PACIFIC JOURNAL OF MATHEMATICS
Vol. 84, No. 2, 1979

ULTRA-HAUSDORFF H-CLOSED EXTENSIONS

JACK R. PORTER AND R. GRANT WOODS

1. Introduction. A Hausdorff topological space X is called ultra-Hausdorff if, given two distinct points p and q of X, there is an open-and-closed (henceforth called "clopen") subset A of X such that $p \in A$ and $q \notin A$. A Hausdorff space X is H-closed if, whenever it is embedded as a subspace of another Hausdorff space Y, it is a closed subset of Y. In this paper we characterize those Hausdorff spaces that have ultra-Hausdorff H-closed extensions and construct, for such spaces, the projective maximum of the set of ultra-Hausdorff H-closed extensions. We compare this projective maximum to the Katětov H-closed extension, and examine when continuous functions can be continuously extended to this projective maximum.

Henceforth all hypothesized topological spaces will be assumed to be Hausdorff.

Let $\mathscr{L}$ be a lattice of subsets of a space X; i.e., suppose $\varnothing \in \mathscr{L}$, $X \in \mathscr{L}$ and that $\mathscr{L}$ is closed under finite unions and intersections. A *filter base* on $\mathscr{L}$ is a subset of $\mathscr{S}$ of $\mathscr{L}$ such that if $\mathscr{G}$ is any finite subset of $\mathscr{S}$ then $\cap \mathscr{G} \neq \varnothing$. (If $\mathscr{G}$ is a collection of sets, $\cap \mathscr{G}$ will denote $\cap \{G: G \in \mathscr{G}\}$; $\cup \mathscr{G}$ is defined similarly.) A *filter* on $\mathscr{L}$ is a subset $\mathscr{F}$ of $\mathscr{L}$ such that: (i) $\varnothing \notin \mathscr{F}$, (ii) if $F_1, F_2 \in \mathscr{F}$ then $F_1 \cap F_2 \in \mathscr{F}$ (iii) if $F \in \mathscr{F}$, $G \in \mathscr{L}$, and $F \subseteq G$ then $G \in \mathscr{F}$. An *ultrafilter* on $\mathscr{L}$ is a filter on $\mathscr{L}$ not contained properly in any other filter on $\mathscr{L}$. By Zorn's lemma each filter base on $\mathscr{L}$ is contained in some ultrafllter on $\mathscr{L}$. The *adherence* of a filter $\mathscr{F}$ on $\mathscr{L}$, denoted ad $(\mathscr{F})$, is defined to be $\cap \{\mathrm{cl}_X F: F \in \mathscr{F}\}$. $\mathscr{F}$ is said to be *free* if ad $(\mathscr{F}) = \varnothing$; otherwise $\mathscr{F}$ is *fixed*. An *open ultrafilter* (filter) on a space X is an ultrafilter (filter) on the lattice of open subsets on X.

We will need the following collection of known facts about H-closed spaces; see Problems 17K and 17L of [12].

PROPOSITION 1.1. (a) *The following are equivalent for a space* X:

(i) *X is H-closed.*

(ii) *Each open filter on X is fixed.*

(iii) *Each open ultrafilter on X is fixed.*

(iv) *If $\mathscr{C}$ is an open cover of X there is a finite subcollection $\mathscr{F}$ of $\mathscr{C}$ such that $\cup \mathscr{F}$ is dense in X.*

(b) *Continuous images of H-closed spaces are H-closed.*

(c) *If U is open in X and X is H-closed then* $\mathrm{cl}_X U$ *is H-closed.*

A space T is an *extension* of a space X if X is a dense subspace of T. Two extensions T_1 and T_2 of X are *equivalent* (as extensions of X) if there exists a homeomorphism $h: T_1 \to T_2$ such that $h|X$ is the identity function on X (henceforth denoted 1_X). We identify equivalent extensions of a space; with this convention, the class of extensions of X is a set (rather than a proper class).

Let $\mathscr{P}$ be a topological property and let $\mathscr{P}(X)$ denote the set of all extensions of a space X that have $\mathscr{P}$. An element T of $\mathscr{P}(X)$ is a *projective maximum* for $\mathscr{P}(X)$ if, whenever $Y \in \mathscr{P}(X)$, there exists a continuous function $f: T \to Y$ such that $f|X = 1_X$. Because of the above-mentioned identification of equivalent extensions, a projective maximum for $\mathscr{P}(X)$, if it exists at all, is unique.

Let $\mathscr{H}(X)$ (respectively $\mathscr{H}_0(X)$) denote the set of H-closed extensions (respectively ultra-Hausdorff H-closed extensions) of X. $\mathscr{H}(X)$ has a projective maximum, the so-called Katětov H-closed extension κX (see [6] or [9]). The space κX has as its underlying set $X \cup \{\mathscr{U}: \mathscr{U}$ is a free open ultrafilter on $X\}$. It is topologized by decreeing that a subset of X is open in κX iff it is open in X, and that if $\mathscr{U} \in \kappa X - X$, then $\{\{\mathscr{U}\} \cup U: U \in \mathscr{U}\}$ is a base of open neighborhoods at $\mathscr{U}$. This is a valid definition of a topology on κX, and it can be proved that κX is the projective maximum for $\mathscr{H}(X)$.

If X is a Tychonoff space, let $\mathscr{K}(X)$ (respectively $\mathscr{K}_0(X)$) denote the set of compactifications (respectively zero-dimensional compactifications) of X. (Recall that a space is zero-dimensional if its clopen sets form a base for its open sets). $\mathscr{K}(X)$ always has a projective maximum, namely the Stone-Čech compactification βX (see Chapter 6 of [4]). If $\mathscr{K}_0(X) \neq \varnothing$ then, as zero-dimensionality is hereditary, X is zero-dimensional. Conversely, suppose X is zero-dimensional, let $\mathscr{B}(X)$ be the Boolean algebra of clopen subsets of X, and let $\beta_0 X$ denote the Stone space of this Boolean algebra (see Chapter 2 of [11] for the definition of Stone spaces). The points of $\beta_0 X$ are precisely the ultrafilters on $\mathscr{B}(X)$, which henceforth we call *clopen ultrafilters*. If $x \in X$, put $\mathscr{U}(x) = \{B \in \mathscr{B}(X): x \in B\}$. The mapping $x \to \mathscr{U}(x)$ is an embedding of X as a dense subspace of $\beta_0 X$, and thus $\beta_0 X$ is a zero-demensional compactification of X. Hence a Tychonoff space X has a zero-dimensional compactification iff X is zero-dimensional. In this case $\beta_0 X$ is the projective maximum for $\mathscr{K}_0(X)$. In general $\beta X \neq \beta_0 X$; in fact $\beta X = \beta_0 X$ (in the sense that βX and $\beta_0 X$ are equivalent extensions of X) iff each disjoint pair of zero-sets of X can be separated by a clopen subset of X (see 16.17 of [11]).

