PACIFIC JOURNAL OF MATHEMATICS
Vol. 84, No. 2, 1979

ON THE SIGNATURE OF GRASSMANNIANS

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1. Introduction. Let $G_{n,k}$ denote the manifold of linear subspaces of $\boldsymbol{R}^n$ of dimension $k > 0$. Then $G_{n,k}$ is compact and has dimension $k(n-k)$. When n is even $G_{n,k}$ is orientable and we may consider the topological invariant $\operatorname{Sign}(G_{n,k})$. The cohomology algebra of $G_{n,k}$ over $\boldsymbol{R}$ was determined by Borel in [3] and thus in principle the problem of computing $\operatorname{Sign}(G_{n,k})$ is a problem in linear algebra. In practice this is very awkward, and it is the purpose of this paper to compute this invariant by a simpler method:

THEOREM. *The signature of $G_{n,k}$ is zero except when n and k are even and $k(n-k) \equiv 0 (\bmod 8)$. In this case (with a conventional orientation)*

$$\operatorname{Sign}(G_{n,k}) = \binom{\left[\frac{n}{4}\right]}{\left[\frac{k}{4}\right]}.$$

REMARK. When n is odd, $G_{n,k}$ is nonorientable and $\operatorname{Sign}(G_{n,k})$ is not defined; however, for odd n $\operatorname{Sign}(\widetilde{G}_{n,k}) = 0$, where $\widetilde{G}_{n,k}$ is the orientation covering of $G_{n,k}$.

2. The Atiyah-Bott formula. We recall a few definitions. Let X be a compact orientable manifold of dimension $4l$. The *signature* of X is defined by

$$\operatorname{Sign}(X) = \dim H^+ - \dim H^-$$

where $H^{2l}(X; \boldsymbol{R}) = H^+ \oplus H^-$ is a decomposition of the middle-dimensional cohomology of X into subspaces on which the cup-product form $B(x, y) = \langle x \cup y, X \rangle$ is positive definite and negative definite, respectively. When $\dim X$ is not divisible by 4 one defines $\operatorname{Sign} X = 0$.

More generally, let $f: X \to X$ be a mapping of X into itself. When the decomposition of $H^{2l}(X, \boldsymbol{R})$ is invariant under f one defines

$$\operatorname{Sign}(f) = \operatorname{tr} f^* | H^+ - \operatorname{tr} f^* | H^-$$

where $f^*: H^{2l}(X; \boldsymbol{R}) \to H^{2l}(X; \boldsymbol{R})$ is the homomorphism induced by f. $\operatorname{Sign}(f)$ is then independent of the choice of H^+ and H^-. When f is homotopic to the identity mapping one obviously has $\operatorname{Sign}(f) = \operatorname{Sign}(X)$.

Now suppose that X is an oriented Riemannian manifold. If $f: X \to X$ is an orientation preserving isometry, then at each isolated fixed point p of f the differential $df_p: T_pX \to T_pX$ is an orthogonal transformation with determinant 1. Let $\theta_1(p), \cdots, \theta_{2l}(p)$ be the $2l$ rotation angles associated with the eigenvalues of df_p. When the fixed point set of f consists of isolated points one has the formula of Atiyah and Bott ([1], p. 473):

$$\operatorname{Sign}(f) = (-1)^l \sum_{\substack{p \\ \text{fixed}}} \prod_{\nu=1}^{\nu=2l} \operatorname{ctn}\left(\frac{\theta_\nu(p)}{2}\right) .$$

We will apply this formula to a certain mapping $f: G_{n,k} \to G_{n,k}$.

REMARK. When f is an element of a compact group acting on X (and this will be the situation in our application) the formula above is also a consequence of the G-signature theorem of Atiyah and Singer. (See [1], p. 582 or [6], §18.)

For simplicity of notation we confine our attention to the case $n = 2s$, $k = 2r$; the remaining cases can be dealt with by minor adjustments in the argument.

