PACIFIC JOURNAL OF MATHEMATICS
Vol. 85, No. 2, 1979

HOMOTOPY THEORY OF RIGID PROFINITE SPACES I.

D. L. RECTOR

This paper is the first part of a study of profinite completion and profinite spaces from a functorial point of view. The study aims at understanding mapping spaces, classifying spaces, and other notions involving higher homotopies in the setting of profinite spaces.

The term *rigid profinite space* will apply to a left filtered (see below) actually commuting diagram of spaces with finite homotopy groups, as distinguished from an Artin-Mazur profinite space which is a homotopy commuting diagram. One advantage in working with actually commuting diagrams is that a *functorial homotopy limit* exists relating diagrams of spaces to spaces. A homotopy limit does not exist in general for diagrams in the homotopy category. The actual relationship between rigid profinite spaces and profinite spaces is not well understood. Theorem 3.4 and Corollary 6.9 below suggest that the relationship is very close. Rigid profinite spaces arise naturally in many contexts; in particular, the étale homotopy type of a nice variety may be made rigid [8], [10].

The main results of this paper are as follows. In § 2 we construct a *functorial profinite completion* $\hat{X}$ for a connected space X taking values in the category of rigid profinite spaces. We employ a construction pioneered by Quillen [13] making use of the profinite completion of a simplicial group. We show that this profinite completion is weakly equivalent to that of Artin-Mazur. A functorial nilpotent p-completion has been studied extensively by Bousfield and Kan [5]; it gives equivalent results on spaces X which are nilpotent and *locally of finite type* — i.e., $H^*(X; M)$ of finite type for all finite local coefficient systems M. For spaces which are not nilpotent, the Bousfield-Kan completion gives very different results.

In § 3 we construct a *functorial discretization* dY for a rigid profinite space Y. We show that *for X a connected space, $d\hat{X}$ represents the Sullivan finite completion of X* [17] in a strong sense — i.e., for all connected spaces Z, the mapping space hom$(Z, d\hat{X})$ has the right higher homotopy groups (see Theorem 3.4). In part II of this paper (in preparation), we will show that when X is locally of finite type, $\hat{X}$ and dY are *homotopy adjoint* in a strong sense — i.e., hom (X, dY) is weakly equivalent to hom $(\hat{X}, Y)$ for a natural definition of hom $(\hat{X}, Y)$.

In § 4 we state a partial generalization of a fibration theorem of Artin-Mazur. We will show that if

$$F \longrightarrow E \longrightarrow B$$

is a nilpotent fibration of connected spaces such that F or B is locally of finite type, then

$$\hat{F} \longrightarrow \hat{E} \longrightarrow \hat{B}$$

is a fibration up to weak equivalence. Artin-Mazur instead require that B be simply connected [2; Thm. 5.9].

In § 5 we begin a study of the homotopy groups of completions. We define the *left derived functors of profinite completion* by a construction reminiscent of Dold-Puppe [7]. We then apply these derived functors to the classical case of a connected space which is *virtually nilpotent and locally of finite type. For X such a space, there are short exact sequences*

$$0 \longrightarrow (\pi_n X)^{\wedge} \longrightarrow \pi_n \hat{X} \longrightarrow \hat{L}_1(\pi_{n-1}X) \longrightarrow 0 \ ,$$

where $\hat{L}_1$ *denotes the first left derived functor of profinite completion.* The calculations for nilpotent spaces are essentially the same as those of Bousfield-Kan [5; Ch. VI]. The proofs given here are new and from a different viewpoint. In Remark 5.6 we indicate a critical idea for studying the homotopy groups of completions of nonnilpotent spaces. Applications will appear in a future paper.

In § 6 we consider the question: when is a rigid profinite space X *functorially realizable as a completion*; i.e., *when is* $(dX)^{\wedge}$ *weakly equivalent to* X? If $(dX)^{\wedge}$ is weakly equivalent to X we call X *intrinsic.* For X *virtually nilpotent,* we show that X *is intrinsic iff* X *is locally of finite type.* In part II of this paper we will show that *if X and Y are connected rigid profinite spaces and X is intrinsic and locally of finite type then* hom (dX, dY) *is weakly equivalent to* hom (X, Y).

2. Rigid profinite spaces and completions. Let $\mathscr{C}$ be a category. A *pro-object* in $\mathscr{C}$ [2] is a functor $X: I \to \mathscr{C}$, where I is a *left filtered* index category, i.e., a small category satisfying

(i) $i, j \in I \Rightarrow \exists k \begin{smallmatrix} \nearrow i \\ \searrow j \end{smallmatrix}$ in I.

(ii) if $j \rightrightarrows i \in I$, $\exists k \to i \in I$ such that the compositions $k \to i \rightrightarrows j$ are equal.

We will use the notations

$$X = \{X_i\}_{i \in I}$$

for $X: I \to \mathscr{C}$, and $X_{i \to j}$ for $X(i \to j): X_i \to X_j$. If $X = \{X_i\}_{i \in I}$ and

$Y = \{Y_j\}_{j \in J}$ then one defines the set of *pro-maps* of X to Y to be

$$\operatorname{Hom}(X, Y) = \varprojlim_{j \in J} \varinjlim_{i \in I} \operatorname{Hom}(X_i, Y_j) .$$

We will denote the category of pro-objects in $\mathscr{C}$ by Pro-$\mathscr{C}$. Pro-$\mathscr{C}$ always has small filtered limits. If $\mathscr{C}$ has finite limits, then Pro-$\mathscr{C}$ has arbitrary small colimits [2; Prop. A4.3].

Let $\mathfrak{G}$ be the category of groups, $\mathfrak{G}_{\mathrm{fin}}$ the subcategory of finite groups, and, for l a set of primes, $\mathfrak{G}_l$ the subcategory of (finite) l-groups — i.e., groups whose order is divisible only by primes in l. Throughout this paper we will denote the identity of a multiplicative group by $*$ and that of an additive group by 0. The category of *profinite groups*, $\hat{\mathfrak{G}} = \text{Pro-}\mathfrak{G}_{\mathrm{fin}}$, is equivalent to the category of compact totally disconnected topological groups and continuous homomorphisms. Denote by $\hat{\mathfrak{G}}_l$ the full subcategory of *pro-l groups*, $\hat{\mathfrak{G}}_l = \text{Pro-}\mathfrak{G}_l$. If $l = \{\text{all primes}\}$, $\hat{\mathfrak{G}}_l = \hat{\mathfrak{G}}$.

If $G \in \mathfrak{G}$, its *l-completion* is constructed as follows. Let I be the category whose objects are the quotients $G \xrightarrow{i} G_i$ such that $G_i \in \mathfrak{G}_l$. A map $i \to j$ is a commuting diagram

$$\begin{array}{ccc} & & G_i \\ & \nearrow & \downarrow \\ G & \searrow & \\ & & G_j \end{array} .$$

Then $\hat{G}_l = \{G_i\}_{i \in I}$. If $l = \{\text{all primes}\}$ $\hat{G}_l$ is the *finite completion* of G and is denoted by $\hat{G}$. The functor $(\ )\hat{}_l$ is left adjoint to the forgetful functor $d: \hat{\mathfrak{G}}_l \to \mathfrak{G}$ obtained by forgetting the topology on $\varprojlim G_i$, where $G = \{G_i\} \in \mathfrak{G}_l$. We call dG the *discretization* of G.

For $\mathscr{C}$ a category, $\Delta\mathscr{C}$ will denote the simplicial objects of $\mathscr{C}$. If $T: \mathscr{C} \to \mathscr{D}$ is a functor, T induces a functor $T: \Delta\mathscr{C} \to \Delta\mathscr{D}$ such that $TX[n] = T(X[n])$, where for $X \in \Delta\mathscr{C}$, $X[n]$ denotes the "n-simplices" of X. Also, for $X, Y \in \Delta\mathscr{C}$, T induces a natural map of function complexes

$$T: \operatorname{hom}(X, Y) \longrightarrow \operatorname{hom}(TX, TY) .$$

In particular T preserves homotopies. Thus the adjoint pair of functors $(\ )\hat{}_l$ and d induce an adjoint pair of functors

$$\Delta\mathfrak{G} \underset{d}{\overset{(\ )\hat{}_l}{\rightleftarrows}} \Delta\hat{\mathfrak{G}}_l$$

preserving group homotopies.

We will make use of the categories of *spaces* (simplicial sets)

$\mathscr{S} = \Delta$ (sets), *simplicial groups* $\Delta\mathfrak{G}$, *simplicial profinite groups* $\Delta\hat{\mathfrak{G}}$, and *simplicial pro-l groups* $\Delta\hat{\mathfrak{G}}_l$. We will also need the category of *reduced spaces* $\mathscr{S}_0$ whose objects are simplicial sets with a single vertex $*$ and the *homotopy category of reduced spaces* $\mathscr{H}_0$ with the same objects but with

$$\mathrm{Hom}\,(X, Y) = [|X|, |Y|] .$$

We will call an object of Pro-$\mathscr{S}$ a *rigid prospace* to distinguish it from an *Artin-Mazur prospace* which is an object of Pro-$\mathscr{H}$, $\mathscr{H}$ the homotopy category of spaces.