It is well-known that a compact Hausdorff space is ultra-Hausdorff iff it is zero-dimensional (see, for example, 29D of [12]). Thus the class of Tychonoff spaces that have ultra-Hausdorff compactifications is precisely the class of zero-dimensional spaces; if X is zero-dimensional then $\mathscr{K}_0(X)$ has a projective maximum, namely $\beta_0 X$. We want to find an H-closed extension of a suitable Hausdorff space X that is to κX as $\beta_0 X$ is to βX; in other words we want to find a projective maximum for $\mathscr{H}_0(X)$. The problem has three parts. First, we must characterize those spaces X for which $\mathscr{H}_0(X) \neq \varnothing$. Second, we want to prove that if $\mathscr{H}_0(X) \neq \varnothing$ then $\mathscr{H}_0(X)$ has a projective maximum $\kappa_0 X$, and we want to give an explicit construction of $\kappa_0 X$. Thirdly, among those X for which $\kappa_0 X$ exists, we want to characterize those X for which $\kappa X = \kappa_0 X$. An additional problem is this: if $\mathscr{H}_0(X) \neq \varnothing \neq \mathscr{H}_0(Y)$, and if $f\colon X \to Y$ is continuous, we want to find necessary and sufficient conditions on f for f to extend continuously to a function $f^0\colon \kappa_0 X \to \kappa_0 Y$.

Our characterization of those spaces X for which $\mathscr{H}_0(X) \neq \varnothing$ will be given in terms of the *semi-regularization* of a topological space. Recall that a subset U of a space X is *regular open* if $U = \mathrm{int}_X \mathrm{cl}_X U$.

DEFINITION 1.2. (a) A space is *semi-regular* if its regular open sets form a basis for its open sets.

(b) Let (X, τ) be a topological space. The *semi-regularization* of (X, τ) is the topological space (X, τ_s) where τ_s is the topology of X for which $\mathscr{A} = \{\mathrm{int}_\tau \mathrm{cl}_\tau V\colon V \in \tau\}$ is an open base. (If we consider more than one topology on a set X, then the closure of a subset S of X with respect to the topology τ is denoted $\mathrm{cl}_\tau S$, and so on.)

The fact that the family $\mathscr{A}$ defined in 1.2(b) is a base for a topology on the set X follows from the easily-verified fact that if $U, V \in \tau$ then $\mathrm{int}_\tau \mathrm{cl}_\tau(U \cap V) = \mathrm{int}_\tau \mathrm{cl}_\tau U \cap \mathrm{int}_\tau \mathrm{cl}_\tau V$. If X is a space with no explicit symbol given for its topology, then X_s denotes its semi-regularization. Let $\mathscr{R}(X)$ denote the set of regular open subsets of X.

PROPOSITION 1.3. (a) (*Ex.* I.8.20, [2]) *If X is a Hausdorff space so is X_s.*

(b) (2.13 [8]) $\mathscr{R}(X) = \mathscr{R}(X_s)$ *and* $\mathscr{B}(X) = \mathscr{B}(X_s)$.

(c) (*Ex.* I.8.20, [2]) *X_s is semi-regular and $X = X_s$ iff X is semi-regular.*

(d) (3.4(i), [9]) *If (X, τ) is dense in a space T then the subspace topology that X inherits from T_s is just τ_s.*

(e) (3.2 [8]) *If X is H-closed and ultra-Hausdorff then X_s is compact.*

2. The main results. We begin by giving a necessary condition for a Hausdorff space to have an ultra-Hausdorff H-closed extension.

THEOREM 2.1. *If X is a Hausdorff space with an ultra-Hausdorff H-closed extension, then X_s is zero-dimensional.*

Proof. Let hX be an ultra-Hausdorff H-closed extension of X. By 1.3 (e) $(hX)_s$ is compact. By 1.3 (b) $(hX)_s$ is ultra-Hausdorff since hX is. But an ultra-Hausdorff compact space is zero-dimensional, as noted earlier. Thus $(hX)_s$ is zero-dimensional. By 1.3 (d) X_s is zero-dimensional as well.

Next we prove the converse to 2.1. We do this by showing that if X_s is zero-dimensional, then $\mathscr{H}_0(X) \neq \varnothing$ and in fact $\mathscr{H}_0(X)$ has a projective maximum.

DEFINITION 2.2. Let (X, τ) be a space such that X_s is zero-dimensional. Let $\kappa_0 X$ denote the set $X \cup \{\mathscr{U}: \mathscr{U}$ is a free clopen ultrafilter on $X\}$. Define a topology τ_0 on $\kappa_0 X$ as follows: $\tau \subseteq \tau_0$ and if $\mathscr{U} \in \kappa_0 X - X$ then $\{\{\mathscr{U}\} \cup U: U \in \tau$ and there exists $A \in \mathscr{U}$ such that $A \subseteq \mathrm{cl}_X U\}$ is a τ_0-neighborhood base at $\mathscr{U}$.

It is straightforward to check that the above is a valid definition of a topology on $\kappa_0 X$, and that (X, τ) is a dense subspace of $(\kappa_0 X, \tau_0)$.

LEMMA 2.3. *Let X be a space for which X_s is zero-dimensional. If $C \in \mathscr{B}(X)$ then $\mathrm{cl}_{\kappa_0 X} C \cap \mathrm{cl}_{\kappa_0 X}(X - C) = \varnothing$. Thus in particular $\mathrm{cl}_{\kappa_0 X} C \in \mathscr{B}(\kappa_0 X)$.*

Proof. Let $\mathscr{U} \in \mathrm{cl}_{\kappa_0 X} C - X$; then each member of $\mathscr{U}$ intersects C. Thus $C \in \mathscr{U}$. Similarly if $\mathscr{U} \in \mathrm{cl}_{\kappa_0 X}(X - C) - X$ then $X - C \in \mathscr{U}$. The lemma follows.