Let $F: R^n \to R^n$ be the linear transformation which rotates the ith coordinate plane $P_i = \operatorname{span}\{e_{2i-1}, e_{2i}\}$ $(i = 1, 2, \cdots, s)$ through the angle α_i, where $0 < \alpha_i < \pi$. The transformation F induces a smooth mapping $f: G_{n,k} \to G_{n,k}$ which is clearly homotopic to the identity mapping. If P_I denotes the k-plane

$$P_I = P_{i_1} \oplus \cdots \oplus P_{i_r}$$

where $I = (i_1, \cdots, i_r)$ is a multi-index with $i_1 < i_2 < \cdots < i_r$ and $1 \leqq i_\nu \leqq s$, then $f(P_I) = P_I$.

PROPOSITION 2.1. *If the angles α_i are all distinct, then the points $P_I \in G_{n,k}$ are the only fixed points of f.*

Proof. Let W be a k-dimensional linear subspace of $\boldsymbol{R}^n$ not equal to any P_I. By regarding W as the row space of a matrix in reduced row echelon form one sees that there exists a $v \in W$ whose orthogonal projections v_i on P_i are nonzero for at least $r + 1$ indices i.

If $F(W) = W$, the vectors $v, F(v), \cdots, F^k(v)$ all belong to W, and hence there is a nontrivial relation

$$\sum_{\nu=0}^{\nu=k} a_\nu F^\nu(v) = 0 .$$

But this implies

$$\sum_{\nu=0}^{\nu=k} a_\nu F^\nu(v_i) = 0$$

for all i. Writing $\lambda_j = \cos(\alpha_j) + i\sin(\alpha_j)$ it follows that the k-degree polynomial $q(x) = a_0 + a_i x + \cdots + a_1 x^k$ has zeros λ_i and $\bar{\lambda}_i$ for each of the $r+1$ indices i for which v_i is nonzero. Since the α_i are all distinct, the coefficients a_ν must all be zero, which contradicts our assumption. Thus when $F(W) = W$, the subspace W must coincide with one of the subspaces P_I.

3. The Normal angles $\theta_\nu(p)$. We wish to show that with respect to an appropriate metric on $G_{n,k}$ the mapping f is an isometry, and then compute the normal angles $\theta_\nu(p)$ at the fixed points p of f. We begin with some remarks about the differentiable structure on $G_{n,k}$.

The smooth structure on $G_{n,k}$ may be defined by identifying $G_{n,k}$ with the left coset space G/H, where $G = O(n)$ is the orthogonal group and $H = O(k) \times O(n-k)$ is the closed subgroup of orthogonal transformations which take span $\{e_1, \cdots, e_k\}$ into itself. The space $O(n)$ may be regarded as the space of orthogonal $n \times n$ matrices (and hence as a subspace of $\boldsymbol{R}^{n^2}$), or, equivalently, as the space of orthonormal n-frames $a = (a_1, \cdots, a_n)$ in $\boldsymbol{R}^n$. We denote the image of an element $a \in G$ under the natural projection $\pi: G \to G/H$ by $\bar{a}$, and the image of a tangent vector $v \in T_aG$ under $d\pi: T_aG \to T_{\bar{a}}G/H$ by $\bar{v}$.

The elements of the tangent space T_eG are determined by smooth curves passing through the identity matrix e. By differentiating the relation $aa^t = e$ one obtains the usual identification of T_eG with the space of skew-symmetric $n \times n$ matrices. As a basis for T_eG we may take the set $\{b_{rs} | r < s\}$ of matrices b_{rs} having -1 in column s and row r, 1 in column r and row s, and 0 everywhere else. The ordering $\{b_{12}, b_{13}, b_{23}, b_{14}, b_{24}, \cdots\}$ then defines a standard orientation for G. More generally, the system of matrices $\{ab_{rs}\}$ may be taken as a basis for the tangent space T_aG at an arbitrary $a \in G$.