DEFINITION 2.1. A *rigid profinite space* is an object $X = \{X_i\}_{i \in I}$ of Pro-$\mathscr{S}$ such that each X_i has finite homotopy sets. Let $\hat{\mathscr{S}} \subseteqq$ Pro-$\mathscr{S}$ be the full subcategory of such objects. Similarly, X is a *rigid pro-l space* if in addition each $\pi_n X_i$, $n \geqq 1$, is a finite l-group. These objects form a category $\hat{\mathscr{S}}_l$. We will call a rigid prospace *reduced* if each $X_i \in \mathscr{S}_0$, and we denote the categories of reduced rigid profinite spaces and rigid pro-l spaces by $\hat{\mathscr{S}}_0$ and $\hat{\mathscr{S}}_{0l}$, respectively. We will usually be concerned only with reduced spaces.

Our immediate object is to extend the l-completion functor $\mathfrak{G} \to \hat{\mathfrak{G}}_l$ to a functor $\mathscr{S}_0 \to \hat{\mathscr{S}}_{0l}$. We will use a construction pioneered by Quillen [13]. In addition we will give in this section a review of some useful properties of the cohomology of simplicial profinite groups.

Consider a simplicial profinite group G. If U is an open normal subgroup of G, then G/U is a simplicial finite group. Indeed, G may be identified with $\varprojlim G/U$, U running over the open normal simplicial subgroups of G, and $\Delta\hat{\mathfrak{G}}$ is equivalent to Pro-$\Delta\mathfrak{G}_{\mathrm{fin}}$ [13; Lemma 2.3]. Denoting by $\bar{W}$ the simplicial classifying space functor, we see that each $\bar{W}(G/U)$ is a reduced space with finite homotopy groups. Thus we may associate with G a rigid profinite space

$$\bar{W}G = \{\bar{W}(G/U)\} .$$

$\bar{W}$ is then a functor from $\Delta\hat{\mathfrak{G}}$ to $\hat{\mathscr{S}}_0$.

For $X \in \mathscr{S}_0$, let GX be its free group loop space [11; Ch. VI]. Let $\hat{G}X = (GX)^\wedge$ and $\hat{G}_l X = (GX)^\wedge_l$. Since G is left adjoint to $\bar{W}$, there is a natural map of rigid prospaces

$$X \longrightarrow \bar{W}\hat{G}_l X$$

where X is considered to be the constant prospace.

DEFINITION 2.2. For $X \in \mathscr{S}_0$, the *rigid finite completion* of X

is $\hat{X} = \bar{W}\hat{G}X$; the *rigid l-completion* is $\hat{X}_l = \bar{W}\hat{G}_lX$.

We will justify these definitions below.

These definitions may be extended to rigid prospaces. If $X \in$ Pro-$\mathscr{S}_0$, $X = \{X_i\}_{i \in I}$, then let

$$\hat{G}_lX = \lim_{\substack{\longleftarrow \\ i \in I}} \hat{G}_lX_i$$

and $\hat{X}_l = \bar{W}\hat{G}_lX$. We write $\hat{X} = \hat{X}_l$ and $\hat{G} = \hat{G}_lX$ when $l = \{\text{all primes}\}$. Clearly, there is a natural map $X \to \hat{X}$. In fact, a straightforward calculation demonstrates that

PROPOSITION 2.3. *$\hat{G}_l$: Pro-$\mathscr{S}_0 \to \Delta\hat{\mathfrak{G}}_l$ is left adjoint to $\bar{W}: \hat{\mathfrak{G}}_l \to$ Pro-$\mathscr{S}_0$.*

For $X \in$ Pro-$\mathscr{S}$, $X = \{X_i\}_{i \in I}$, the homotopy progroups of X are

$$\pi_nX = \{\pi_nX_i\} .$$

If $X \in \hat{\mathscr{S}}_l$, then $\pi_nX \in \mathfrak{G}_l$, $n \geqq 1$. If $X, Y \in \hat{\mathscr{S}}_0$ and $f: X \to Y$, we say that f is a *weak equivalence* if

$$f_*: \pi_nX \longrightarrow \pi_nY$$

is an isomorphism for each $n \geqq 1$. For $X \in$ Pro-$\mathscr{S}_0$, M a continuous π_1X-module, the *continuous cohomology of X with coefficients in M is*

$$H^*(X; M) = \lim_{\substack{\longrightarrow \\ i \in I}} H^*(X_i; M^U) ,$$

where $U = \ker(\pi_1X \to \pi_1X_i)$. We will say that the coefficient system M is an *l-coefficient system* if M is a finite abelian l-group and the π_1X action $\pi_1X \to \mathrm{Aut}(M)$ factors through an l-subgroup of $\mathrm{Aut}(M)$.

PROPOSITION 2.4. (*Artin-Mazur* [2; Thm. 4.3]). *Let $X, Y \in \mathscr{S}_{0l}$, $f: X \to Y$. The following are equivalent.*

(i) *f is a weak equivalence.*

(ii) *$\pi_1X \xrightarrow{\sim} \pi_1Y$, and for every l-coefficient system M,*

$$H^*(Y; M) \xrightarrow{\sim} H^*(X; M) .$$

(iii) *f is a ♮-isomorphism* [2; § 4].

Let G be a simplicial profinite group. Then we may define homotopy groups $\pi_n G$ for G in the usual way [11; § 17] as the homology of the non-abelian chain complex with

$$\tilde{G}_n = \bigcap_{i=1}^{n} (\ker d_i : G_n \longrightarrow G_{n-1})$$

and with boundary $d_0 : \tilde{G}_n \to \tilde{G}_{n-1}$. $\pi_n G$ will thus be a profinite group. Since filtered inverse limits of finite groups are exact,

$$\pi_n G = \varprojlim \pi_n(G/U) \ ,$$

U running over the open normal subgroups of G. Furthermore, $\pi_n G \approx \pi_{n+1} \bar{W} G$. If M is a continuous $\pi_0 G$ module, the *continuous cohomology of G with coefficients in M* is given by

$$H^*(G; M) = \varinjlim H^*(\bar{W}(G/U); M^{\pi_0 U}) \ ,$$

where the limit is taken over all open normal subgroups of G. This coincides with the usual notion of cohomology of a profinite group in case G is a constant simplicial group. For both profinite and discrete simplicial groups we have the following properties of cohomology.

PROPOSITION 2.5. (*Quillen* [13; 2.1, 2.2]). *Let G, H, and R be simplicial (profinite) groups, and let M be a (continuous) $\pi_0 G$-module.*

(a) *If $f: H \to G$ is a weak equivalence, then*

$$H^*(f; M): H^*(G; M) \xrightarrow{\sim} H^*(H; M) \ .$$

(b) *If f and g are homotopic then $H^*(f; M) = H^*(g; M)$.*

(c) *There is a canonical spectral sequence*

$$E_2^{p,q} = \pi^p \mathscr{H}^q(G; M) \Longrightarrow H^{p+q}(G; M)$$

where $\mathscr{H}^q(G; M)$ is the cosimplicial abelian group whose n-simplices are $H^q(G_n; M)$.

(d) *There is a canonical isomorphism*

$$H^0(G; M) = M^{\pi_0 G} \ ,$$

and $H^1(G; M)$ is the (continuous) crossed homomorphisms from $\pi_0 G$ to M modulo principal crossed homomorphisms.

(e) *If $* \to R \to G \to H \to *$ is exact then there is a Serre spectral sequence*

$$E_2^{p,q} = H^p(H; H^q(R; M)) \Longrightarrow H^{p+q}(G; M) \ .$$

(f) *$H^*(G; M)$ is a cohomological functor of M.*

Recall [15] that a group G is *l-good*, (*good* when $l = \{\text{all primes}\}$) if

$$H^*(\hat{G}_l; M) \xrightarrow{\sim} H^*(G; M)$$

for all continuous l-coefficient systems M. We will say that a simplicial group G is *l-good* if each G_n is *l-good*. A group is *l-virtually nilpotent* if it is the extension of a finite l-group by a nilpotent group.

LEMMA 2.6. *Free groups are l-good; finitely generated l-virtually nilpotent groups are l-good.*

Proof. The argument of [13; 3.5] generalizes directly using [15; Ch I, Prop. 16].

PROPOSITION 2.7. *If G is a simplicial group and l a set of primes, then*

$$\pi_0 \hat{G}_l = (\pi_0 G)\hat{}_l \ .$$

Proof. Let $\tilde{G}_1 = \ker\ d_1 \colon G_1 \to G_1$, $\bar{G}_1 = \ker\ \hat{d}_1 \colon (G_1)\hat{}_l \to (G_0)\hat{}_l$. The diagram

$$\begin{array}{ccccccccc}
* & \longrightarrow & \tilde{G}_1 & \longrightarrow & G_1 & \xrightarrow{d_1} & G_0 & \longrightarrow & * \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & (\tilde{G}_1)\hat{}_l & \longrightarrow & (G_1)\hat{}_l & \xrightarrow{\hat{d}_1} & (G_0)\hat{}_l & \longrightarrow & *
\end{array}$$

commutes, and since l-completion is right exact [1; Prop. 2], both rows are exact. Therefore, the natural map $(\tilde{G}_1)\hat{}_l \to \bar{G}_1$ is epic. Similarly, the diagram

$$\begin{array}{ccccccccc}
 & & & \swarrow & (\tilde{G}_1)\hat{}_l & & & & \\
 & & & & \downarrow{\scriptstyle d_0} & & & & \\
* & \longrightarrow & \bar{G}_1 & \xrightarrow{\hat{d}_0} & (G_0)\hat{}_l & \longrightarrow & \pi_0(\hat{G}_l) & \longrightarrow & * \\
 & & & & \downarrow & \nearrow & & & \\
 & & & & (\pi_0 G)\hat{}_l & & & & \\
 & & & & \downarrow & & & & \\
 & & & & * & & & &
\end{array}$$

commutes and has exact row and column. Thus $(\pi_0 G)\hat{}_l \xrightarrow{\sim} \pi_0 \hat{G}_l$.

