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VALENCE PROPERTIES OF THE SOLUTION OF A DIFFERENTIAL EQUATION

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Libera proved that the first order linear differential equation $F(z) + zF'(z) = 2f(z)$ has a convex, starlike or close-to-convex solution in $|z| < 1$ if the driving term $f(z)$ is convex, starlike, or close-to convex in $|z| < 1$. It was an open question whether the solution would be univalent if $f(z)$ were spiral-like or univalent. The paper shows the relation of Libera's question to the Mandelbrojt — Schiffer conjecture and the class M defined by S. Ruscheweyh. The paper proves there are spiral-like functions $f(z)$ for which the solution of the above differential equation is of infinite valence. The paper closes with four open problems.

Libera [6] proved that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ maps $|z| < 1$ onto a convex, starlike, or close-to-convex domain, then so does $F(z) = 2z^{-1}\int_0^z f(t)dt = z + \sum_{n=2}^{\infty} 2a_n z^n/(n+1)$. Bernardi [1] then proved that if $f(z)$ maps $|z| < 1$ onto a convex, starlike, or close-to-convex domain, then for any positive integer c, $F_c(z) = (c+1)z^{-c}\int_0^z t^{c-1}f(t)dt = \sum_{n=1}^{\infty}(c+1)a_n z^n/(n+c)$ does also. Lewandowski, Miller, and Zlotkiewicz noted that Bernardi's result could be rephrased as, for any positive integer c, the first order linear differential equation

$$cF(z) + zF'(z) = (c+1)f(z) \tag{1}$$

with convex, starlike, or close-to-convex driving term $f(z)$ has a convex, starlike, or close-to-convex solution. They then proved [5] that (1) has a starlike univalent solution for any starlike driving function $f(z)$ for any complex c with $\operatorname{Re} c \geqq 0$.

Libera [8, Problem 2.3] asked whether the differential equation (1) would have this geometric invariance property if $f(z)$ were univalent or if $f(z)$ were spiral-like. Before we answer both of these questions in the negative, let us see how his question is connected with the Mandelbrojt-Schiffer conjecture for univalent functions.

Mandelbrojt and Schiffer conjectured that if $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ are univalent in $|z| < 1$, then so also are the functions $H^* = \{f * g(z) : f * g(z) = \sum a_n b_n z^n/n\}$. This was settled negatively (it would have implied the Bieberbach conjecture) in three separate papers. Hayman [4] exhibited a univalent function $f(z)$ such that $f * f(z)$ grows too fast for z near 1. His analysis shed no light on the valence of functions in H^*. Epstein and Schoenberg [2] exhibited a starlike univalent polynomial whose composition with a non-elementary univalent function was not even locally univalent (but was at most three valent). Finally, Loewner and Netanyahu [7] exhibited two close-to-convex functions whose Hadamard composition is not even locally univalent (but again they were unable to determine if H^* contains functions of infinite valence). Loewner and Netanyahu's counterexample is to be contrasted with Ruscheweyh and Sheil-Small's [13] proof of the Polya-Schoenberg conjecture that $f*g$ is starlike if f and g are starlike.

In 1968 A. W. Goodman [3, p. 1046] in his survey paper on univalent functions raised the question of determining the maximum valency for functions in H^*.

Using Libera's result for starlike functions, we see that

$$\frac{2}{z}\int_0^z \frac{t}{(1-t)^2}dt = \sum_{n=1}^{\infty} \frac{2n}{n+1} z^n \equiv g(z)$$

is starlike and for this $g(z)$ the Hadamard composition

$$(f*g)(z) = \sum_{n=1}^{\infty} \frac{2n}{n+1} \frac{a_n}{n} z^n = \frac{2}{z}\int_0^z f(t)dt ,$$

so that Libera's question is a special case of the Mandelbrojt-Schiffer conjecture with g(z) a specific slow-growing starlike function. (Libera's result that $2z^{-1}\int_0^z f(t)dt$ is starlike if $f(z)$ is starlike follows, therefore, from the Polya-Schoenberg theorem.) Libera's question is, therefore, also related to the class M defined by Ruscheweyh [10], $M = \{\text{univalent } f: f*g \text{ is univalent for all starlike } g\}$. Ruscheweyh proved that the Bieberbach conjecture holds for M and remarked that the close-to-convex functions are a subset of M, while M is a proper subset of the univalent functions by Epstein-Schoenberg's example. Since Epstein-Schoenberg's example has no known obvious geometric properties (it is constructed via the Loewner differential equation) it is of interest to exhibit univalent functions with known geometric properties that are not in M.