THEOREM 2.4. *Let X be a space for which X_s is zero-dimensional. Then $\kappa_0 X$ is an ultra-Hausdorff H-closed extension of X that is a projective maximum in the set of all ultra-Hausdorff H-closed extensions of X.*

Proof. We first prove that $\kappa_0 X$ is ultra-Hausdorff (and therefore in particular Hausdorff). Let x and y be distinct points of X. As X is Hausdorff find an open subset V of X such that $x \in V$ and $y \in X - \mathrm{cl}\, V$. As X_s is zero-dimensional find $C \in \mathscr{B}(X_s) = \mathscr{B}(X)$ such that $x \in C \subseteqq \mathrm{int}_X \mathrm{cl}_X V$. Then $y \notin C$. By 2.3 $\mathrm{cl}_{\kappa_0 X} C$ is a $\kappa_0 X$-clopen set separating x and y in $\kappa_0 X$.

If $x \in X$ and $\mathscr{U} \in \kappa_0 X - X$, then as $\mathscr{U}$ is a free clopen ultrafilter on X there exists $B \in \mathscr{U}$ such that $x \notin B$. Obviously $\mathscr{U} \in \mathrm{cl}_{\kappa_0 X} B$, so by 2.3 $\mathrm{cl}_{\kappa_0 X} B$ is a $\kappa_0 X$-clopen set separating x and $\mathscr{U}$ in $\kappa_0 X$.

If $\mathscr{U}$ and γ are distinct points of $\kappa_0 X - X$, then there exists $B \in \mathscr{B}(X)$ such that $B \in \mathscr{U}$ and $X - B \in \gamma$. Then $\mathrm{cl}_{\kappa_0 X} B$, as above, separates $\mathscr{U}$ and γ in $\kappa_0 X$. Thus $\kappa_0 X$ is ultra-Hausdorff.

We now show that $\kappa_0 X$ is H-closed. Let $\mathscr{W}$ be an open ultrafilter on $\kappa_0 X$; by 1.1(a) it suffices to show that $\mathrm{ad}_{\kappa_0 X} \mathscr{W} \neq \varnothing$. This will be the case if $\cap \{\mathrm{cl}_X(W \cap X) \colon W \in \mathscr{W}\} \neq \varnothing$, so we will assume that $\cap \{\mathrm{cl}_X(W \cap X) \colon W \in \mathscr{W}\} = \varnothing$. Let $\mathscr{U} = \mathscr{B}(X) \cap \mathscr{W}$. We claim that $\mathscr{U}$ is a free clopen ultrafilter on X. Obviously it is a filter as $\mathscr{W}$ is a filter and $\mathscr{B}(X)$ is a lattice of subsets of X. Now suppose $C \in \mathscr{B}(X)$ and $C \notin \mathscr{U}$. Then $C \notin \mathscr{W}$ so there exists $W \in \mathscr{W}$ such that $C \cap W = \varnothing$. As X is dense and open in $\kappa_0 X$, $W \cap X \in \mathscr{W}$. As $W \cap X \subseteqq X - C$, it follows that $X - C \in \mathscr{W} \cap \mathscr{B}(X) = \mathscr{U}$. Hence $\mathscr{U}$ is a clopen ultrafilter on X. If $x \in X$, by assumption there exists $W(x) \in \mathscr{W}$ such that $x \notin \mathrm{cl}_X(W(x) \cap X)$. As X_s is zero-dimensional, by 1.3(c) there exists $B(x) \in \mathscr{B}(X)$ such that $x \in B(x) \subseteqq X - \mathrm{cl}_X(W(x) \cap X)$. Then $W(x) \cap X \subseteqq X - B(x)$ so $X \backslash B(x) \in \mathscr{U}$; hence $\mathscr{U}$ is free, and hence a point of $\kappa_0 X \backslash X$. Now suppose V is open in X and that there exists $C \in \mathscr{U}$ such that $C \subseteqq \mathrm{cl}_X V$ – i.e., suppose $\{\mathscr{U}\} \cup V$ is a basic open neighborhood of $\mathscr{U}$. If $W \in \mathscr{W}$ then as $C \in \mathscr{W}$, $W \cap C \neq \varnothing$. As $C \subseteqq \mathrm{cl}_X V$, $W \cap C \cap V \neq \varnothing$. It follows that $\mathscr{U} \in \mathrm{cl}_{\kappa_0 X} W$ and so $\mathscr{U} \in \mathrm{ad}_{\kappa_0 X} \mathscr{W}$. Hence $\kappa_0 X$ is H-closed.

Finally, we show that $\kappa_0 X$ is a projective maximum in $\mathscr{H}_0(X)$. Let hX be any ultra-Hausdorff H-closed extension of X, and let $\mathscr{U} \in \kappa_0 X - X$. Put $S(\mathscr{U}) = \cap \{\mathrm{cl}_{hX} U \colon U \in \mathscr{U}\}$. We will show that $|S(\mathscr{U})| = 1$. First note that if $U \in \mathscr{U}$ then there is an open set U' of hX such that $U' \cap X = U$. Thus as $\{U' \colon U \in \mathscr{U}\}$ is an open filter base on hX, since hX is H-closed by 1.1(a) $\cap \{\mathrm{cl}_{hX} U' \colon U \in \mathscr{U}\} \neq \varnothing$. As $\mathrm{cl}_{hX} U' = \mathrm{cl}_{hX} U$ for each $U \in \mathscr{U}$, it follows that $|S_{,}| \geqq 1$. If y and z were distinct points of $S(\mathscr{U})$, as hX is ultra-Hausdorff there is a clopen subset A of hX such that $y \in A$ and $z \in hX - A$. As $y \in S(\mathscr{U})$ it follows that $A \cap U \neq \varnothing$ for each $U \in \mathscr{U}$; hence $A \cap X \in \mathscr{U}$. Similarly $X - A \in \mathscr{U}$, which is a contradiction. Thus $|S(\mathscr{U})| = 1$. Let $S(\mathscr{U}) = \{y(\mathscr{U})\}$. We now define a function $f \colon \kappa_0 X \to hX$ as follows: $f | X = 1_X$ and $f(\mathscr{U}) = y(\mathscr{U})$ for each $\mathscr{U} \in \kappa_0 X \backslash X$. Then f is well-defined; we claim that f is continuous. If $x \in X$ and $f(x) \in W$, where W is open in hX, then $x \in W \cap X \subseteqq$

$f^{\leftarrow}[W]$, and $W \cap X$ is open in $\kappa_0 X$ as X is open in $\kappa_0 X$. Hence f is continuous at x.