PROPOSITION 2.8. *Let l be a set of primes and G an l-good simplicial group. Then for all continuous l-coefficient systems M,*

$$H^*(\hat{G}_l; M) \xrightarrow{\sim} H^*(G; M) .$$

Proof. The natural map $G \to \hat{G}_l$ induces a map of spectral sequences

$$\begin{array}{ccc} E_2^{p,q} = \pi^p \mathscr{H}^q(\hat{G}_l; M) & \longrightarrow & E_2^{pq} = \pi^p \mathscr{H}^q(G; M) \\ \Downarrow & & \Downarrow \\ H^{p+q}(\hat{G}_l; M) & \longrightarrow & H^{p+q}(G; M) \end{array}$$

which is an isomorphism on $E_2^{p,q}$. Hence $H^*(\hat{G}_l; M) \xrightarrow{\sim} H^*(G; M)$.

COROLLARY 2.9. *If $X \in \mathscr{S}_0$, then $X \to \hat{X}_l$ induces isomorphisms*

$$(\pi_1 X)\hat{}_l \xrightarrow{\sim} \pi_1 \hat{X}_l ,$$

$$H^*(\hat{X}_l; M) \xrightarrow{\sim} H^*(X; M)$$

for all l-coefficient systems M.

COROLLARY 2.10. *If $X \in \hat{\mathscr{S}}_{0l}$, then*

$$X \longrightarrow \hat{X}_l$$

is a weak equivalance of profinite spaces.

Proof. First, for $X = \{X_i\}_{i \in I}$,

$$\pi_1 \hat{X}_l = \varprojlim \pi_1(X_i)\hat{}_l = \varprojlim (\pi_1 X_i)\hat{}_l = \varprojlim \pi_1 X_i = \pi_1 X .$$

Second, for M a continuous l-coefficient system.

$$H^*(\hat{X}_l; M) = \varinjlim_i H^*((X_i)\hat{}_l; M) = \varinjlim_i H^*(X_i; M) = H^*(X; M) .$$

Thus by 2.4, $X \to \hat{X}_l$ is a weak equivalence.

COROLLARY 2.11. *If $X, Y \in \mathscr{S}_0$, $f: X \to Y$ a weak equivalence, then $\hat{f}: \hat{X}_l \to \hat{Y}_l$ is a weak equivalence.*

COROLLARY 2.12. *If $X \in \mathscr{S}_0$ is n-connected, then $\hat{X}_l$ is n-connected and*

$$\pi_{n+1} \hat{X}_l = (\pi_{n+1} X)\hat{}_l .$$

Proof. By (2.11) we may assume $X_i = *$, $i \leqq n$. The argument of (2.7) then generalizes.

THEOREM 2.13. (a) *The functor* $(\ )\hat{}_l$: Pro-$\mathscr{S}_0 \to \mathscr{S}_{0l}$ *preserves homotopies.*

(b) *If* $X \in$ Pro-$\mathscr{S}_0$, $Y \in \hat{\mathscr{S}}_{0l}$, *and* $f: X \to Y$, *there is a functorial commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & \hat{X}_l \\ \downarrow f & & \downarrow \hat{f} \\ Y & \longrightarrow & \hat{Y}_l \end{array}$$

such that $Y \to \hat{Y}_l$ *is a weak equivalence.*

(c) *If* $X \in$ Pro-$\mathscr{S}_0$ *and* $PF(X)$ *denotes the Artin-Mazur pro-l completion of* [2], *then there is a natural* ♮*-isomorphism in* Pro-$\mathscr{H}_0$

$$PF(X) \longrightarrow \hat{X}_l\ .$$

Proof. Only (c) requires comment. By the universal properties of $PF(X)$ there is a natural commuting diagram

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ PF(X) & \longrightarrow & \hat{X}_l \end{array}$$

in Pro-$\mathscr{H}_0$. By 2.7, 2.8, and [2; 3.7, 4.3]

$$\pi_1 PF(X) \xrightarrow{\sim} \pi_1(\hat{X}_l)$$

and for all finite l-coefficient systems M,

$$H^*(\hat{X}_l; M) \xrightarrow{\sim} H^*(PF(X); M)\ .$$

Thus $PF(X) \to \hat{X}_l$ is a ♮-isomorphism.

COROLLARY 2.14. *Let* $F \in \mathscr{S}_0$ *have finite homotopy dimension and suppose each* $\pi_n F \in \mathfrak{G}_l$. *If* $X \in \mathscr{S}_0$, *then the homotopy classes of maps of prospaces* $\hat{X}_l \to F$ *are in one-to-one correspondence with* $[X; F]$.

Proof. [2; p. 40].

We have established that $(\ )\hat{}_l$ is a good rigid profinite completion. We will now see that the completions for various sets of primes l are compatible.

PROPOSITION 2.15. *Let l and l' be two sets of primes, $l' \subseteq l$. Then $\mathscr{S}_{0l'} \subseteq \mathscr{S}_{0l}$, and for $X \in \hat{\mathscr{S}}_{0l'}$, the natural map*

$$\hat{X}_l \longrightarrow \hat{X}_{l'}$$

is a weak equivalence.

Proof. First $\pi_1\hat{X}_l = \pi_1\hat{X}_{l'}$, since the l-completion of a finite l'-group π is again π. Next, let M be a continuous l-coefficient group. Since M is a finite abelian group it may be written

$$M = M_{l'} \oplus N$$

where $M_{l'}$ is a Sylow l'-group and N is a Sylow $(l - l')$-group. Since for abelian groups, Sylow l-groups are characteristic subgroups, the decomposition is compatible with the action of $\pi_1 X$ on M. Thus

$$H^*(\hat{X}_l; M) = H^*(\hat{X}_l; M_{l'}) \oplus H^*(X; N) .$$

But $H^*(X; N) = \varinjlim H^*(X_i; N)$ where $X = \{X_i\}_{i \in I}$, and $\pi_n X_i \in \mathfrak{G}_{l'}$ for all i, n. Thus $\tilde{H}^*(X; N) = 0$. By the same arguments, $\tilde{H}^*(\hat{X}_{l'}; N) = 0$. Thus

$$\tilde{H}^*(\hat{X}_l; M) = \tilde{H}^*(\hat{X}_l; M_{l'}) = \tilde{H}^*(\hat{X}_{l'}; M_{l'}) = \tilde{H}^*(\hat{X}_{l'}; M) .$$

3. Discretization and the Sullivan completion. In this section we will extend the discretization functor defined on profinite groups to a functor $d: \hat{\mathscr{S}}_0 \to \mathscr{S}_0$ and show that, for $X \in \mathscr{S}_0$, $d\hat{X}_l$ represents the Sullivan l-completion of X [17].

Let G be a simplicial profinite group. Then dG is a (discrete) simplicial group, and the functor d is right adjoint to finite completion. As we remarked in § 2, the homotopy groups of G and dG may be computed from the homology of certain non-abelian chain complexes; also, inverse limits of profinite groups are exact. Thus

PROPOSITION 3.1. (a) $\pi_n dG = d\pi_n G$.

(b) *If $* \to H \to G \to G/H \to *$ is an exact sequence in $\Delta\hat{\mathfrak{G}}$ then $* \to dH \to dG \to d(G/H) \to *$ is exact in $\Delta\mathfrak{G}$.*

Since finite products of sets commute with inverse limits, we also see that

$$\bar{W}dG = \varprojlim \bar{W}(G/U) ,$$

U running over the open normal subgroups of G.

PROPOSITION 3.2. *Let $G \in \Delta\hat{\mathfrak{G}}$. Then $\bar{W}dG$ is fibrant* (i.e., *a*

Kan complex) and the natural map

$$\bar{W}dG \longrightarrow \underset{\longleftarrow}{\text{holim}}\ \bar{W}(G/U)\ ,$$

U running over the open normal subgroups of G, is a homotopy equivalence [5; Ch. XI].

Proof. Since each $\pi_n \bar{W}(G/U)$ is finite,

$$\pi_n \underset{\longleftarrow}{\text{holim}}\ \bar{W}(G/U) = \underset{\longleftarrow}{\lim}\ \pi_n \bar{W}(G/U) = \pi_n \bar{W}dG\ .$$

For $X \in \hat{\mathscr{S}}_0$, we define the *discretization* of X to be

$$d\hat{X} = \bar{W}d\hat{G}X\ .$$

If $X = \{X_i\}_{i \in I}$, there is a natural map

$$\underset{\overleftarrow{i \in I}}{\text{holim}}\ X_i \longrightarrow \underset{\substack{\overleftarrow{U \lhd GX} \\ \text{open}}}{\text{holim}}\ \bar{W}(\hat{G}X/U)$$

which is a homotopy equivalence if each X_i is fibrant. More generally, for $X \in \text{Pro-}\mathscr{S}_0$, we may define the *discrete l-completion* of X to be

$$d\hat{X}_l = \bar{W}d\hat{G}_l X\ .$$

PROPOSITION 3.3. (a) *For* $X \in \hat{\mathscr{S}}_0$, $\pi_n d\hat{X} = d\pi_n X$.

