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THERE ARE 2^c NONHOMEOMORPHIC CONTINUA IN $\beta R^n - R^n$

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In this paper it is shown that for $n \geqq 3, \beta R^n - R^n$ contains 2^c nonhomeomorphic continua. In the proof we will also construct c continua in $\beta R^3 - R^3$ with nonisomorphic first Cech cohomology groups and 2^c compacta in $\beta R^3 - R^3$ no two of which have the same shape.

Introduction. Much work has been done in the study of the Stone-Čech compactification of the natural numbers. Some of these results have been applied to the study of $\beta X - X$ for other topological spaces X, as in the proof of Frolik's result that $\beta X - X$ is not homogeneous for a nonpseudocompact space X (see [9]). Shape theory has offered new methods for examining βX and $\beta X - X$ that utilize the intrinsic topological properties of βX, as is illustrated in this paper in the case of βR^n. Using the fact that shape factors through Čech cohomology, we will construct c continua in $\beta R^3 - R^3$, no two of which have the same shape. Then, a particular embedding of subsets of the continua into βR^3 will exhibit 2^c compacta in $\beta R^3 - R^3$ with different shapes. An easy modification of the compacta will yield 2^c nonhomeomorphic continua in $\beta R^3 - R^3$, the proof of which utilizes the properties of shape dimension as developed by J. Keesling [5]. From this it follows that for $n \geqq 3$ there are 2^c nonhomeomorphic continua in $\beta R^n - R^n$.

Preliminaries. Let βX denote the Stone-Čech compactification of a space X. For references, see Gillman and Jerison [2], or Walker [9]. $H^n(X)$ will denote the n-dimensional Čech cohomology of X with coefficients in Z based on the numerable covers of X. Also, $[X, S^1]$ will denote all homotopy classes of maps from X into S^1, with the group structure induced by the group structure on S^1. Since S^1 is a $K(Z, 1)$, $H^1(X)$ is isomorphic to $[X, S^1]$. Finally, let $\tilde{\prod} A_i$ be the group $\prod_{i \in Z} A_i / \sum_{i \in Z} A_i$.

The following theorems will be used in this paper:

THEOREM 1 (*Lemma* 1.7 *of* [1]). *For X normal and connected, there is an exact sequence $0 \to C(X)/C^*(X) \to [\beta X, S^1] \to [X, S^1] \to 0$ where $C(X)$ is the additive group of real valued continuous functions on X, and $C^*(X)$ is the subgroup of bounded real continuous functions.*

THEOREM 2 (*Theorem* 1.6 *of* [5]). *Let $n \geqq 1$ be an integer. Let*

X be a locally compact, σ-compact space such that for every compact set $K \subseteqq X$ there is a compact set $L \subseteqq X - K$ such that $\dim L \geqq n$. *Then the shape dimension of βX,* $\operatorname{Sd} \beta X \geqq n$ *and* $\operatorname{Sd}(\beta X - X) \geqq n$.

THEOREM 3 (*Corollary* 1.9 *of* [5]). *Let X be a Lindelöf space and let K be a compact set contained in $\beta X \quad X$. Then* $\dim K = \operatorname{Sd} K$.

THEOREM 4 (*Theorem* 1.12 *of* [4]). *Suppose that X is realcompact and that K is a continuum contained in $\beta X - X$. Then if $f(K) = Y$ is any continuous maps which is a shape equivalence, f is a homeomorphism.*

Main Theorems.

THEOREM 5. *There are c subcontinua of $\beta R^3 - R^3$ which have nonisomorphic first Čech cohomology groups.*

Proof. Consider the collection $\{P_a : a \in \mathscr{A}\}$, where each P_a is a sequence of prime numbers such that there are an infinite number of distinct primes in P_a, and each prime occurs an infinite number of times; if $a, b \in \mathscr{A}$ with $a \neq b$, then there is a prime occuring in P_a which is not in P_b, or a prime in P_b which is not in P_a; and card $\mathscr{A} = c$. Let $\sum_a$ be the solenoid corresponding to the sequence P_a, and let $B_a = H^1(\sum_a)$. We know that B_a is isomorphic to $\{m/p_1p_2 \cdots p_k : m \in Z,\ p_i \in P_a\}$.

The solenoid $\sum_a$ may be described as follows: let $P_a = \{p_1, p_2, p_3, \cdots\}$. $\sum_a$ is the intersection of a decreasing tower of solid tori $\{T_n\}$ in R^3 with the properties that (i) $T_{n+1} \subseteqq T_n$ for every $n \in Z^+$; (ii) $\lim_{n\to\infty}$ [length of cross section of T_n] $= 0$; and (iii) T_{n+1} is wrapped p_n times around the hole of T_n. Also, let $p, q \in T_1$ so that the distance from p to q is maximal, and specify that T_n passes through p and q for every n.