Let $\mathscr{U} \in \kappa_0 X - X$ and let W be an hX-neighborhood of $y(\mathscr{U})$. Since hX is an ultra-Hausdorff H-closed space, by 1.3(e) $(hX)_s$ is compact and zero-dimensional. Now $\operatorname{int}_{hX}\operatorname{cl}_{hX} W$ is an $(hX)_s$-neighborhood of $y(\mathscr{U})$, so there is an $(hX)_s$-clopen set C such that $y(\mathscr{U}) \in C \subseteqq \operatorname{int}_{hX}\operatorname{cl}_{hX} W$. By 1.3(b) C is hX-clopen, and so $C \cap X$ is clopen in X. Since $y(\mathscr{U}) \in C$, $(C \cap X) \cap U \neq \varnothing$ for each $U \in \mathscr{U}$; thus $C \cap X \in \mathscr{U}$. But $C \cap X \subseteqq X \cap (\operatorname{int}_{hX}\operatorname{cl}_{hX} W) = \operatorname{int}_X \operatorname{cl}_X (W \cap X)$, so $\{\mathscr{U}\} \cup (W \cap X)$ is a $\kappa_0 X$-neighborhood of $\mathscr{U}$ mapped into W by f. Thus f is continuous at $\mathscr{U}$. Hence f is continuous, and so $\kappa_0 X$ is a projective maximum for $\mathscr{H}_0(X)$.

COROLLARY 2.5. *Let X be a Hausdorff space. The following conditions are equivalent*:

(a) *X_s is zero-dimensional..*

(b) *X has an ultra-Hausdorff H-closed extension.*

If either condition holds then the set of ultra-Hausdorff H-closed extensions of X has a projective maximum.

Proof. This follows from 2.1 and 2.4.

Let X be a space for which X_s is zero-dimensional. We next give necessary and sufficient conditions for κX to be the "same as" $\kappa_0 X$ – i.e., for κX and $\kappa_0 X$ to be equivalent as extensions of X. Obviously $\kappa X = \kappa_0 X$ in this sense iff κX is ultra-Hausdorff. A proof of 2.6 below can be obtained by using Theorem 6.4 of [8], but we will give a more direct, self-contained proof. Recall that the boundary of a subset S of X, denoted $\operatorname{bd}_X S$, is defined to be $\operatorname{cl}_X S\text{-}\operatorname{int}_X S$.

THEOREM 2.6. *Let X be a space for which X_s is zero-dimensional. The following are equivalent*:

(a) *$\kappa X = \kappa_0 X$.*

(b) *Each regular open subset of X_s has a compact boundary in X_s.*

(c) *If B is the boundary in X of a regular open subset of X, then every cover of B by members of $\mathscr{B}(X)$ has a finite subcover.*

Proof. First note that by 1.3(b) if U is a regular open subset of X then $\operatorname{bd}_{X_s} U = \operatorname{bd}_X U$ as sets.

(b) $\Leftrightarrow$ (c): Obviously (b) $\Rightarrow$ (c) by 1.3(b) and the above remark. Obviously (c) $\Rightarrow$ (b) because $\mathscr{B}(X)$ is a base for the topology of X_s.

(b) $\Rightarrow$ (a): Let x and y be distinct points of κX. If either x or y is in X, then the argument used in 2.4 and the fact that X_s is zerodimensional imply that x and y can be separated in κX by a κX-clopen set. Hence suppose $x = \mathscr{U}$ and $y = \gamma$ where $\mathscr{U}$ and γ are distinct free open ultrafilters on X. As $\mathscr{U} \neq \gamma$ there is a regular open set U of X such that $U \in \mathscr{U}$ and $X - \text{cl}_X U \in \gamma$. If $x \in \text{bd}_{X_s} U$ find $W(x) \in \mathscr{U}$ such that $x \notin \text{cl}_X W(x)$. As X_s is zero-dimensional there exists $B(x) \in \mathscr{B}(X)(=\mathscr{B}(X_s))$ such that $x \in B(x) \subseteq X - \text{cl}_X W(x)$. As $\text{bd}_{X_s} U$ is compact, there exists a finite subset $\{x_1, \cdots, x_n\}$ of $\text{bd}_{X_s} U$ such that $\text{bd}_{X_s} U \subseteq \bigcup_{i=1}^n B(x_i) = B$. Thus $B \in \mathscr{B}(X)$ and as $\bigcap_{i=1}^n W(x_i) \subseteq X - B$, it follows that $X - B \in \mathscr{U}$. Thus $(X - B) \cap U \in \mathscr{U}$. Now $(X - B) \cap U$ is closed in X; for if $p \in \text{cl}_X[(X - B) \cap U]$, then $p \in X - B$ and $p \in \text{cl}_X U$. But $p \notin \text{bd}_{X_s} U$ as $\text{bd}_{X_s} U \subseteq B$, so $p \in \text{int}_X \text{cl}_X U = U$. Thus $p \in (X - B) \cap U$ so $(X - B) \cap U \in \mathscr{B}(X)$. It is now easy to verify that $\text{cl}_{\kappa X}[(X - B) \cap U]$ is a κX-clopen set separating $\mathscr{U}$ and γ.