(b) *For* $X \in \hat{\mathscr{S}}_{0l}$, *the natural map* $d\hat{X} \to d\hat{X}_l$ *is a homotopy equivalence.*

(c) *If* $F \to X \to B$ *is a fibration in* $\hat{\mathscr{S}}_0$, *then* $d\hat{F} \to d\hat{X} \to d\hat{B}$ *is a quasi fibration in* $\mathscr{S}_0$.

Proof. Parts (a), (c) follows from (3.1). Part (b) is just (2.15).

We will see that $d\hat{X}_l$ is just a *rigid Sullivan l-completion* of X.

For $X \in \mathscr{S}_0$, the Sullivan l-completion of X, $X \to \hat{S}_l X$ is characterized as follows [17]. Let $PF_l(X) = \{F_i\}_{i \in I}$ be the Artin-Mazur l-completion. I is the category whose objects are maps in $\mathscr{H}_0$,

$$X \longrightarrow F_i\ ,$$

where F_i has homotopy groups which are finite l-groups. A map $i \to j$ in I is a class of (homotopy) commuting diagrams

$$X \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} F_i \\ \downarrow \\ F_j \end{array} .$$

Without loss of generality, one may assume I is small. $\hat{S}_l X$ is characterized by

$$[Y, \hat{S}_l X] = \varprojlim_{PF_l(X)} [Y, F_i]$$

where [,] denotes pointed homotopy classes of maps and $Y \in \mathscr{S}_0$.

More generally, if $X, Y \in \mathscr{S}_0$ are fibrant, let $\hom_*(X, Y)$ be the pointed simplicial function space [5; VIII, 4.8], and put

$$[X, Y]_n = \pi_n \hom_*(X, Y) .$$

If $Y \rightrightarrows Y'$ are two homotopic maps, they induce the same map $[X, Y]_n \to [X, Y']_n$. Furthermore, a weak equivalence $Y \to Y'$ induces an isomorphism $[X, Y]_n \xrightarrow{\sim} [X, Y']_n$. Thus $[\ ,\ ]_n$ may be made a functor of two variables on the homotopy category $\mathscr{H}_0$. We will show that $d\hat{X}_l$ represents $\hat{S}_l X$ in the following strong sense.

THEOREM 3.4. *If $X \in$ Pro-$\mathscr{S}_0$, then for $Y \in \mathscr{S}_0$,*

$$[Y, d\hat{X}_l]_n = \varprojlim_{PF_l(X)} [Y, F_i]_n .$$

The main ingredient of the proof is the following.

PROPOSITION 3.5. *Let I be a left filtered index category and $F: I \to \mathscr{S}$ a functor such that*

(a) *I is a partial order.*

(b) *for each $i \in I$, there are only finitely many $i' \in I \ni i \to i'$.*

(c) *each F_i is fibrant.*

(d) *for $i \in I$, let $i//I$ be the objects strictly under i* — i.e., *$i \to i' \in i//I$ if $i \neq i'$. Then*

$$F_i \longrightarrow \varprojlim_{i' \in i/I} F_{i'}$$

is a fibration.

(e) *each F_i has finitely many simplices in each degree. Then $F = \varprojlim_{i \in I} F_i$ is fibrant and*

$$\pi_* F = \varprojlim_{i \in I} \pi_* F_i .$$

Proof. We use the notation of [5; Ch. VIII]. First, let $\bar{\alpha}: \Delta[n, k] \to F$. We must show that each $\bar{\alpha}_i$ may be extended compatibly to $\alpha_i: \Delta[n] \to F_i$. We will use Zorn's lemma. Consider compatible families $\{\alpha_i\}_{i\in I'}$, α_i extending $\bar{\alpha}_i$, where $I' \subseteqq I$ is such that $i \in I'$ and $i \to i' \Rightarrow i' \in I'$. Choose a maximal such family I'. If $I' = I$ we are done. Suppose not. Let $i \in I$ be such that $i//I \subseteqq I'$ but $i \notin I'$. Then the α_j, $j \in i//I$ define

$$\alpha\colon \Delta[n] \longrightarrow \varprojlim_{j\in i//I} F_j$$

extending the $\bar{\alpha}_j$. By (d) $\exists \alpha_i\colon \Delta[n] \to F_i$ covering α and extending $\bar{\alpha}_i$. Thus we have extended $\{\alpha_i\}$ to $I' \cup \{i\}$, contradiction. Suppose now that each F_i is reduced. Let $\{h_i \in \pi_n F_i\}_{i\in I}$ be a compatible family representing $h \in \varprojlim \pi_n F_i$. We must construct a compatible family $\{\alpha_i\}_{i\in I}$, $\alpha_i \in F_i[n]$, such that $d_k\alpha_i = *$ for all k, and α_i represents h_i. For each $i \in I$, let $A_i = \{\alpha \in F_i[n] \mid d_k\alpha = *, \forall k,$ and α represents $h_i\}$. Then for $i \to i' \in I$,

$$F_{i\to i'}(A_i) \subseteqq A_{i'} \; .$$

Note that each A_i is nonempty and finite. Thus $\varprojlim A_i$ is nonempty [3; Ch. I, App., Th. 1]. If $\{\alpha_i\} \in \varprojlim A_i$, $\{\alpha_i\}$ represents an element of $\pi_n F$ mapping to h. Thus $\pi_n F \to \varprojlim \pi_n F_i$ is epic. A similar argument shows this map to be monic.

To complete the proof, note that, by the last argument, $\pi_0 F = \varprojlim \pi_0 F_i$, and apply the above result to each component.

REMARK 3.6. More generally it may be shown that if (a) through (d) are satisfied, then

$$\varprojlim F_i \longrightarrow \operatorname{holim}_{\longleftarrow} F_i$$

is a weak equivalence.

Proof of 3.4. We first show that we may restrict the limit on the right hand side of (3.4) to the subcategory $PF_l'(X) \subseteqq PF_l(X)$ whose objects are classes of maps $X \xrightarrow{i} F_i$ such that F_i has finite homotopy dimension. For let F be a fibrant simplicial set with finitely many simplices in each degree. Then the inverse system

$$\{\hom_*(Y_\alpha, F)\} \; ,$$

where Y_α runs over the finite subcomplexes of Y, satisfies the hypotheses of (3.5). Furthermore, $\hom_*(Y, F) = \varprojlim_\alpha \hom_*(Y_\alpha, F)$. Thus

$$[Y, F]_n = \varprojlim_\alpha [Y_\alpha, F]_n \; .$$

Similarly, if $F^{(k)}$ denotes the k-th Postnikov complex of F, then

$$[Y_\alpha, F]_n = \varprojlim_k [Y_\alpha, F^{(k)}]_n .$$

Thus we see that

$$\varprojlim_{PF_l(X)} [Y, F_i]_n = \varprojlim_{\alpha} \varprojlim_{PF'_l(X)} [Y_\alpha, F_i]_n .$$

But by (2.14), $PF'_l(X) = PF'_l(\hat{X})$; so

$$\varprojlim_{\alpha} \varprojlim_{PF(X)} [Y_\alpha, F_i]_n = \varprojlim_{\alpha} \varprojlim_{U} [Y_\alpha, \bar{W}(\hat{G}_l X/U)]_n ,$$

where U runs over the open normal subgroups of $\hat{G}_l X$.

Now, consider the inverse system

$$\{\hom_*(Y_\alpha, \bar{W}(\hat{G}_l X/U))\}_{(\alpha,U)} .$$

Generalizing [11; 7.16], we see that this system satisfies the hypotheses of (3.5). Thus

$$[Y, d\hat{X}_l]_n = \varprojlim_{(\alpha, U)} [Y_\alpha, \bar{W}(\hat{G}_l X/U)]$$

as required.

COROLLARY 3.6. *If* $X \in \hat{\mathscr{S}}_0$, $Y \in \mathscr{S}_0$, $[Y, d\hat{X}]_n = \varprojlim [Y, X_i]_n$, *where* $X = \{X_i\}_{i \in I}$.

Proof. If $X \in \hat{\mathscr{S}}_0$, X is cofinal in $PF(X)$.

4. Fibration theorems and nilpotent spaces. In this section we will discuss some circumstances under which finite completion preserves fibrations up to homotopy. Two types of restrictions arise — conditions on the action of fundamental groups and finite type conditions. Throughout this section, l will be a fixed set of primes.

DEFINITION 4.1. Recall that a (profinite) group G has property (F) if for any finite group F there are only finitely many (continuous) homomorphisms $G \to F$. We will say that a connected space X is *locally of finite type* if $(\pi_1 X)\hat{}$ has property (F) and, for each finite l-coefficient system M, $H^n(X; M)$ is finite for all n. We will say G is locally of finite type if $K(G, 1)$ is.

A fibration

$$F \longrightarrow E \longrightarrow B$$

is *l-virtually nilpotent* if for each $N \geqq 1$ there is a normal subgroup $\pi \subseteqq \pi_1 E$ such that $\pi_1 E/\pi$ is a finite l-group and π acts nilpotently on $\pi_n F$ for $n \leqq N$. A space X is l-virtually nilpotent if the trivial fibration $X \to X \to *$ is.