To obtain an oriented basis for the tangent space $T_{\bar{a}}G/H$ we simply restrict ourselves to vectors in T_aG which are orthogonal, as vectors in $\boldsymbol{R}^{n^2}$, to $T_a(aH)$. It is easily shown that the vectors ab_{ij} with $1 \leqq i \leqq k$ and $k+1 \leqq j \leqq n$ provide such a system. The coherence of the orientations will follow from the proof of Proposition 3.1. Note that even when a and a' represent the same coset in G/H, the bases $\{\overline{ab_{ij}}\}$ and $\{\overline{a'b_{ij}}\}$ will in general be different bases.

These facts all have simple interpretations in terms of curves in $O(n)$ and $G_{n,k}$. For example, the tangent vector $\overline{ab_{ij}}$ may be viewed as the infinitesimal motion of the k-plane span $\{a_1, \cdots, a_k\}$

towards its orthogonal complement obtained by rotating the vector a_i toward complementary vector a_j.

PROPOSITION 3.1. *There is a unique Riemannian metric on $G_{n,k}$ for which the standard bases $\{\overline{ab_{ij}}\}$ are all orthonormal. The mapping $f: G_{n,k} \to G_{n,k}$ is an orientation preserving isometry with respect to this metric. Moreover, the system of normal angles $\{\theta_\nu(p)\}$ is the same at each fixed point p of f.*

Proof. To prove the first assertion it will be enough to show that for arbitrary n-frames a and a' in $SO(n)$ the matrix of transition between the bases $\{\overline{ab_{ij}}\}$ and $\{\overline{a'b_{ij}}\}$ is orthogonal. Let $a' = ah$, where $h \in O(k) \times O(n-k)$. Then $\overline{a'b_{ij}} = \overline{a'b_{ij}h^{-1}} = \overline{ahb_{ij}h^{-1}}$.

Let $hb_{ij}h^{-1} = \sum_{\nu,\mu} q_{ij,\nu\mu} b_{\nu\mu}$. Clearly $q = [q_{ij,\nu\mu}]$ is the required transition matrix. Writing

$$h = \begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}, \quad E \in O(k),\ F \in O(n-k)\ ,$$

we obtain $q_{ij,\nu\mu} = e_{\nu i} f_{\mu j}$, that is, $q = E \otimes F$. Hence

$$\begin{aligned} \sum_{i,j} q_{ij,\nu\mu} q_{ij,\nu'\mu'} &= \sum_{i,j} e_{\nu i} f_{\mu j} e_{\nu' i} f_{\mu' j} \\ &= \sum_{i,j} e_{\nu i} e_{\nu' i} f_{\mu j} f_{\mu' j} = \delta_{\nu\nu'} \delta_{\mu\mu'}\ , \end{aligned}$$

which proves that $qq^t = e$. Moreover, it follows from $\det q = (\det E)^{n-k} (\det F)^k = 1$ that the various bases are coherently oriented.

To see that f is an isometry it is enough to observe that $df_{\bar a}(\overline{ab_{ij}}) = \overline{F(a)b_{ij}}$.

Finally, let $p = \bar a$ be any fixed point of f. We will compare the normal angles at $\bar a$ with those $\bar e$.

Denoting $F(e)$ by c we have

$$df_{\bar e}(\overline{b_{ij}}) = \overline{cb_{ij}} = \overline{cb_{ij}c^{-1}}\ ,$$

since $c \in O(k) \times O(n-k)$. On the other hand, $f(\bar a) = \bar a$ implies that $F(a) = ah$ for some $h \in O(k) \times O(n-k)$. Thus $ca = ah$ and hence

$$df_{\bar a}(\overline{ab_{ij}}) = \overline{F(a)b_{ij}} = \overline{ab_{ij}a^{-1}ca}\ .$$

Writing out the matrices D and D' of $df_{\bar e}$ and $df_{\bar a}$ with respect to the appropriate bases we have

$$(1) \qquad \overline{cb_{ij}c^{-1}} = df_{\bar e}(\overline{b_{ij}}) = \sum_{\nu,\mu} d_{ij,\nu\mu} \overline{b_{\nu\mu}}\ ,$$

$$(2) \qquad \overline{cab_{ij}a^{-1}c^{-1}a} = df_{\bar a}(\overline{ab_{ij}}) = \sum_{\nu,\mu} d'_{ij,\nu\mu} \overline{ab_{\nu\mu}}\ .$$