Position $\sum_a$ in R^3 so that $p = (0, 0, 0)$ and $q = (0, 0, 1)$. Define $f : R^3 \to R^3$ by $f(x, y, z) = (x, y, z + 1)$, and let $A = \bigcup_{n\geqq 0} f^n(\sum_a)$. Hence, A is the union of a countable number of copies of $\sum_a$ placed end to end. Now $H^1(A) = \prod_{n\geqq 0} H^1(f^n(\sum_a)) = \prod H^1(\sum_a)$ (the countable infinite product of copies of $H^1(\sum_a)$), and so we have $H^1(A) = \prod B_a$.

Let $A_n = \bigcup_{i\geqq n} f^i(\sum_a)$, i.e., A_n is the closure of A with the first n copies of $\sum_a$ deleted. Since A and A_n are closed subsets of R^3, βA and βA_n are contained in βR^3. Also, A_n is connected implies that βA_n is connected. Hence, $\beta A - A = \bigcap_{n\geqq 0} \beta A_n$ is a continuum in $\beta R^3 - R^3$. Let $A^* = \beta A - A$. We now wish to compute $H^1(A^*)$.

By Theorem 1, there is an exact sequence $0 \to C(X)/C^*(X) \to$

$[\beta X, S^1] \to [X, S^1] \to 0$, where $C(X)$ is the additive group of real continuous functions on X, and $C^*(X)$ is the subgroup of bounded functions. Since $A^* = \bigcap_{n \geq 0} \beta A_n$, by the continuity of Čech cohomology, $H^1(A^*) = \varinjlim H^1(\beta A_n)$, where the bonding maps are induced by inclusion, $i_n^*: H^1(\beta A_n) \to H^1(\beta A_{n+1})$. For each n, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(A_n)/C^*(A_n) & \longrightarrow & [\beta A_n, S^1] & \longrightarrow & [A_n, S^1] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i_n^*} & & \downarrow{\scriptstyle i_n^*} & & \downarrow{\scriptstyle i_n^*} & & \\ 0 & \longrightarrow & C(A_{n+1})/C^*(A_{n+1}) & \longrightarrow & [\beta A_{n+1}, S^1] & \longrightarrow & [A_{n+1}, S^1] & \longrightarrow & 0 . \end{array}$$

This diagram gives rise to the following exact sequence: $0 \to \varinjlim C(A_n)/C^*(A_n) \to \varinjlim [\beta A_n, S^1] \to \varinjlim [A_n, S^1] \to 0$. Since $[X, S^1] \cong H^1(X)$, we have $\varinjlim [\beta A_n, S^1] \cong \varinjlim H^1(\beta A_n) \cong H^1(A^*)$, and $\varinjlim [A_n, S^1] \cong \varinjlim H^1(A_n)$, where the bonding maps are i_n^*. Hence, we have the following exact sequence:

$$0 \longrightarrow \varinjlim C(A_n)/C^*(A_n) \longrightarrow H^1(A^*) \longrightarrow \varinjlim H^1(A_n) \longrightarrow 0 .$$

We will now evaluate these direct limits.

Since A_n differs from A_{n+1} by a set of compact closure, $i_n^*: C(A_n)/C^*(A_n) \to C(A_{n+1})/C^*(A_{n+1})$ is an isomorphism. Hence, $\varinjlim C(A_n)/C^*(A_n)$ is isomorphic to $C(A_1)/C^*(A_1)$. Since $C(A_1)/C^*(A_1)$ is a torsion free divisible group, $C(A_1)/C^*(A_1)$ is isomorphic to a direct sum of copies of Q, the rational numbers. Therefore, $\varinjlim C(A_n)/C^*(A_n) \cong \bigoplus_c Q$.

Now consider $\varinjlim H^1(A_n)$. As before, $H^1(A_n)$ is isomorphic to $\prod B_a$, the countable infinite product of copies of B_a. The bonding map $i_n^*: H^1(A_n) \to H^1(A_{n+1})$ is defined by

$$i_n^*((x_1, x_2, x_3, \cdots)) = (x_2, x_3, \cdots) \quad (x_i \in B_a) .$$

Now $\varinjlim H^1(A_n)$ is isomorphic to $(\sum H^1(A_n))/S = (\sum (\prod B_a))/S$, where S is the subgroup generated by $i_n^*(y_n) - y_n$, $y_n \in H^1(A_n)$. (See [7], page 29.) Define a map $g: \prod B_a \to (\sum(\prod B_a))/S$ by $g(a) = (a, 0, 0, \cdots) + S$. One can verify that g is an onto homomorphism with kernel $\sum B_a$. Hence, g induces an isomorphism $(\sum(\prod B_a))/S \cong (\prod B_a)/(\sum B_a) = \tilde{\prod} B_a$, and so $\varinjlim H^1(A_n) \cong \tilde{\prod} B_a$.