Let $F \to E \to B$ be a fibration in $\mathscr{S}_0$, and let $\bar{F}$ be the fibre of the induced map $\hat{E}_l \to \hat{B}_l$,

$$\bar{F} = \bar{W}(\ker(\hat{G}_l E \longrightarrow \hat{G}_l B)) .$$

Then there is a natural map $\hat{F}_l \to \bar{F}$.

THEOREM 4.2. (*Artin-Mazur* [2; Th. 5, 9]). *If B is simply connected and either F or B is locally of finite type relative to l, then $\hat{F}_l \to \bar{F}$ is a weak equivalence.*

THEOREM 4.3. *If $F \to E \to B$ is an l-virtually nilpotent fibration and either F or B is locally of finite type relative to l, then $\hat{F}_l \to \bar{F}$ is a weak equivalence.*

The proof of this theorem is not needed for the next two sections and is defered to § 7.

REMARK 4.4. The finite type restrictions in Theorems 4.2 and 4.3 are necessary. For let A be a free abelian group on infinitely many generators, n an integer, $F = B = K(A, n)$, and $E = F \times B$. Then a calculation with the Künneth theorem shows that

$$H^{2n}(\hat{E}; \mathbf{Z}/p) \neq H^{2n}(\hat{F} \times \hat{B}; \mathbf{Z}/p)$$

for any p.

The notion of a p-sylow subgroup G_p of a finite group G generalizes to profinite groups [15]. A profinite group G is pro-nilpotent if it is the product of its p-Sylow subgroups. We say that a G module A is *pro-nilpotent* if G_p acts trivially on A_q when $p \neq q$. Note that since the p-Sylow subgroups of an abelian group are characteristic, the action of a group on A respects the decomposition

$$A = \prod_p A_p .$$

We will say a profinite space X is *pro-nilpotent* if $\pi_1 X$ is pro-nilpotent and each $\pi_n X$ is a pro-nilpotent $\pi_1 X$ module.

THEOREM 4.5. *If $X \in \hat{\mathscr{S}}_0$ is pro-nilpotent, the natural map*

$$X \longrightarrow \prod_p \hat{X}_p ,$$

is a weak equivalence.

Proof. Note that $\pi_1 X = \pi_1 \pi \hat{X}_p$ by assumption. We must show that for any finite $\pi_1 X$ module M the cohomology map is an isomorphism. We may assume M is a simple $\pi_1 X$ module and is therefore a $\boldsymbol{Z}/q$ vector space for some prime q. Indeed, by [16; I. 3.2, III. 4.3] we may assume

$$M = \bigotimes_p M_p$$

where M_p is a simple $(\pi_1 X)_p$ module which is isomorphic to the trivial one dimensional representation for almost all p and whenever $p = q$. Thus, by the Künneth theorem,

$$H^*(\prod_p \hat{X}_p; M) \simeq M^{\pi_1 X} \otimes H^*(\hat{X}_q; \boldsymbol{Z}/q) .$$

Now, assuming X is of the form $\bar{W}\hat{G}Y$, we have a fibration

$$\bar{X} \longrightarrow X \longrightarrow K(N, 1)$$

where $N = \prod_{p \neq q} (\pi_1 X)_p$. Then there is a spectral sequence

$$H^*(N; H^*(\bar{X}; M)) \Longrightarrow H^*(X; M) .$$

Note that $\pi_n \bar{X} = \pi_n X$, $n \geqq 2$, and $\pi_1 \bar{X} = (\pi_1 X)_q$. The action of N on $H^*(\bar{X}; M)$ is determined by that on $(\pi_* \bar{X})_q$ which is trivial; thus

$$H^*(N; H^*(\bar{X}; M)) = H^*(N; M \otimes H^*(\bar{X}; \boldsymbol{Z}/q)) = M^N \otimes H^*(\bar{X}; \boldsymbol{Z}/q) .$$

But, by the same argument, $H^*(\bar{X}; \boldsymbol{Z}/q) = H^*(X; \boldsymbol{Z}/q)$. Thus

$$H^*(X; M) = M^{\pi_1 X} \otimes H^*(X; \boldsymbol{Z}/q)$$

as required.

COROLLARY 4.6. *Let $X \in \mathscr{S}_0$ be nilpotent, then $\hat{X}_l$ is pro-nilpotent.*

Proof. Apply the above argument to X noting that $\pi_1 X$ is dense in $\pi_1 \hat{X}_l$.

COROLLARY 4.7. *Let $X \in \mathscr{S}_0$ be nilpotent and l-locally of finite type, then there is a weak equivalence*

$$\hat{X}_l \longrightarrow \prod_{p \in l} (\boldsymbol{Z}/p)_\infty X$$

where $(\boldsymbol{Z}/p)_\infty$ ( ) is the nilpotent completion of Bousfield-Kan [5; Ch. VI].

We will conclude this section by elaborating on the local finite type condition of Theorem 4.3. In particular, we wish to show that a virtually nilpotent space is locally of finite type iff each of its homotopy groups is. We first characterize abelian and nilpotent groups which are locally of finite type.

PROPOSITION 4.8. *Let A be an abelian group, $n \geqq 1$ an integer and l a set of primes. For $p \in l$ put ${}_pA = \{a \in A \mid pa = 0\}$. Then the following are equivalent:*

(a) *A is locally of finite type relative to l.*
(b) *$H^n(A, n; \boldsymbol{Z}/p)$ and $H^{n+1}(A, n; \boldsymbol{Z}/p)$ are finite for each $p \in l$.*
(c) *$K(A, n)$ is locally of finite type relative to l.*
(d) *${}_pA$ and A/pA are finite for each $p \in l$.*

Proof. By the computations of Cartan [6; § 9, § 10], $H^*(A, n; \boldsymbol{Z}/p)$ is generated as an algebra over the Steenrod algebra by $H^n(A, n; \boldsymbol{Z}/p)$ and $H^{n+1}(A, n; \boldsymbol{Z}/p)$. Thus (b) and (c) are equivalent. Also by Cartan's computations, $H^n(A, n; \boldsymbol{Z}/p)$ and $H^{n+1}(A, n; \boldsymbol{Z}/p)$ are finite iff A/pA and ${}_pA$ are finite. Part (a) is a special case for $n = 1$.

LEMMA 4.9. *If A and B are abelian groups such that $H^1(A; \boldsymbol{Z}/p)$ and $H^1(B; \boldsymbol{Z}/p)$ are finite, then the same condition is true for $A \oplus B$ and $A \otimes B$.*

Proof. $H^1(A; \boldsymbol{Z}/p)$ is finitely generated iff A/pA is finitely generated. Note that

$$(A \oplus B)/p(A \oplus B) = (A/pA) \oplus (B/pB) ,$$

and

$$\begin{aligned}(A \otimes B)/p(A \otimes B) &= A \otimes B \otimes \boldsymbol{Z}/p \\ &= (A \otimes \boldsymbol{Z}/p) \otimes_{\boldsymbol{Z}/p} (B \otimes \boldsymbol{Z}/p) \\ &= A/pA \otimes B/pB .\end{aligned}$$

LEMMA 4.10. *If $A \to B$ is an epimorphism of abelian groups and $H^1(A; \boldsymbol{Z}/p)$ is finite, so is $H^1(B; \boldsymbol{Z}/p)$.*

Proof. $H^1(B; \boldsymbol{Z}/p) \to H^1(A; \boldsymbol{Z}/p)$ *is monic.*

PROPOSITION 4.11. *If A is a nilpotent group with $H^1(A; \boldsymbol{Z}/p)$ and $H^2(A; \boldsymbol{Z}/p)$ finite for all $p \in l$, then*

(a) *A is locally of finite type relative to l.*
(b) *There is a filtration by normal subgroups,*

$$0 \subseteqq A^n \subseteqq \cdots \subseteqq A^i \subseteqq \cdots \subseteqq A^1 = A ,$$

such that each A^i/A^{i+1} is a trivial A module and is locally of finite type relative to l. Furthermore, the lower central series of A may be chosen.

Proof. Let A^i be the ith term of the lower central series of A, $A^1 = A$, and

$$A^{i+1} = [A, A^i] .$$

Then $\sum_{i=1}^{n} A^i/A^{i+1}$ is a Lie ring generated by A^1/A^2 [9; Ch. 10, Ch. 11]. Thus by (4.9) and (4.10), $H^1(A^i/A^{i+1}; Z/p)$ is finite for $p \in l$.

We will induct on the degree of nilpotence n. Proposition 2.8 establishes the result for $n = 1$.