Let us first consider

$$f(z) = (1-bi)^{-1}[(1-z)^{-1+bi} - 1] , \quad 0 < b \leqq 1 ,$$

which is obviously univalent upon considering the geometry of $\exp((-1+bi)\log(1-z))$ in $|z| < 1$. However,

$$\frac{2}{z}\int_0^z f(t)dt = \frac{2}{-1+bi}\left[1 + \frac{1}{bi}\left(\frac{(1-z)^{bi}-1}{z}\right)\right] \equiv F(z) ,$$

and $F(z_n) = 2/(-1+bi)$ at $z_n = 1 - \exp(-2\pi n/b)$, $n = 1, 2, \cdots$. Thus, this univalent $f(z)$ has an infinite valent $F(z)$. Although $f(z)$ maps onto a domain which "spirals," it is not spiral-like, and so we turn to a second example of a function not in M which provides an infinite valent function in H^*.

Let us consider $f(z) = z(1+z)^{-1+i}$ which satisfies

$$\operatorname{Re}\,[e^{i\pi/4}zf'(z)/f(z)] = (1-|z|^2)\sqrt{2}\,|1+z|^2 > 0$$

and is, therefore, spiral-like. Since

$$\frac{2}{z}\int_0^z f(t)dt = \frac{2}{z(i-1)}[(1+z)^i(iz-1)+1]\,,$$

it therefore suffices to show that $(1+z)^i(iz-1)$ equals -1 countably often in the disc. We will prove this in the context of the following useful theorem which guarantees that a complex number is in the range set of an analytic function if it is in the range set of the analytic function times a "well-behaved part."

THEOREM. *Let $k(z)$ be analytic in $|z|<1$ and N be a simply connected region in $|z|<1$ with $\partial N \cap \{|z|=1\} = e^{i\theta}$. Let $f(z)$, $g(z)$ be analytic in $|z|<1$ and satisfy*

(1) *$k(z) = f(z)\cdot g(z)$.*

(2) *$\lim g(z) = c \neq 0$ as $z \to e^{i\theta}$ within N.*

(3) *f has no asymptotic values within N at $e^{i\theta}$,* i.e., *for every path γ in N ending at $e^{i\theta}$, $f(z)$ does not tend to a finite or infinite limit as $z \to e^{i\theta}$ along γ.*

(4) *w_0 is in the range set of f on N,* i.e., *for every $r > 0$, there is a point z in $\{|z-e^{i\theta}|<r\} \cap N$ with $f(z) = w_0$.*

(5) *w_0 is not in the cluster set of f on ∂N,* i.e., *there is no sequence z_n on ∂N for which $f(z) \to w_0$.*

Then cw_0 is in the range set of $k(z)$ on $\bar{N}$.

Proof. Suppose cw_0 were not in the range set of $k(z)$ on $\bar{N}$. Then there is an $r > 0$ such that $k(z) \neq cw_0$ on $N^* = \{|z-e^{i\theta}| \leqq r\} \cap \{\bar{N} - e^{i\theta}\}$. Let

$$d = \inf_{z \in \partial N^*} |k(z) - cw_0|\,.$$

If d were 0, then there would be a sequence of points $z_n \in \partial N^*$ with $k(z_n) - w_0c \to 0$. If z_n accumulated inside $|z|<1$, then by continuity this would violate $k(z) \neq w_0c$ on N^*. If z_n accumulated on $|z|=1$, then since $\partial N \cap \{|z|=1\} = e^{i\theta}$, we would have z_n eventually on ∂N and $z_n \to e^{i\theta}$. However by (2) and $c \neq 0$, this would imply $f(z_n) \to w_0$ on ∂N which would contradict (5). Therefore $d > 0$.