By these two evaluations, we get the following exact sequence: $0 \to \bigoplus_c Q \to H^1(A^*) \to \tilde{\prod} B_a \to 0$. Since $\bigoplus_c Q$ is divisible, the sequence splits (see [7]), and $H^1(A^*) \cong \tilde{\prod} B_a \oplus (\bigoplus_c Q)$. Thus we have constructed a continuum A^* in $\beta R^3 - R^3$ with $H^1(A^*) \cong \tilde{\prod} B_a \otimes (\bigoplus_c Q)$.

Now for $a, b \in A$, $a \neq b$, $H^1(A^*)$ is not isomorphic to $H^1(B^*)$. This follows from the fact every element of $H^1(A^*)$ is divisible by a prime p if and only if $p \in P_a$. Hence, we have constructed c continua in $\beta R^3 - R^3$ with nonisomorphic first Čech cohomology groups. Since two spaces with nonisomorphic Čech cohomology groups have different shapes, we have the following corollary.

COROLLARY 1. *There are c continua in $\beta R^3 - R^3$, no two of which have the same shape.*

THEOREM 6. *There are 2^c compacta in $\beta R^3 - R^3$, no two of which have the same shape.*

Proof. Theorem 6 is a continuation of Theorem 5. Suppose $\mathscr{A}$, A, and A^* are as in the proof of Theorem 5. For each $a \in \mathscr{A}$, we have constructed a continuum A^* in $\beta R^3 - R^3$ such that for $a \neq b$, $\mathrm{Sh}(A^*) \neq \mathrm{Sh}(B^*)$. Now for each subset of $\mathscr{A}$ of cardinality c, we will construct a compactum in $\beta R^3 - R^3$ such that if $S_1, S_2 \subseteqq \mathscr{A}$, $S_1 \neq S_2$, and $\mathrm{card}\, S_1 = \mathrm{card}\, S_2 = c$, then the corresponding compacta will have different shapes. Since there are 2^c subsets of $\mathscr{A}$ of cardinality c, this will exhibit 2^c nonshape equivalent compacta in $\beta R^3 - R^3$.

Let $S \subseteqq \mathscr{A}$ such that $\mathrm{card}\, S = c$. There is a one-to-one correspondence between elements of S and real numbers r such that $0 \leqq r < 2\pi$. So each element a of S corresponds to a unique $r_a \in [0, 2\pi)$. Let $h_{r_a}\colon R^3 \to R^3$ be a rotation of the $y - z$ plane r_a radians. Define $A_r = h_{r_a}(A)$, where A is as defined above. As before, $H^1(A_r^*) = \tilde{\prod} B_a \oplus (\bigoplus_c Q)$, where $A_r^* = \beta A_r - A_r$. Let $C_S = \overline{\bigcup_{a \in S} A_r^*}$. Then C_S is a compact subset of $\beta R^3 - R^3$.

Claim. A_r^* is an isolated component of C_S.

Proof of Claim. Let N_i, $i = 1, 2$, be a neighborhood of the ray $h_{r_a}(\{(0, 0, z)\colon z \in R^+\})$ of radius 2, 3, respectively. By construction, $A_r \subseteqq N_1$. Define a function $f\colon \bar{N}_1 \cup (R^3 - N_2) \to [0, 1]$ by $f(\bar{N}_1) = 0$ and $f(R^3 - N_2) = 1$. Since R^3 is normal, there is a continuous extension of f, say $\bar{f}$, to all of R^3. Then $\bar{f}$ has a continuous extension, $\beta\bar{f}$, to all of βR^3. Since $\beta\bar{f}(A_r) = f(A_r) = 0$, we have $\beta\bar{f}(\bar{A}_r) = 0$, and so $\beta\bar{f}(A_r^*) = 0$. For $b \in S$, $b \neq a$, $\beta\bar{f}(B_r^*) = 1$, since for some neighborhood about the origin, points in B_r not in this neighborhood are in $R^3 - N_2$. Thus, $\beta\bar{f}(\overline{\bigcup_{b \in S-\{a\}} B_r^*}) = 1$. By normality, there exist open sets U and V in βR^3 with $U \cap V = \emptyset$, $A_r^* \subseteqq U$, and $(\overline{\bigcup_{b \in S-\{a\}} B_r^*}) \subseteqq V$. Hence, A_r^* is an isolated component of $C_S = (\overline{\bigcup_{b \in S-\{a\}} B_r^*}) \cup A_r^*$.