Let $ab_{ij}a^{-1} = \sum_{\nu,\mu} m_{ij,\nu\mu} b_{\nu\mu}$, and $m = [m_{ij,\nu\mu}]$. Then (2) becomes

$$\sum_{\nu,\mu} m_{ij,\nu\mu}\overline{cb_{\nu\mu}c^{-1}} = \sum_{\nu,\mu}\sum_{s,t} d'_{ij,\nu\mu} m_{\nu\mu,st}\overline{b_{st}} \,.$$

Substituting (1) we obtain

$$\sum_{\nu,\mu} m_{ij,\nu\mu} d_{\nu\mu,st}\overline{b_{st}} = \sum_{\nu,\mu} d'_{ij,\nu\mu} m_{\nu\mu,st}\overline{b_{st}}$$

for each i and j. Thus $md = d'm$. Since m is nonsingular this means that d' is similar to d, and hence the normal angles of f at p are the same as those at $\bar{e}$.

PROPOSITION 3.2. *At each fixed point p of $f\colon G_{2s,2r} \to G_{2s,2r}$ the normal angles $\{\theta_\nu(p)\}$ are the $2r(s-r)$ angles $\{\alpha_j \pm \alpha_i\}$ with $1 \leqq i \leqq r$ and $r+1 \leqq j \leqq s$.*

Proof. It is enough to compute the matrix m of $df_{\bar{e}}$ relative to the basis $\{\overline{b_{ij}}\}$. Since $c = F(e) \in O(k) \times O(n-k)$,

$$df_{\bar{e}}(\overline{b_{ij}}) = \overline{F(e)b_{ij}} = \overline{cb_{ij}c^{-1}}$$

for $1 \leqq i \leqq r$ and $r+1 \leqq j \leqq s$. Hence, as above, we have

$$m_{i'j',ij} = c_{ii'}c_{jj'} \,.$$

It follows that m is a sum of disjoint 4×4 blocks

$$\begin{bmatrix} \cos(\alpha_j)B & -\sin(\alpha_j)B \\ \sin(\alpha_j)B & \cos(\alpha_j)B \end{bmatrix}$$

where $B = \begin{bmatrix} \cos(\alpha_i) & -\sin(\alpha_i) \\ \sin(\alpha_i) & \cos(\alpha_i) \end{bmatrix}$. Each such block is the image of the matrix $e^{i\alpha_j}B$ under the standard monomorphism $U(2) \to SO(4)$. Since the eigenvalues of $e^{i\alpha_j}B$ are $e^{i(\alpha_j \pm \alpha_i)}$, the proposition follows.

4. Computation of the signature. We apply the Atiyah-Bott formula to the mapping $f\colon G_{n,k} \to G_{n,k}$ described above. Since f is homotopic to the identity mapping we obtain

$$\operatorname{Sign}(G_{n,k}) = (-1)^l \sum_{\substack{p \\ \text{fixed}}} \prod_{\substack{i \in I \\ j \in J}} \operatorname{ctn}\frac{(\alpha_j \pm \alpha_i)}{2} \,.$$

Here $I = (i_1 \cdots, i_r)$ is the multi-index which corresponds to the fixed point $P_I = P_{i_1} \oplus \cdots \oplus P_{i_r}$ and J is the complementary multi-index.

With the aid of the formula for the cotangent of a sum the right-hand side may be written in the form

$$\sum_{\substack{p \\ \text{fixed}}} \prod_{\substack{i \in I \\ j \in J}} \frac{1 - x_j x_i}{x_j - x_i}$$

where $x_\nu = \operatorname{ctn}^2(\alpha_\nu/2)$. Since the formula is true for all systems of distinct angles between 0 and π (noninclusive), it is true in particular when the angles $\alpha_1, \alpha_3, \cdots$ are taken between 0 and $\pi/2$ and the angles $\alpha_2, \alpha_4, \cdots$ are chosen to be their supplements.