Consider the exact sequence

$$0 \longrightarrow A^n \longrightarrow A \longrightarrow A/A^n \longrightarrow * .$$

A acts trivially on A^n, so the Hochschild-Serre spectral sequence of this exact sequence takes the form shown in figure 4.12. Since

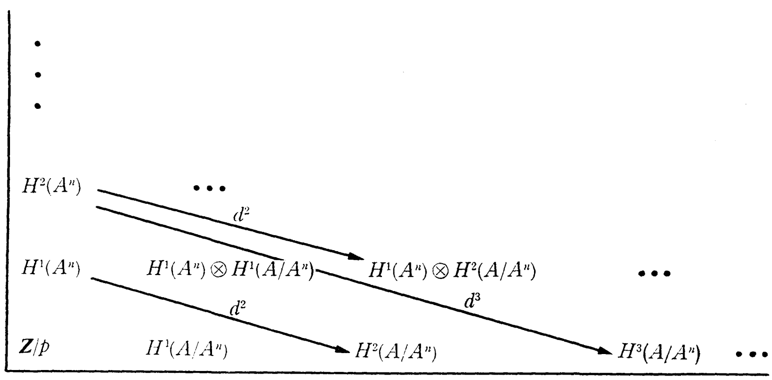

FIGURE 4.12

$H^1(A)$ is finite, so is $H^1(A/A^n)$. Since $H^2(A)$ is finite, so is $E_\infty^{02} = H^2(A/A^n)/d^2(H^1(A^n))$. But $H^1(A^n)$ is finite; therefore so is $H^2(A/A^n)$. By induction we may now assume that $H^i(A/A^n)$ is finite for all i. Thus $H^3(A/A^n)$ and $H^1(A^n) \otimes H^2(A/A^n)$ are finite, and consequently $H^2(A^n)$ is finite. By (4.8), $H^i(A)$ is finite for all i; thus, $E_2^{p,q}$ is finite for all p, q and $H^i(A; Z/p)$ is finite for all i.

A similar argument yields

PROPOSITION 4.13. *Let G be a group and A a nilpotent G module. If for all $p \in l$*

(i) *$H^1(G; Z/p)$ is finite.*

(ii) *$H^1(A; Z/p)$ and $H^2(A; Z/p)$ are finite,* *then there exists a G-invariant filtration*

$$0 \subseteqq A^n \subseteqq \cdots \subseteqq A^i \subseteqq \cdots \subseteqq A^1 = A$$

such that

(a) *Each A^i/A^{i+1} is a trivial G-module.*

(b) *For each i, j, $H^j(A^i; Z/p)$ is finite $\forall p \in l$.*

Furthermore, the filtration may be chosen to be the lower central series $A^1 = A$,

$$A^{i+1} = [G, A^i] ,$$

where for $g \in B$, $a \in A$, $[g, a] = a - {}^g a$.

LEMMA 4.14. *Let G be a group and let G act nilpotently on a $\boldsymbol{Z}/p$ vector space A. If $H^1(G; \boldsymbol{Z}/p)$ is finite and $H^0(G; A)$ is finite, then A is finite.*

Proof. Let $0 \subseteqq A^n \subseteqq \cdots \subseteqq A^1 = A$ be a filtration of A as in (4.13). We will induct on the length of such a filtration. If $n = 1$, G acts trivially on A and $H^0(G; A) = A$, so A is finite. In general, consider the exact sequence

$$0 \longrightarrow A^n \longrightarrow A \longrightarrow A/A^n \longrightarrow 0 .$$

This induces a long exact sequence

$$0 \longrightarrow H^0(G; A^n) \longrightarrow H^0(G; A) \longrightarrow H^0(G; A/A^n) \longrightarrow H^1(G; A^n) \longrightarrow \cdots .$$

G acts trivially on A^n, so $H^0(G; A^n) = A^n$. $H^0(G; A)$ is finite, so $A^n \subseteqq H^0(G; A)$ is finite. Thus $H^1(G; A^n)$ is finite, and consequently, $H^0(G; A/A^n)$ is finite. Then, by induction, A/A^n is finite implying that A is finite.

PROPOSITION 4.15. *An l-virtually nilpotent space X is locally of finite type relative to l iff each $\pi_n X$ is locally of finite type relative to l.*

Proof. If each $\pi_n X$ is locally of finite type relative to l, then so is X by a straightforward Serre spectral sequence argument. Suppose then that X is locally of finite type relative to l. Then so is any l-finite regular covering space of X. Thus we may assume that $\pi_1 X$ acts nilpotently on $\pi_n X$ for $n \leqq N$, N any chosen integer. Let $\tilde{X}$ be the universal cover of X. We assert that $H^n(\tilde{X}; \boldsymbol{Z}/p)$ is finite for all $p \in l$ and $n \leqq N$. To see this, consider the spectral sequence of the universal cover $\tilde{X}$,

$$E_r^{p,q} \Longrightarrow H^{p+q}(X; \boldsymbol{Z}/p)$$
$$E_2^{p,q} = H^p(\pi_1 X; H^q(\tilde{X})) ,$$

shown in Figure 4.16. Since each $H^n(X; \boldsymbol{Z}/p)$ is finite, so are $H^1(\pi_1 X; \boldsymbol{Z}/p)$ and $H^2(\pi_1 X; \boldsymbol{Z}/p)$ for $p \in l$. Thus $\pi_1 X$ is locally of finity type with respect to l, and $H^n(\pi_1 X; \boldsymbol{Z}/p)$ is finite for all n. Consequently, $H^0(\pi_1 X; H^2(\tilde{X}))$ and $H^1(\pi_1 X; H^2(\tilde{X}))$ are finite. By (4.14), $H^2(\tilde{X})$ is finite; thus, so is $H^i(\pi_1 X; H^2(\tilde{X}))$ for all i. Continuing by induction, $H^n(\tilde{X}; \boldsymbol{Z}/p)$ is finite for $n \leqq N$. To complete the proof, we apply a similar argument to the fibrations

$$E_{m+1}X \longrightarrow E_m X \longrightarrow K(\pi_m X, m)$$

where $E_m X$ denotes the $(m-1)$-connected cover of X.

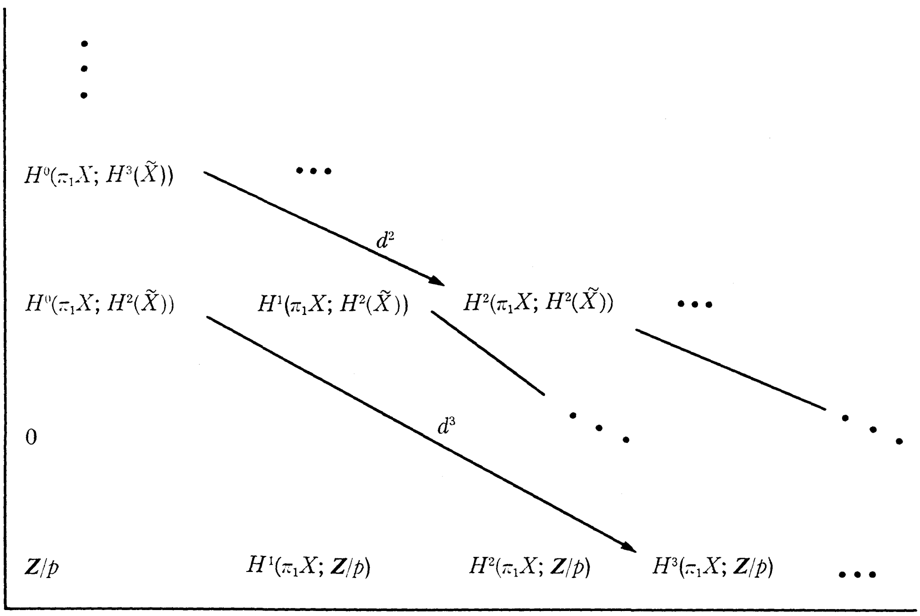

FIGURE 4.16

5. **Homotopy groups of completions.** In this section we define derived functors of finite completion and use them to calculate the homotopy groups of finite completions of virtually nilpotent spaces. In view of (4.7), this computation is essentially the same as that of Bousfield-Kan [5; Ch. VI].

Let $n \geqq 0$ be an integer, G a group which is abelian if $n > 0$. We define the *left derived functors of finite completion* on G by

$$\hat{L}_i(G; n) = \pi_{n+i+1}(K(G, n+1))\hat{} \; .$$

Note that $GK(G, n+1)$ is a free simplicial group of type (G, n)

and that $\hat{L}_i(G; n) = \pi_{n+i}\hat{G}K(G, n+1)$; thus the profinite groups $\hat{L}_i(G; n)$ are indeed derived functors of completion on the category of groups by analogy with Dold-Puppe [7]. If l is a set of primes we also define the *derived functors of l-completion* by

$$\hat{L}_{l,i}(G; n) = \pi_{n+i+1}(K(G, n+1))\hat{}_l \ .$$

To simplify the notation in the following discussion, we will denote $\hat{L}_{l,i}$ by $\hat{L}_i$ when l is clear from context.

PROPOSITION 5.1. *For l a set of primes*

$$\hat{L}_0(G; n) = \hat{G}_l \ ,$$

and $\hat{L}_i(G; n)$ is abelian for $i > 0$.

Proof. Immediate from (2.12).

PROPOSITION 4.2. *If G is abelian and locally of finite type with respect to l, then*

$$\hat{L}_i(G; n) = \hat{L}_i(G, n+1) \ .$$

Proof. Apply (4.2) to the fibration

$$K(G, n) \longrightarrow E \longrightarrow K(G, n+1)$$

with contractible total space.

In case G is locally of finite type or $n = 0$, we will denote $\hat{L}_i(G; n)$ by $\hat{L}_i(G)$.

PROPOSITION 5.3. *Let l be a set of primes. Then $\hat{L}_i(G) = 0$ for all $i > 0$ iff G is l-good.*

Proof. We have a natural map

$$(K(G, 1))\hat{}_l \longrightarrow K(\hat{G}_l, 1)$$

in $\hat{\mathscr{S}}_0$ which is a weak equivalence iff G is l-good.