(a) $\Rightarrow$ (b): As κX has an ultra-Hausdorff H-closed extension (i.e., itself), by 2.1 $(\kappa X)_s$ is zero-dimensional. By 1.6(b)$(\kappa X)_s$, being a continuous image of κX, is H-closed, so by 1.6(c) $(\kappa X)_s$ is a compact space. Let U be a regular open set in X_s (and hence in X by 1.3(b)) and let $U^\# = U \cup \{\mathscr{U} \in \kappa X \backslash X \colon U \in \mathscr{U}\}$. It is straightforward to verify that $\text{int}_{\kappa X} \text{cl}_{\kappa X} U^\# = U^\#$, so $U^\#$ is open in $(\kappa X)_s$. As $U \cup (X - \text{cl}_X U)$ is dense in X, it follows that $\kappa X \backslash X \subseteq U^\# \cup (X - \text{cl}_X U)^\#$. Thus $\text{bd}_{X_s} U = (\kappa X)_s - [U^\# \cup (X - \text{cl}_X U)^\#]$, so $\text{bd}_{X_s} U$ is a compact subset of $(\kappa X)_s$. By 1.6(a) $\text{bd}_{X_s} U$ is a compact subset of X_s.

Recall that a subset $A \subseteq X$ is *nowhere dense* if $X - A$ is an open, dense subset. In particular, it follows that the set of nowhere dense sets is precisely the set of boundaries of open, dense sets and, hence, is also the set of boundaries of open sets. We now answer the question of when $\kappa_0 X$ and $\beta_0 X$ are equivalent extensions of X. For $\beta_0 X$ to exist, our initial hypothesis must be that X is zero-dimensional.

THEOREM 2.7. *Suppose X is zero-dimensional. Then $\kappa_0 X = \beta_0 X$ if and only if X is compact.*

Proof. If X is compact, then $\kappa_0 X = X = \beta_0 X$. Conversely, suppose $\kappa_0 X = \beta_0 X$. Since X is open in $\kappa_0 X$ and, hence, in $\beta_0 X$, then X is locally compact and $\beta_0 X - X$ is compact. But $\beta_0 X - X$ is homeomorphic to $\kappa_0 X - X$ which is discrete. So, $\beta_0 X - X$ is finite. Let A be a nowhere dense subset of X and $U = X - A$. Since $\text{cl}_X U = X$, then $(\beta_0 X - X) \cup U$ is open in $\kappa_0 X$ and, hence, in

$\beta_0 X$. But $A = \beta_0 X - ((\beta_0 X - X) \cup U)$ is compact. By Lemmas 5 and 6 in [7], X is the topological sum of a compact subspace X_1, with a discrete subspace X_2, i.e., $X = X_1 + X_2$. By 2.3, $\kappa_0 X = \kappa_0 X_1 + \kappa_0 X_2 = X_1 + \kappa_0 X_2$ and, similarly, $\beta_0 X = \beta_0 X_1 + \beta_0 X_2 = X_1 + \beta_0 X_2$. Thus, $\kappa_0 X_2 = \beta_0 X_2$ (as extensions of X_2), and since X_2 is discrete, $\beta_0 X_2 = \beta X_2$. Since $\beta_0 X_2 - X_2 = \beta_0 X - X$ is finite and $\beta X_2 - X_2$ is infinite whenever X_2 is infinite, then it follows that X_2 is finite. Thus, $X = X_1 + X_2$ is compact.

EXAMPLES 2.8. (a) There are separable metrizable ultra-Hausdorff spaces that are not zero-dimensional—for example, the set of rational points in real separable Hilbert space (see, for example, Problem 16L of [4]). Thus a completely regular ultra-Hausdorff space need not have an ultra-Hausdorff H-closed extension.

(b) Let X be zero-dimensional and suppose $\kappa X = \kappa_0 X$. Then $\beta X = \beta_0 X$; for suppose Z_1 and Z_2 are disjoint zero-sets of X. Then there is a real-valued continuous function f on X such that $f[Z_1] = \{0\}$ and $f[Z_2] = \{1\}$ (see 1.15 of [4]). Let $V = \mathrm{int}_X \mathrm{cl}_X f^{\leftarrow}(-1/2, 1/2)$. Then $\mathrm{bd}_X V$ is compact by 2.6. For each $p \in \mathrm{bd}_X V$ find a clopen set $B(p)$ of X such that $p \in B(p) \subseteqq X - Z_2$ (X is zero-dimensional). As $\mathrm{bd}_X V$ is compact there exist $p_1, \cdots, p_n \in \mathrm{bd}_X V$ such that $\mathrm{bd}_X V \subseteqq \bigcup_{i=1}^n B(p_i) = B$. Then $B \cup V$ is a clopen set of X separating Z_1 and Z_2, so $\beta X = \beta_0 X$ (see § 1).

However, there are lots of zero-dimensional spaces X for which $\beta X = \beta_0 X$ but $\kappa X \neq \kappa_0 X$. In 16.17 of [4] it is proved that $\beta X = \beta_0 X$ whenever X is a zero-dimensional Lindelof space. Let Z denote the integers, regarded as a subspace of the space Q of rational numbers. Obviously Z is not compact and $Z = \mathrm{bd}_Q[\cup \{(2n, 2n+1) \cap Q : n \in Z\}]$. Thus $\beta Q = \beta_0 Q$ but by 2.6 $\kappa Q \neq \kappa_0 Q$. Many other such examples can be found.

(c) Recall that a space is *extremally disconnected* if open sets have open closures (see 1H of [4]). Thus regular open subspaces of extremally disconnected spaces are clopen, and therefore have empty boundaries. Hence by 2.6 if X is extremally disconnected then $\kappa X = \kappa_0 X$. There is a large variety of extremally disconnected spaces; see, for example, [4] and [10].

(d) There are lots of nonsemi-regular spaces X for which X_s is zero-dimensional. For example, suppose $\underline{N} \subseteqq T \subseteqq \kappa \underline{N}$, where $\underline{N}$ denotes the countable discrete space. By Theorem E of [5], $\kappa T = \kappa \underline{N}$, so T has an ultra-Hausdorff H-closed extension (namely $\kappa \underline{N}$). Thus T_s is zero-dimensional. Note that *as sets*, $\kappa \underline{N} = \beta \underline{N}$; it is easy to verify that T is semi-regular iff $T \backslash \underline{N}$ is a discrete subspace of $\beta \underline{N}$. Thus there are lots of such spaces T for which T is not semi-regular and T_s is zero-dimensional.