Consider first the case s even, r even. Then the indicated choice of angles gives

$$\begin{aligned} x_2 &= x_1^{-1} \, , \\ x_4 &= x_3^{-1} \, , \\ &\cdots\cdots\cdots \\ x_s &= x_{s-1}^{-1} \, . \end{aligned}$$

For such a choice most of the terms in the sum vanish, since if there exists an $i \in I$ for which $x_j = x_j^{-1}$ for some $j \in J$, then

$$(1 - x_j x_i)(x_j - x_i)^{-1} = (1 - x_i^{-1} x_i)(x_i^{-1} - x_i)^{-1} = 0 \, .$$

The only terms which survive are those for which no x_i^{-1} can be an x_j; for such I, the factors may be grouped in pairs of the form

$$[(1 - x_j x_i)(x_j - x_i)^{-1}][(1 - x_j x_i^{-1})(x_j - x_i^{-1})^{-1}] = 1 \, ,$$

and to evaluate the sum we need only count the number of such multi-indices I. Since these are precisely those multi-indices which are a disjoint union of pairs (odd, odd + 1) the sum in question is $\binom{s/2}{r/2}$.

If s is even and r is odd, some x_i^{-1} must be an x_j; thus in this case no terms survive and the sum is 0.

When s is odd x_s is not the inverse of any other x_ν. For even r the contributing multi-indices are then exactly as in the first case, giving a value of $\binom{(s-1)/2}{r/2}$ for the sum. For odd r the contributing multi-indices are obtained from those already mentioned by adjoining the index s. The extra factors then occur in pairs of the form

$$[(1 - x_j x_s)(x_j - x_s)^{-1}][(1 - x_j^{-1} x_s)(x_j^{-1} - x_s)^{-1}] = 1 \, ,$$

giving a sum of $\binom{(s-1)/2}{(r-1)/2}$.

As for the sign preceding the sum, $(-1)^l = (-1)^{r(s-r)} = 1$ for those cases in which the sum is nonzero.

This completes the proof of the theorem stated at the beginning of the paper.

5. Further remarks.

1. A similar argument may be used to compute the signature of the complex Grassmannian $G_{n,k}(C)$ of complex k-dimensional sub-

spaces of C^n. The normal angles at a fixed point in this case have the form $\alpha_j - \alpha_i$.

One obtains

$$\operatorname{Sign}(G_{n,k}(C)) = \begin{cases} \begin{pmatrix} \left[\frac{n}{2}\right] \\ \left[\frac{k}{2}\right] \end{pmatrix} & k(n-k) \text{ even} \\ 0 & k(n-k) \text{ odd} \end{cases}$$

(For a different approach to the computation of $\operatorname{Sign} G_{n,k}(C)$ see Connolly and Nagano [4] (their formula contains a minor error due to a counting mistake).) [Added in proof; see also Mong [5]].

2. The same line of argument used here to compute the signature of $G_{n,k}$ may be used to compute the Euler characteristic $E(G_{n,k})$. The Lefschetz fixed point theorem is used in place of the theorem of Atiyah and Bott, and instead of computing the normal angles $\theta_\nu(p)$ one need only determine the fixed-point indices $\operatorname{Ind}_p(f)$. Since f is an isometry, these must necessarily be 1. One obtains

$$E(G_{n,k}) = \begin{cases} \begin{pmatrix} \left[\frac{n}{2}\right] \\ \left[\frac{k}{2}\right] \end{pmatrix} & k(n-k) \text{ even} \\ 0 & k(n-k) \text{ odd} \end{cases} .$$

3. The assumption that the angles α_i used in the definition of the transformation F are all distinct was necessary to obtain a mapping f with *isolated* fixed points. When coincidences $\alpha_{i_1} = \alpha_{i_2} = \cdots$ are permitted the fixed point sets become submanifolds of $G_{n,k}$ of positive dimension. The G-signature theorem of Atiyah and Singer (see [2] or [6]) may then be used to obtain information about the normal bundles of these submanifolds.

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Received October 3, 1978. This paper was written while the author was a visitor at the Mathematical Institute, Oxford.

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