PROPOSITION 5.4. *For l a set of primes, $\hat{L}_i(\ ; n)$ is a functor from groups (abelian groups if $n > 0$) to pro-l groups.*

Proof. If $G \longrightarrow H$ is a homomorphism, there is induced a map, unique up to homotopy,

$$f\colon K(G, n) \longrightarrow K(H, n) \ .$$

By (2.13a) f induces a map unique up to homotopy

$$K(G, n)_{\hat{l}} \longrightarrow K(H, n)_{\hat{l}} \; .$$

THEOREM 5.5. *Let*

$$* \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow *$$

be a short exact sequence of groups such that G acts l-virtually nilpotently on H and either

(i) *H is locally of finite type, or*

(ii) *G/H is locally of finite type.*

Then there is a natural long exact sequence

$$\cdots \longrightarrow L_i(H) \longrightarrow L_i(G) \longrightarrow L_i(G/H) \longrightarrow L_{i-1}(H) \longrightarrow \cdots$$
$$\cdots \longrightarrow L_1(G/H) \longrightarrow \hat{H}_l \longrightarrow \hat{G}_l \longrightarrow (G/H)_{\hat{l}} \longrightarrow * \; .$$

Proof. Apply (4.3) to the fibration

$$K(H, n) \longrightarrow K(G, n) \longrightarrow K(G/H, n) \; .$$

REMARK 5.6. To generalize (5.5) to extensions which are not nilpotent, it is necessary to define the notion of relative finite completion and its derived functors. If H is a group with G action, the *finite completion of H relative to G* is the profinite group

$$\hat{H}_G = \{F_i\}$$

where i runs over the family of G-epimorphisms $H \xrightarrow{i} F_i$, where F_i is a finite G-group. Then $\hat{H}_G$ has a natural $\hat{G}$ action. In case the action of G on H is virtually nilpotent and either G or H has property (F), $\hat{H}_G = \hat{H}$, reducing relative finite completion to finite completion. If $P = H \int G$ is the semi-direct product of H and G, the sequence

$$* \longrightarrow \hat{H}_G \longrightarrow \hat{P} \longrightarrow \hat{G} \longrightarrow *$$

is always split exact. Let F be the fibre of $K(P, 1)^\wedge \to K(G, 1)^\wedge$. Then $\pi_1 F = \hat{H}_G$ and one may define the derived functors of G-completion as

$$\hat{L}_i(H, G) = \pi_{i+1} F \; .$$

It may be shown that there is always a long exact sequence

$$\cdots \longrightarrow L_i(H, G) \longrightarrow H_i(G) \longrightarrow L_i(G/H) \longrightarrow L_{i-1}(H, G) \longrightarrow \cdots$$
$$\cdots \longrightarrow L_1(G/H) \longrightarrow \hat{H}_G \longrightarrow \hat{G} \longrightarrow (G/H)^\wedge \longrightarrow * \; .$$

Proof of these facts and applications will appear in a future paper.

EXAMPLE 5.7. Let $G = \boldsymbol{Q}$, $H = \boldsymbol{Z}$, $G/H = \boldsymbol{Q}/\boldsymbol{Z}$. Then since $\tilde{H}^*(\boldsymbol{Q}, M) = 0$ for all finite abelian groups M, $\hat{L}_*(\boldsymbol{Q}) = 0$. Thus

$$\hat{L}_{i+1}(\boldsymbol{Q}/\boldsymbol{Z}) = \hat{L}_i(\boldsymbol{Z}) = \begin{cases} \hat{\boldsymbol{Z}}, & i = 0 \\ 0 & i > 0 , \end{cases}$$

and $\hat{L}_0(\boldsymbol{Q}/\boldsymbol{Z}) = 0$. These groups may be computed directly by noting that the map of $\boldsymbol{Q}/\boldsymbol{Z}$ to the circle group $U(1)$ given by $x \mapsto e^{2\pi i x}$ induces isomorphisms

$$H^*(CP^\infty; \boldsymbol{Z}/p) \longrightarrow H^*(K(\boldsymbol{Q}/\boldsymbol{Z}, 1); \boldsymbol{Z}/p)$$

for all p. We shall see that this example is typical.

THEOREM 5.8. *If A is an l-virtually nilpotent group which is locally of finite type with respect to l, then*

$$\hat{L}_n(A) = 0$$

for $n \geqq 2$.

Proof. Using (5.5) and (4.11) we are reduced to the case when A is abelian. By (4.5) we may assume $l = \{p\}$.

Suppose A is torsion free. Then ${}_pA = 0$. Let $e_1, \cdots e_n \in A$ be elements whose images in A/pA are a basis. Let A' be the free abelian group on symbols $[e_i]$, $i = 1, \cdots, n$ and let

$$f: A' \longrightarrow A$$

send $[e_i]$ to e_i. Then f induces isomorphisms $A'/pA' \xrightarrow{\sim} A/pA$. Hence it induces an isomorphism

$$H^*(A; \boldsymbol{Z}/p) \xrightarrow{\sim} H^*(A'; \boldsymbol{Z}/p) .$$

Therefore $K(A', 1)\hat{{}_p}$ and $K(A, 1)\hat{{}_p}$ are weakly equivalent. But then $\hat{L}_i(A) = \hat{L}_i(A') = 0$ for $i > 0$.

Now since an abelian group is the extension of a torsion free abelian group by a torsion abelian group, we may suppose A is all torsion. Again, since A/pA is finite, there is a finite subgroup J of A such that A/J is p-divisible. Since J is good, we may assume A is a p-divisible torsion group. Now ${}_pA$ is finite, so let $e_1, \cdots, e_m \in {}_pA$ be a basis. Since A is p-divisible, there is a group A' of the form

$$A' = \prod_{i=1}^{m} \boldsymbol{Z}/p^\infty$$

and a map $A' \to A$ inducing an isomorphism ${}_pA' \to {}_pA$. As before,

$$H^*(A; \boldsymbol{Z}/p) \longrightarrow H^*(A'; \boldsymbol{Z}/p)$$

is an isomorphism. By example (5.7), $\hat{L}_i(A) = 0$, $i \geqq 2$.

We mention the following characterization.

PROPOSITION 5.9. (*Kan-Bousfield* [5; VI 2.2]). *If A is abelian and locally of finite type relative to l, then*

$$d\hat{L}_0(A) = \operatorname{Ext}(\boldsymbol{Z}/l^\infty, A)$$
$$d\hat{L}_1(A) = \operatorname{Hom}(\boldsymbol{Z}/l^\infty, A)$$

where $\boldsymbol{Z}/l^\infty = \bigoplus_{p \in l} \boldsymbol{Z}/p^\infty$ and d denotes discretization.

For G not nilpotent, $\hat{L}_*(G)$ may be very complicated. For example, if $G = \sum_\infty$, the automorphisms of $\boldsymbol{N}$ with finite support, $\hat{L}_i(G)$ is the $(i+1)$-st stable homotopy group of S^0[12].

PROPOSITION 5.10.

(a) *If A is nilpotent and locally of finite type relative to l, then $\hat{A}$ acts nilpotently on $\hat{L}_*(A)$.*

(b) *If A is abelian, locally of finite type relative to l, G is a group acting nilpotently on A, and $H^1(G; Z/p)$ is finite for all $p \in l$, then $\hat{G}$ acts nilpotently on $\hat{L}_*(A)$.*

Proof. Note that if G acts nilpotently on A, then applying (4.3) to the fibration

$$K(A, 1) \longrightarrow K\left(A \int G, 1\right) \longrightarrow K(G, 1) ,$$

where $A \int G$ is the semi-direct product of A and G, shows that $\hat{G}$ acts naturally on $\hat{L}_*(A)$. Note also that if $0 \subseteqq A^n \subseteqq \cdots \subseteqq A^1 \subseteqq A$ is a filtration of A by G-subgroups which are locally of finite type, then the action of $\hat{G}$ respects the long exact sequences of the resulting tower of fibrations. By (4.13) such a filtration exists with G acting trivially on A^i/A^{i+1}, $1 \leqq i \leqq n$. Part (b) then follows easily. The proof of part (a) is similar.

Our main result is the following.

THEOREM 5.11. *Let $X \in \mathscr{S}_0$ be l-virtually nilpotent and locally of finite type relative to l. Then there are natural short exact sequences*

$$0 \longrightarrow (\pi_n X)\hat{}_l \longrightarrow \pi_n \hat{X}_l \longrightarrow \hat{L}_1(\pi_{n-1} X) \longrightarrow 0$$

for $n \geqq 1$.

COROLLARY 5.12. *If* X *is as above,* $\hat{X}_l$ *is virtually nilpotent.*

Proof of 5.11. Consider the Moore-Postnikov tower of X,

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & P^{n+1}X & \longrightarrow & P^nX & \longrightarrow \cdots \longrightarrow & P^2X & \longrightarrow & P^1X \\
& \nearrow & & \nearrow & & \nearrow & & \| \\
& K(\pi_{n+1}X, n+1) & & K(\pi_nX, n) & & K(\pi_2X, 2) & & K(\pi_1X, 1)\ .
\end{array}$$

By (4.15) each of the spaces above is locally of finite type with respect to l, and each fibration is l-virtually nilpotent. Thus, by (4.3), l-completion yields a tower of fibrations with fibres $K(\pi_nX, n)\hat{{}_l}$. The homotopy exact couple of the tower of completions then yields a strongly convergent spectral sequence

$$E^1_{p,q} \Longrightarrow \pi_{p+q}X$$

with

$$E^2_{p,q} = \hat{L}_p(\pi_qX)\ .$$

Now $\hat{L}_p(\pi_qX) = 0$ for $p > 1$. Thus the spectral sequence collapses to the exact sequence of 4.11.