Note that these are the only isolated components, for if $X \subseteqq C_s - \bigcup_{a \in S} A_r^*$, then any open set containing X also contains points of $\bigcup_{a \in S} A_r^*$, since every point of X is a limit point of $\bigcup_{a \in S} A_r^*$.

Now, for $S_1, S_2 \subseteqq \underline{A}$ with $S_1 \neq S_2$ and $\operatorname{card} S_1 = \operatorname{card} S_2 = c$, the shape of C_{S_1} is different from the shape of C_{S_2}. This follows from the fact that if $\operatorname{Sh}(C_{S_1}) = \operatorname{Sh}(C_{S_2})$, then each isolated component of C_{S_1} is shape equivalent to an isolated component of C_{S_2}. Either $S_1 - S_2 \neq \emptyset$, or $S_2 - S_1 \neq \emptyset$, so without loss of generality assume that $S_1 - S_2 \neq \emptyset$, and let $a \in S_1 - S_2$. Then A_r^* is an isolated component of C_{S_1} which is not shape equivalent to any isolated component of C_{S_2}, which implies that $\operatorname{Sh}(C_{S_1}) \neq \operatorname{Sh}(C_{S_2})$.

Hence, there are 2^c compacta in $\beta R^3 - R^3$ no two of which have the same shape. Since there are at most 2^c compacta in βR^3, there are exactly 2^c compacta in $\beta R^3 - R^3$ no two of which have the same shape.

COROLLARY 2. *For $n \geqq 3$, there are 2^c compacta in $\beta R^n - R^n$, no two of which have the same shape.*

THEOREM 7. *There are 2^c nonhomeomorphic continua in $\beta R^3 - R^3$.*

Proof. As in the proof of Theorem 6, let $S \subseteqq \mathscr{A}$ such that $\operatorname{card} S = c$; $A_r = h_{r_a}(A)$; and $C_S = \overline{\bigcup_{a \in S} A_r^*}$.

Consider a plane P tangent to each solenoid of $\bigcup_{a \in S} A_r$, and let $P^* = \beta P - P \subseteqq \beta R^3 - R^3$. Let $X = C_S \cup P^*$. One can easily verify that X is a continuum. Now suppose $C_T = \overline{\bigcup_{b \in T} B_r^*}$ is the result of a collection of solenoids corresponding to the subset T of $\mathscr{A}$, where $\operatorname{card} T = c$ and $T \neq S$. Then $Y = C_T \cup P^*$ is a continuum of $\beta R^3 - R^3$.

We will show that X and Y are not homeomorphic. The method will be as follows. If h is a homeomorphism from X onto Y, then $h(C_S) = C_T$ which implies that C_S and C_T are homeomorphic, contradicting the fact that C_S and C_T have different shapes by Theorem 6, and therefore are not homeomorphic.

Claim 1. Let $x \in \beta R^2 - R^2$, and V an open set of $\beta R^2 - R^2$ containing x. Then there exists a closed set F containing x, such that $F \subseteqq V$ and F has dimension 2.

Proof of Claim 1. Since V is an open set in $\beta R^2 - R^2$, $V = U \cap (\beta R^2 - R^2)$, where U is open in βR^2. There is a set W, open in βR^2, such that $x \in W$ and $\bar{W} \subseteqq U$. Let $D = \operatorname{cl}_{R^2}(W \cap R^2)$. Now

$$\operatorname{Cl}_{\beta R^2}(\operatorname{Cl}_{R^2}(W \cap R^2)) = \bar{W} \subseteqq U ,$$

so that the set $\beta D - D = \operatorname{Cl}_{\beta R^2}(\operatorname{Cl}_{R^2}(W \cap R^2)) - \operatorname{Cl}_{R^2}(W \cap R^2)$ is a closed subset of V in $\beta R^2 - R^2$.

For any compact subset C of D, $D - C$ is open in $D = \operatorname{Cl}_{R^2}(W \cap R^2)$.

Since $W \cap R^2$ is open in R^2, $D - C$ contains a subset Z that is open in R^2. Let N be a basic open set in R^2 such that $\bar{N} \subseteqq Z$. Since $\dim \bar{N} = 2$, by Theorem 2 $\mathrm{Sd}(\beta D - D) \geqq 2$. By Theorem 3,

$$\dim(\beta D - D) \geqq 2 .$$

(See also [8].) Since $\dim(\beta D - D) \leqq 2$, it follows that $\dim(\beta D - D) = 2$. Hence, $F = \beta D - D$ is a closed subset of V containing x of dimension 2.