Then $h(z) = (k(z) - w_0c)^{-1}$ is analytic in N^*, bounded by $1/d$ on

$\partial N^* - \{e^{i\theta}\}$ and, by (2) and (4), unbounded in N^*. Choose a point z_0 in N^* such that $|h(z_0)| > 1/d$. Lift the ray $t \cdot h(z_0)$, $t \geqq 1$, that is, let $\gamma(t) = \{h^{-1}(t \cdot h(z_0)) : t \geqq 1\}$. The path $\gamma(t)$ lies in N^* and cannot go to $\partial N^* - \{e^{i\theta}\}$ since $|h(z)| < 1/d$ on $\partial N^* - \{e^{i\theta}\}$. Thus $\gamma(t)$ must approach $e^{i\theta}$. Consequently $h(z)$ must have an asymptotic value at $e^{i\theta}$. But by (2) this implies that $f(z)$ has an asymptotic value at $e^{i\theta}$ which contradicts (3). Therefore the assumption that cw_0 is not in the range set of $k(z)$ on $\bar{N}$ cannot hold. This concludes the proof of the theorem.

We apply this theorem to

$$k(z) = (1+z)^i(iz-1),\ f(z) = (1+z)^i,\ g(z) = (iz-1)\ ,$$
$$w_0 = 1/(1+i)\ ,$$

N an appropriately large Stolz angle at $z = -1$.

We remark that an identical theorem holds four an additive version and we no longer need to restrict c.

We close the paper with a remark and four related open questions. Loewner and Netanyahu [7, p. 286] claimed "We should like to remark that one can also obtain another disproof of Conjecture I (the Mandelbrojt-Schiffer conjecture) by composing (Convolution)

$$f_\theta(z) = [z - z^2(1-e^{i\theta})/2](1-z)^{-2}$$

with itself and checking the well known inequality $|a_2^2 - a_3| \leqq 1$ for schlicht mappings. One easily computes that this inequality is not satisfied for instance for $\theta = i$." This remark is false. A computation shows $a_2 = (3+e^{i\theta})^2/8$, $a_3 = (2+e^{i\theta})^2/3$, and

$$\left|a_3 - a_2^2\right| = \left|\frac{3e^{4i\theta} + 36e^{3i\theta} + 98e^{2i\theta} + 68e^{i\theta} - 13}{192}\right| = h(\theta)$$

and therefore

$$\begin{aligned}192^2h^2(\theta) = 3^2 + 36^2 + 98^2 + 68^2 + 13^2 &+ (e^{i\theta} + e^{-i\theta})\\ &\times(3\cdot 36 + 36\cdot 98 + 68\cdot 98 + 68\cdot -13)\\ &+ (e^{2i\theta} + e^{-2i\theta})(98\cdot 3 + 36\cdot 68 + 98\cdot -13)\\ &+ (e^{3i\theta} + e^{-3i\theta})(3\cdot 68 + 36\cdot -13)\\ &+ (e^{4i\theta} + e^{-4i\theta})(3\cdot -13)\ .\end{aligned}$$

Consequently

$$\begin{aligned}(192)^2h(\theta)\frac{dh}{d\theta} &= -9416\sin\theta - 2936\sin 2\theta + 792\sin 3\theta + 156\sin 4\theta\\ &= -\sin\theta(7040 + 3168\sin^2\theta + 5248\cos\theta + 1248\sin^2\theta\cos\theta)\ .\end{aligned}$$

Since $7040 + 3168 \sin^2\theta + 5248 \cos\theta + 1248 \sin^2\theta \cos\theta \geqq 544$, we see that the maximum of $h(\theta)$ occurs for $\theta = 0$ and is 1.

Problem 1. Do there exist univalent functions f and g such that the coefficients of $F(z) = f*g$ satisfy $|a_3 - a_2^2| > 1$.

Problem 2. Do there exist close-to-convex univalent functions f and g such that the coefficients of $F(z) = f*g$ satisfy $|a_3 - a_2^2| > 1$.

Problem 3. If $f(z)$ is univalent must $2z^{-1}\int_0^z f(t)\,dt$ at least be normal in the sense of Lehto-Virtanen? (If so then all the integrals of a univalent function are normal.)

Problem 4. If $f(z)$ and $g(z)$ are univalent must $f*g$ be in H^p for all $p < 1/2$? A negative answer would provide an entirely different type of disproof of the Mandelbrojt-Schiffer conjecture since all univalent functions are in H^p, $p < 1/2$.

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