Let X and Y be two spaces. In [5] Harris characterizes those maps from X to Y that can be extended continuously to maps from κX to κY. A *p-cover* of a space Y is an open cover of Y which has a finite subfamily whose union is dense in Y. Harris proves that the map $f: X \to Y$ extends continuously to $f^{\kappa}: \kappa X \to \kappa Y$ iff whenever $\mathscr{C}$ is a p-cover of Y then $\{f^{\leftarrow}[C]: C \in \mathscr{C}\}$ is a p-cover of X. We now characterize those maps from X to Y that can be extended continuously to maps from $\kappa_0 X$ to $\kappa_0 Y$ (in the case where X_s and Y_s are zerodimensional). Our characterization is similar to that of Harris, with "po-covers" (defined below) playing the role that p-covers play in Harris's result.

DEFINITION 2.9. A *po-cover* of a space X is an open cover $\mathscr{C}$ of X with the following property: there is a finite subcollection $\{C_i: i = 1 \text{ to } n\}$ of $\mathscr{C}$, and a finite clopen cover $\{B_i: i = 1 \text{ to } n\}$ of X, such that $B_i \subseteq \mathrm{cl}_X C_i$, $i = 1$ to n.

THEOREM 2.10. *Let X and Y be spaces such that X_s and Y_s are zero-dimensional. Let $f: X \to Y$ be a continuous function. The following are equivalent:*

(a) *f can be extended continuously to $f^0: \kappa_0 X \to \kappa_0 Y$.*

(b) *If $\mathscr{C}$ is a po-cover of Y then $\{f^{\leftarrow}[C]: C \in \mathscr{C}\} = f^{\leftarrow}[\mathscr{C}]$ is a po-cover of X. (We will call such an f a "po-map").*

Proof. (a) $\Rightarrow$ (b): Let $\mathscr{V}$ be an open cover of Y and suppose that $f^{\leftarrow}[\mathscr{V}]$ is not a po-cover of X. We will show that $\mathscr{V}$ is not a po-cover of Y.

Let $\mathscr{F} = \{B \in \mathscr{B}(X)$: there is a $V \in \mathscr{V}$ such that $X - B \subseteq \mathrm{cl}_X f^{\leftarrow}[V]\}$. We verify that $\mathscr{F}$ is a free clopen filter base on X. Let $\{B_1, \cdots, B_n\}$ be a finite subcollection of $\mathscr{F}$. Find $V_i \in \mathscr{V}$ such that $X - B_i \subseteq \mathrm{cl}_X f^{\leftarrow}[V_i]$ $(i = 1$ to $n)$. If $\bigcap_{i=1}^n B_i = \varnothing$ then $\bigcup_{i=1}^n X - B_i = X$, contradicting the assumption that $f^{\leftarrow}[\mathscr{V}]$ is not a po-cover. Thus $\mathscr{F}$ is a clopen filter base. Now suppose $x \in X$; as $\mathscr{V}$ covers Y there exists $V \in \mathscr{V}$ such that $x \in f^{\leftarrow}[V]$. As X_s is zero-dimensional, find $A_x \in \mathscr{B}(X)$ such that $x \in A_x \subseteq \mathrm{cl}_X f^{\leftarrow}[V]$. Thus $X - A_x \in \mathscr{F}$, and so $\mathscr{F}$ is free. Thus $\mathscr{F}$ is contained in a free clopen ultrafilter $\mathscr{U}$, which is a point of $\kappa_0 X - X$.

Suppose $f^0(\mathscr{U}) \in Y$. Then there exists $V \in \mathscr{V}$ such that $f^0(\mathscr{U}) \in V$. As f^0 is continuous it follows easily that $\{\mathscr{U}\} \cup f^{\leftarrow}[V]$ is a $\kappa_0 X$-neighborhood of $\mathscr{U}$. Hence there exists $B \in \mathscr{U}$ such that $B \subseteq \mathrm{cl}_X f^{\leftarrow}[V]$. Thus $X - B \in \mathscr{F}$, which is a contradiction. Thus $f^0(\mathscr{U}) \in \kappa_0 Y - Y$, so $f^0(\mathscr{U})$ is a free clopen ultrafilter $\mathscr{U}'$ on Y.

Now suppose $\mathscr{V}$ were a po-cover of Y. Then there would exist $V_1, \cdots, V_k \in \mathscr{V}$ and $A_1, \cdots, A_k \in \mathscr{B}(Y)$ such that $\bigcup_{i=1}^k A_i = Y$ and

$A_i \subset \mathrm{cl}_Y V_i$ for $i = 1$ to k. As $\mathscr{U}'$ is prime, one of $A_1, \cdots, A_k$—say A_m—belongs to $\mathscr{U}'$. Then $\{\mathscr{U}'\} \cup V_m$ is a $\kappa_0 Y$-neighborhood of $\mathscr{U}'$. As f^0 is continuous, it follows that $\{\mathscr{U}\} \cup f^{\leftarrow}[V_m]$ is a $\kappa_0 X$-neighborhood of $\mathscr{U}$, and we obtain the same contradiction as above. Hence $\mathscr{V}$ cannot be a *po*-cover of Y.

(b) $\Rightarrow$ (a): If $\mathscr{U} \in \kappa_0 X - X$, define $\mathscr{U}'$ to be $\{A \in \mathscr{B}(Y) : f^{\leftarrow}[A] \in \mathscr{U}\}$. It is easy to show that $\mathscr{U}'$ is a clopen ultrafilter on Y. If $\cap \mathscr{U}' \neq \varnothing$, then as Y_s is zero-dimensional there is a unique point $y(\mathscr{U})$ in $\cap \mathscr{U}'$. Define $f^0(\mathscr{U})$ to be $y(\mathscr{U})$. If $\mathscr{U}'$ is free, define $f^0(\mathscr{U})$ to be $\mathscr{U}'$, a point of $\kappa_0 Y - Y$. Let $f^0 | X = f$; we show that if (b) holds, the function $f^0 \colon \kappa_0 X \to \kappa_0 Y$ so defined is continuous.