Claim 2. If $x \in A_r^*$ such that $x \notin P^*$, then $h(x) \in C_T$.

Proof of Claim 2. The claim follows from the fact that any neighborhood of a point in $Y - C_T \subseteqq P^*$ has dimension 2, by Claim 1, while x has neighborhoods of dimension $\leqq 1$.

Claim 3. If $x \in A_r^* \cap P^*$, then $h(x) \in C_T$.

Proof of Claim 3. We will show that x is a limit point of $A_r^* \cap (X - P^*)$. Then by Claim 2, since $h(A_r^* \cap (X - P^*)) \subseteqq C_T$, it follows that $h(x) \in C_T$.

Let U be an open set in $\beta R^3 - R^3$ containing x. There is a set W, open in $\beta R^3 - R^3$ such that $x \in W \subseteqq \bar{W} \subseteqq U$. Now, $W = (\beta R^3 - R^3) \cap V$, where V is open in βR^3. Since V is an open set containing $x \in A_r^* = \beta A_r - A_r$, $V \cap A \neq \varnothing$. This implies that V intersects an infinite number of solenoids of A_r.

Let $x_n \in A_r \cap V \cap (R^3 - P)$ such that $|x_n| \to \infty$ as $n \to \infty$. This is possible since $V \cap R^3$ is open, and $A_r \cap P$ is a countable set. Let $y \in \beta(\{x_n : n \geqq 1\}) - \{x_n : n \geqq 1\} \subseteqq \beta R^3 - R^3$. Since $x_n \in A_r$ for every n, $y \in \beta A_r - A_r$. Now, define f on $P \cup \{x_n : n \geqq 1\}$ by $f(P) = 0$ and $f(x_n) = 1$ for every n. Since $P \cup \{x_n : n \geqq 1\}$ is closed in R^3, there is a continuous extension of f to all of R^3, say $\bar{f}$. Then $\bar{f}$ can be extended continuously to βR^3, say by $\beta\bar{f}$. Now $\beta\bar{f}(x_n) = 1$ for every n implies that $\beta\bar{f}(y) = 1$. Since $\beta\bar{f}(P) = 0$, $\beta\bar{f}(\bar{P}) = 0$. Hence, $y \notin \beta P$. Also, $x_n \in V$ for every n, which implies that $y \in \bar{V} - V$, and hence $y \in \bar{W} \subseteqq U$. Therefore, $U \cap (A_r^* - P^*) \neq \varnothing$, which implies that x is a limit point of $A_r^* \cap (X - P^*)$. Hence, $h(x) \in C_T$.

By Claim 2 and Claim 3, $h(A_r^*) \subseteqq C_T$ for every A_r^*. Then $h(\cup A_r^*) \subseteqq C_T$, which implies $h(\overline{\cup A_r^*}) \subseteqq \bar{C}_T = C_T$, and $h(C_S) \subseteqq C_T$. Similarly, $h^{-1}(C_T) \subseteqq C_S$, which implies $C_T \subseteqq h(C_S)$. Therefore, $h(C_S) = C_T$ and C_S and C_T are homeomorphic. This contradicts Theorem 6, since $\mathrm{Sh}(C_S) \neq \mathrm{Sh}(C_T)$. Hence, X and Y are not homeomorphic.

By Theorem 6, there are 2^c choices for X, and since no two of them are homeomorphic, there are 2^c nonhomeomorphic continua in $\beta R^3 - R^3$.

COROLLARY 3. *For* $n \geqq 3$, $\beta R^n - R^n$ *contains* 2^c *nonhomeomorphic continua.*

COROLLARY 4. *Let* X *and* Y *be as in the proof of Theorem* 7. *Then there does not exist a continuous map* $f: X \rightarrow Y$ *that is a shape equivalence. In particular,* X *and* Y *are not homotopic.*

Proof. By Theorem 4, if f is a continuous map, $f: X \rightarrow Y$, which is a shape equivalence, then f is a homeomorphism, contradicting Theorem 7.

Note that Corollary 4 does not imply that X and Y are not shape equivalent, since there are shape morphisms that are not induced by continuous functions.

The problem appears much more nontrivial in the cases $n = 1, 2$. Since solenoids cannot be embedded in R^2, the same argument fails in the case $n = 2$. In fact, the method of Theorem 5 fails in general for R^2, since the cohomology of a continuum in the plane is either 0 or a direct sum of copies of Z, the integers. The solution in the case of $n = 1$ appears even more difficult, and is yet unsolved.

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