: \hat{X} \to W$. Since $\ker \psi \circ j = Y$, there is a map $\theta: Z \to W$ so that $\theta \circ \phi = \psi \circ j: X \to W$. Thus $\theta \circ \hat{\phi} \circ j = \psi \circ j$, but since $j(X)$ is dense in $\hat{X}$ and since $\theta \circ \hat{\phi}$ and ψ are continuous, we have $\theta \circ \hat{\phi} = \psi$. Therefore θ and thus j are surjective maps, so that X is complete.

LEMMA 2. *A Fréchet space X satisfies the conclusions of the theorem if X has a closed subspace Y which satisfies the conclusions of the theorem.*

Proof. Let ξ be the topology on X with neighborhood basis of the origin $\mathscr{U}$. Let η be a topology on Y with $\sigma(Y, Y') \leqq \eta < \xi|_Y$ and so that the space (Y, η) is complete. Let $\mathscr{V}$ be the neighborhood basis of the origin for (Y, η). Let $\mathscr{W} = \{V + U: V \in \mathscr{V}, U \in \mathscr{U}\}$. It is straightforward to check that $\mathscr{W}$ is a neighborhood basis of the origin for a topology ζ on X with the properties:

(i) $\sigma(X, X') \leqq \zeta < \xi$,

(ii) $\zeta|_Y = \eta$, and

(iii) $(X, \zeta)/Y \equiv (X, \xi)/Y$.

Thus by (ii), (iii) and Lemma 1, (X, ζ) is complete and by (i) it satisfies the conclusion of the theorem.

Proof of the theorem. Let (X, ξ) be a Fréchet space $\neq \omega$. It follows that ξ is not the weak topology on X. First, we show there is a separable closed subspace Y of X, so that ξ, restricted to Y has a continuous norm. Since ξ is strictly stronger than $\sigma(X, X')$, there exists a continuous semi-norm on (X, ξ) which is not a linear combination of semi-norms $x \to |\langle x, x' \rangle|$ with $x' \in X'$, and thus from Schaefer [8], corollary on p. 124 it follows that X has a continuous semi-norm ρ so that the dimension of $X/\ker \rho$ is infinite. Let E be the normed space $X/\ker \rho$ with ρ norm and let $\psi: X \to E$ be the quotient map. Let $\{e_n\} \subset E$ be a linearly independent sequence. Let $\{x_n\} \subset X$ be so that $\psi(x_n) = e_n$, and let Y be the closed linear span of $\{x_n\}$ in (X, ξ). Since $\rho(\sum_1^n \alpha_i x_i) = \rho(\sum_1^n \alpha_i e_i)$, for all scalar sequences $\{\alpha_i\}$, ψ, restricted to Y, is an isometry of Y with semi-norm ρ into a subspace of E with norm ρ. Thus by Lemma 2, we assume that (X, ξ) is separable and has a continuous norm.

Suppose (X, ξ) is a Banach space. In the notation of Bellenot and Ostling [3], since X is separable and complete, we have $\xi = \xi_M$. Furthermore, Theorem 3.1 of that same paper shows (X, ξ_{SW}) is complete, where ξ_{SW} is the topology of uniform convergence on ξ-equicontinuous $\sigma(X', X)$-null sequences. Clearly, $\sigma(X, X') < \xi_{SW} \leqq \xi$, and if $\xi_{SW} < \xi$, then we are done. If $\xi_{SW} = \xi$ and since (X, ξ) is a Banach space, there must be a $\sigma(X', X)$-null sequence $\{a'_n\} \subset X'$, whose polar in X is contained in the unit ball of X. It is easy to check that the map, $T: X \to c_0$, which sends $x \in X$ to the sequence $\{a'_n(x)\} \in c_0$, is an isomorphism of X onto a closed subspace of c_0. (These results are known, see the author [1].) A classical result of Banach (see Lindenstrauss and Tzafriri [6, p. 53]) says that X must have a subspace isomorphic to c_0. An application of Lemma 2 and Case I completes the proof if (X, ξ) is a Banach space.

If (X, ξ) is not a Banach space, then X is not a subspace of $B \oplus \omega$, for any Banach space B. Thus a result of Bessaga, Pelczyński and Rolewicz [5] show that (X, ξ) has a nuclear subspace Y. Thus Case II and Lemma 2 completes the proof of the theorem.

REMARKS. It is possible that the following statement is true:

(*) Each complete space (X, ξ) with $\xi \neq \sigma(X, X')$, has another complete topology η with $\sigma(X, X') < \eta < \xi$.

There are three places in the proof of the theorem where metrizability was used. The most subtle use of the metric was in Lemma 2. If (X, ξ) is not Fréchet, it is possible that X/Y is not complete (Schaefer [6, Ex. 11, p. 192]) and hence Lemma 1 cannot be used to show (X, ζ) is complete. (The author thanks E. G. Ostling for pointing this out to the author.) Thus it is possible that (*) could be true for separable X, but false in general.

If (X, ξ) is separable and complete, then, as in the proof of the theorem (X, ξ_{SW}) is complete (see Bellenot and Ostling [3]). In this case $\xi = \xi_{SW}$ implies that (X, ξ) is a closed subspace of a product of copies of the Banach space c_0. In order to handle this in the manner of Case I, one must extend this case to include each (X, ξ) which is not inductively semi-reflexive, but for which $\xi = \xi_{SW}$. Examples of spaces which fall into this extended case and which may fail (*) are the spaces (X, ξ_{SW}) where (X, ξ) is any separable nonreflexive Banach space.

The proof that the topology constructed in Case II is complete works for any inductive semi-reflexive space. However, to show that this constructed topology was different from the given topology made strong use of the metrizability. In fact, if $(X, \xi) = \phi$, then for any positive nonincreasing null-sequence, $\{\mu_n\}$, the topology constructed in Case II will be the ξ-topology. It is open question if ϕ is the only such exception among complete separable spaces with a continuous norm. (Weak topologies are also exceptions.) In any case the space ϕ is perhaps the most likely counter-example (among the inductively semi-reflexive spaces) to (*).

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