Suppose that $f^0(\mathscr{U}) = y(\mathscr{U}) \in Y$. Let V be open in Y with $y(\mathscr{U}) \in V$. As Y_s is zero-dimensional, there exists $A \in \mathscr{B}(Y)$ such that $y(\mathscr{U}) \in A \subseteq \mathrm{cl}_Y V$. For each $y \in A - V$, choose $C(y) \in \mathscr{B}(Y)$ such that $y \in C(y)$ and $y(\mathscr{U}) \notin C(y)$. Then $\{C(y) : y \in A - V\} \cup \{V, Y - A\}$ is a *po*-cover $\mathscr{C}$ of Y. Thus by hypothesis $f^{\leftarrow}[\mathscr{C}]$ is a *po*-cover of X. Therefore there is a finite subcollection $\mathscr{G}$ of $f^{\leftarrow}[\mathscr{C}]$ whose closures are refined by members of a finite clopen cover of X. But $C(y)$ and $Y - A$ are not in $\mathscr{U}'$, so $f^{\leftarrow}[C(y)]$ and $f^{\leftarrow}[Y - A]$ are not in $\mathscr{U}$. Hence if $f^{\leftarrow}[V] \notin \mathscr{G}$, then $\mathscr{G}$ is a finite clopen cover of X no member of which belongs to the clopen ultrafilter $\mathscr{U}$. This is impossible, so $f^{\leftarrow}[V] \in \mathscr{G}$. Hence there exists $B \in \mathscr{B}(X)$ such that $B \subseteq \mathrm{cl}_X f^{\leftarrow}[V]$ and $B \cup [\cup \{G \in \mathscr{G} : G \neq f^{\leftarrow}[V]\}] = X$. As no member of $\mathscr{G} - \{f^{\leftarrow}[V]\}$ is in $\mathscr{U}$, $B \in \mathscr{U}$. Thus $\{\mathscr{U}\} \cup f^{\leftarrow}[V]$ is a $\kappa_0 X$-neighborhood of $\mathscr{U}$ contained in $(f^0)^{\leftarrow}[V]$, so f^0 is continuous at $\mathscr{U}$.

Suppose that $f^0(\mathscr{U}) = \mathscr{U}' \in \kappa_0 Y - Y$. Let V be open in Y, and suppose that $A \in \mathscr{U}'$ and $A \subseteq \mathrm{cl}_y V$. Then $\{\mathscr{U}'\} \cup V$ is a basic $\kappa_0 Y$-neighborhood of $\mathscr{U}$. Since $\mathscr{U}'$ is free, for each $y \in A - V$ we can choose $C(y) \in \mathscr{B}(Y)$ such that $y \in C(y)$ and $C(y) \notin \mathscr{U}'$. Then $\{C(y) : y \in A - V\} \cup \{V, Y - A\}$ is a *po*-cover of Y, and we can repeat the above argument to show that $\{\mathscr{U}\} \cup f^{\leftarrow}[V]$ is a $\kappa_0 X$-neighborhood of $\mathscr{U}$ contained in $(f^0)^{\leftarrow}[\{\mathscr{U}'\} \cup V]$. Thus again f^0 is continuous at $\mathscr{U}$.

We next give some examples of continuous functions that are not *p*-maps nor *po*-maps. First we need a general result.

PROPOSITION 2.11. *Let X be a space, let $D(X)$ be the discrete space of the some cardinality as X, and let $j \colon D(X) \to X$ be a bijection. The following are equivalent*:

(a) *j extends continuously to $j^\kappa \colon \kappa D(X) \to \kappa X$.*

(b) *Each closed nowhere dense subset of X is compact.*

If X_s is zero-dimensional each of the above conditions is equivalent to:

(c) *j extends continuously to $j^0 \colon \kappa D(X) \to \kappa_0 X$.*

Proof (a) $\Rightarrow$ (b): Suppose (b) fails. Let S be a noncompact closed nowhere dense subset of X. Let $\mathscr{C}$ be a cover of S, by open subsets of X, with no finite subcover. Let $\mathscr{C}' = \{X-S\} \cup \mathscr{C}$; then $\mathscr{C}'$ is a p-cover of X. But $j^{\leftarrow}[\mathscr{C}'](=\{j^{\leftarrow}[V]: V \in \mathscr{C}'\})$ is not a p-cover of $D(X)$ for if it were then it would have a finite subcover as $D(X)$ is discrete. This would imply that $\mathscr{C}$ has a finite subfamily covering S, contrary to hypothesis. Thus j is not a p-map and so by Theorem A of [5] (see the remarks preceding 2.9), (a) fails.

(b) $\Rightarrow$ (a): Let $\mathscr{C}$ be a p-cover of X; then there exist $C_1, \cdots, C_n \in \mathscr{C}$ such that $\bigcup_{i=1}^n C_i$ is dense in X. By hypothesis $X - \bigcup_{i=1}^n C_i$ is compact, and it follows that $\mathscr{C}$ has a finite subcover. Thus $j^{\leftarrow}[\mathscr{C}]$ has a finite subcover, so j is a p-map and so (a) holds by Theorem A of [5].

Now suppose X_s is zero-dimensional. Replace "p-cover" and "p-map" by "po-cover" and "po-map" in the above argument, and use 2.10 instead of Harris's theorem, and one obtains a proof of the equivalence of (a) and (c).

EXAMPLES 2.12. (a) Let Q denote the space of rationals. No bijection $j: D(Q) \to Q$ can be continuously extended to $j^{\kappa}: \kappa D(Q) \to \kappa Q$ (or to $j^0: \kappa D(Q) \to \kappa_0 D(Q)$) since Q contains noncompact closed nowhere dense subsets (e.g., the set of integers).

(b) Let D be an infinite discrete space. Then $\kappa D - D$ is a closed, nowhere dense, noncompact subset of κD, so a bijection $j: D(\kappa D) \to \kappa D$ cannot be continuously extended to $j^{\kappa}: \kappa D(\kappa D) \to \kappa D$.

(c) If X is compact then a bijection $j: D(X) \to X$ can be continuously extended to $j^{\kappa}: \kappa D(X) \to X$.

REMARKS 2.13. (a) If X_s is zero-dimensional and Condition 2.11(b) holds for X, then by 2.6 $\kappa X = \kappa_0 X$. Thus there cannot exist a situation in which X_s is zero-dimensional, $\kappa X \neq \kappa_0 X$, and a bijection $j: D(X) \to X$ extends to $j^0: \kappa D(X) \to \kappa_0 X$ but not to j^{κ}: $\kappa D(X) \to \kappa X$.

(b) There are examples of spaces S and T, and a continuous bijection $f: S \to T$, for which T but not S satisfies 2.11(b). In such a situation a bijection $j: D(T) \to T$ can be extended to $j^{\kappa}: \kappa D(T) \to \kappa T$ but j^{κ} cannot be "factored through" κS, even though j factors through S. First note that there is a compactification $\gamma \underline{N}$ of the countably infinite discrete space N such that $\gamma \underline{N} - \underline{N}$ is homeomorphic to the one-point compactification of $\underline{N}$. Thus there is a continuous function $g: \beta N \to \gamma N$ such that $g[\beta N - N] = \gamma N - N$ (see

Chapter 6 of [4]). Then $\cup\{g^{\leftarrow}(x): x$ is isolated in $\gamma N - N\}$ is a proper cozero-set C of $\beta N - N$ and therefore not dense in $\beta N - N$ (see 6S of [4]). Thus there is an infinite discrete subset S of $\beta N - N$ such that $\mathrm{cl}_{\beta N} S \cap \mathrm{cl}_{\beta N} C = \varnothing$. Choose $p_x \in g^{\leftarrow}(x)$ for each isolated point x of $\gamma \underline{N} - \underline{N}$, and let $H = \underline{N} \cup S \cup \{p_x: x$ is isolated in $\gamma N - N\}$. Then $g|H: H \to \gamma N$ is a bijection, each closed nowhere dense subset of γN is compact, but S is a noncompact closed nowhere dense subset of H.

Next we exhibit an important class of continuous functions that are p-maps (and, in certain cases, po-maps). Recall that a continuous closed surjection $f: X \to Y$ is called *irreducible* if, whenever A is a proper closed subset of X, then $f[A]$ is a proper closed subset of Y.

PROPOSITION 2.14. *Each continuous irreducible closed surjection is a p-map. If the domain is extremally disconnected and the range admits an ultra-Hausdorff H-closed extension, then it is a po-map.*

Proof. Let f be a continuous irreducible closed surjection. It is well-known, and straightforward to verify, that if S is a dense subset of the range then $f^{\leftarrow}[S]$ is a dense subset of the domain. This immediately implies that f is a p-map. If the domain X is extremally disconnected then by 2.8(c) $\kappa X = \kappa_0 X$ so f extends continuously to $f^0: \kappa_0 X \to \kappa Y$. As κY is a projective maximum for $\mathscr{H}(Y)$, there is a map $g: \kappa Y \to \kappa_0 Y$ such that $g|Y = 1_Y$. Thus $g \circ f^0$ extends f to a map from $\kappa_0 X$ to $\kappa_0 Y$, so f is a po-map.

EXAMPLE 2.15. If X is a regular space then there is an extremally disconnected space $E(X)$, called the *absolute* or *projective cover* of X, and a perfect irreducible surjection $k_X: E(X) \to X$; $E(X)$ is unique up to homeomorphism (see [10] or Chapter 10 of [11]). By 2.14 k_X is a p-map (and also a po-map if X is zero-dimensional). Thus k_X extends continuously to $k_X^{\kappa}: \kappa E(X) \to \kappa X$.

REMARK 2.16. We do not know if, whenever $f: X \to Y$ is a closed irreducible surjection with X_s and Y_s zero-dimensional, f must be a po-map.

REMARK 2.17. We conclude this paper by noting that it is possible to define our ultra-Hausdorff H-closed extension $\kappa_0 X$ by another method. Flachsmeyer [3] has developed a method of generating many of the H-closed extensions of a given Hausdorff space; in fact, he develops a Katětov-like, H-closed extension $\kappa_{\mathscr{B}} X$ for each

π-basis $\mathscr{B}$ on a Hausdorff space X ($\mathscr{B}$ is π-basis means $\mathscr{B}$ is open basis for X that is closed under finite intersections and for each $V \in \mathscr{B}$, $X - \mathrm{cl}_X V \in \mathscr{B}$). If X is zero-dimensional and $\mathscr{B} = \{V: V$ open, $\mathrm{cl}_X V$ clopen$\}$, then $\mathscr{B}$ is π-basis for X and it is straightforward to show that $\kappa_0 X$ and $\kappa_{\mathscr{B}} X$ are equivalent extensions of X.

Added in proof. The authors and E. von Douwen have independently shown that the answer to 2.16 is "yes".

REFERENCES

1. P. Bankston, *Ultraproducts in topology*, General Topology and Appl., **7** (1977), 283–308.
2. N. Bourbaki, *General Topology Part I*, Addison-Wesley, Reading, Mass., 1966.
3. J. Flachsmeyer, *On the theory of H-closed extensions*, Math. Zeit., **94** (1966), 349–381.
4. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton. N. J., 1960.
5. D. Harris, *Katětov extension as a functor*, Math. Ann., **193** (1971), 171–175.
6. M. Katětov, *Über H-abgeschlossene und bikompakte Räume*, Časopis Pěst. Mat Fys., **69** (1940), 36–49.
7. ———, *On the equivalence of certain types of extensions of topological spaces*, ibid., **72** (1947), 101–106.
8. J. Porter and J. D. Thomas, *On H-closed and minimal Hausdorff spaces*, Trans. Amer. Math. Soc., (1969), 157–170.
9. J. Porter and C. Votaw, *H-closed extensions I*, General Topology and Appl., **3** (1973), 211–224.
10. D. P. Strauss, *Extremally disconnected spaces*, Proc. Amer. Math. Soc., **18** (1967), 305–309.
11. R. C. Walker, *The Stone-Čech Compactification*, Springer, New York, 1974.
12. S. Willard, *General Topology*, Addison-Wesley, Reading, Mass., 1970.

Received October 16, 1978. The research of the first-named author was partially supported by the University of Kansas General Research Fund. The research of the second-named author was partially supported by a grant (No. A7592) from the National Research Council of Canada.

THE UNIVERSITY OF KANSAS
LAWRENCE, KS
AND
THE UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA, CANADA