6. Realizing profinite spaces as completions.

Let X be a rigid profinite space. Then the natural map $d\hat{X} \to \hat{X}$ induces a map of rigid profinite spaces

$$(d\hat{X})\hat{} \longrightarrow \hat{X}\ .$$

We will say X is *intrinsic* if this map is a weak equivalence. In this section we attempt to characterize intrinsic rigid profinite spaces. A complete answer will be given for the nilpotent case.

In order that a rigid profinite space be intrinsic, its homotopy groups must have a similar property. Let G be a profinite group. We say that G is *intrinsically topologized* if the natural map

$$(dG)\hat{} \longrightarrow G$$

is an isomorphism. This notion first appeared in Sullivan [18]. It is easy to see that G is intrinsically topologized iff every normal subgroup of finite index in G is open.

EXAMPLE 6.1 [18]. Let $G = \prod_{i \in N} \boldsymbol{Z}/p$, N the natural numbers. Let $H = \bigoplus_{i \in N} \boldsymbol{Z}/p$ be considered an abstract subgroup of G. Then H is dense in G, and G/H is a $\boldsymbol{Z}/p$ vector space. Hence there is a nontrivial homomorphism $G/H \to \boldsymbol{Z}/p$. Let K be the kernel of the composition $G \to G/H \to \boldsymbol{Z}/p$. Then if K is open, $K = G$ since $H \subseteqq K$ and H is dense. But $K \neq G$, contradiction. Thus G is not intrinsic.

This example is typical.

THOREM 6.2. *Let G be a pro-p group. Then G is intrinsically topologized iff G is finitely generated.*

Proof. Let G^* be the Frattini subgroup of G [14; p. 70]. If G is intrinsic, so is G/G^*. Now G/G^* is an abelian pro-p group dual to $H^1(G; \boldsymbol{Z}/p\boldsymbol{Z})$. Thus for some index set X,

$$G/G^* = \prod_X \boldsymbol{Z}/p \ ;$$

and, by Example 6.1, G/G^* is intrinsic only if X is finite. But if G/G^* is finitely generated so is G [14; Cor. 1, p. 72].

To prove the converse, it suffices to show that the p-completion of a finitely generated free group is intrinsic. But that is just [4; Thm. 13.3].

THEOREM 6.3. *A pro-nilpotent group $G \in \hat{\mathfrak{G}}$ is intrinsically topologized iff each p-Sylow subgroup is finitely generated.*

Proof. This theorem follows from (6.2) and the following more general fact.

LOCALIZATION LEMMA 6.4. *If G is a profinite group, $N \subseteqq G$ a normal subgroup of finite index n, then N is open iff $N \cap G_p$ is open in G_p for all primes $p|n$, where G_p is any p-Sylow subgroup.*

Proof. If N is open, then $N \cap G_p$ is open in G_p. Conversely, suppose $N \cap G_p$ is open in G_p for $p|n$. Then for each such p, there are open normal subgroups $K_p \subseteqq N \cap G_p$. Let $K = \bigcap_{p|n} K_p$. Then K is open. If $K \subseteqq N$, N will be open. Let $p_1, \cdots, p_r$ be the prime factors of n. We claim that each $g \in K$ may be written

$$g = g_0 g_1 \cdots g_r$$

where $g_i \in G_{p_i} \cap K$ for $i = 1, \cdots, r$, G_{p_i} some p_i-Sylow subgroup, and g_0 is n-divisible. Since K and N are normal, $g_1, \cdots, g_r \in N$. Since g_0 is n-divisible, $g_0 \in N$. Thus $g \in N$ and N is open. The claim is the following lemma applied to K.

LEMMA 6.5. *Let G be a profinite group, $g \in G$, $l = \{p_1, \cdots, p_r\}$ a finite set of primes, and n an integer divisible only by the primes* *in l. Then g may be written*

$$g = g_0 g_1 \cdots g_r ,$$

where g_0 *is n-divisible and, for* $1 \leqq i \leqq r$, $g_i \in G_{p_i}$ *for some* p_i*-Sylow subgroup* G_{p_i}.

The proof is a lengthy but straightforward calculation using elementary number theory.

Consider a finitely generated pro-abelian p-group A. Then A has a natural structure as a Z_p-module, Z_p the p-adic integers. Since Z_p is a principal ideal domain,

$$A = \underbrace{Z_p \oplus \cdots \oplus Z_p}_{r\text{-copies}} \oplus F$$

where F is a finite p-group. Thus $H^*(A; Z/p)$ has finite type. Theorem 6.3 now implies

COROLLARY 6.6. *A nilpotent profinite group G is intrinsically topologized iff it is locally of finite type.*

PROPOSITION 6.7. *If G is a nilpotent profinite group which is locally of finite type, then dG is good.*

Proof. First let A be a finitely generated pro-abelian p-group. By the remarks above, to see that A is good, it suffices to see that Z_p is good. But Z_p is n-divisible for $p \neq n$, torsion free, and $Z_p/pZ_p = Z/p$. Thus

$$H^*(dZ_p; Z/q) = H^*(Z_p; Z/q)$$

for all primes q. In particular, if $q \neq p$, $H^*(dZ_p; Z/q) = 0$.

Now in general, if G nilpotent, $G = \Pi G_p$ and each G_p is a finitely generated nilpotent pro-p group. Simple calculations with the Künneth theorem and the Hochschild-Serre spectral sequence show that G is good.

Our main result now follows from the results of § 5.

THEOREM 6.8. *Let X be a virtually nilpotent rigid profinite space. Then X is intrinsic iff it is locally of finite type.*

It is not known if (6.8) is true for pro-nilpotent X. For it is not known whether the p-completion of a finitely generated free group has a good discretization.

COROLLARY 6.9. *If X is an Artin-Mazur profinite space which is virtually nilpotent and locally of finite type, then X is weakly equivalent to a rigid pro-finite space.*

Proof. By Sullivan's argument [17; p. 35–38] the functor $\varprojlim [\ ;X_i]$ on spaces is representable by a space $\hat{S}X$. Then $(\hat{S}X)^\wedge$ will be weakly equivalent to X by the same argument which proves 6.8.

7. Proof of Theorem 4.3.

We will follow the proof by Artin-Mazur [2; pp. 60–68] of Theorem 4.2, paying careful attention to the actions of fundamental groups.

Let

$$* \longrightarrow A \longrightarrow D \xrightarrow{f} C \longrightarrow *$$

be an exact sequence of simplicial groups. Let $\mathscr{D}$ be the category of diagrams i of the form

$$\begin{array}{ccc} D & \xrightarrow{f} & C \\ \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle \gamma_i} \\ D_i & \xrightarrow{f_i} & C_i \end{array} \tag{7.1}$$

where D_i and C_i are simplicial l-groups; a map $i \to j$ between objects is a map of diagrams which is the identity on C and D. Such a map of diagrams is necessarily unique.

LEMMA 7.2. *Let $\mathscr{C}$ be the full subcategory of $\mathscr{D}$ consisting of objects* (7.1) *satisfying* $f(\ker \beta_i) = \ker \gamma_i$. *Then:*

(1) *For each object of $\mathscr{C}$,*

(i) $\beta_i(\ker f) = \ker f_i$,

(ii) *the kernels of* $\ker f \to \ker f_i$ *and* $\ker \beta_i \to \ker \gamma_i$ *are equal, and*

(iii) *the natural map $D \to D_i \times_{C_i} C$ is epic.*

(2) *$\mathscr{C}$ is cofinal in $\mathscr{D}$.*

(3) *The pro-simplicial finite groups represented by $\{D_i\}_{i \in \mathscr{C}}$ and $\{C_i\}_{i \in \mathscr{C}}$ are equivalent to $\hat{D}$ and $\hat{C}$ respectively.*

Proof. (1) is an elementary verification. (2) follows by noting that, for any object i of $\mathscr{D}$, $C/f(\ker \beta_i)$ is a simplicial finite l-group; therefore, the object $j \in \mathscr{C}$ with $D_j = D_i$, $C_j = C/f(\ker \beta_i)$, maps to i. For a similar reason, $\{D_i\}_{i \in \mathscr{C}}$ represents $\hat{D}$. To finish (3), let $C \xrightarrow{\gamma} F$, F a finite l-group. Then the object i with $C_i = F$, $D_i =$

$D/f^{-1}(\ker \gamma)$ is an object of $\mathscr{C}$; thus, $\{C_i\}_{i\in\mathscr{C}}$ represents $\hat{C}$.

Let

$$F \longrightarrow E \longrightarrow B$$

be a fibration of reduced spaces satisfying the hypotheses of (4.3). We may assume that $E \to B$ is of the form $\bar{W}f\colon \bar{W}D \to \bar{W}C$, where $f\colon D \to C$ is a map of free simplicial groups. Applying $\bar{W}$ to the category $\mathscr{C}$ of (7.2), we construct a family of commuting diagrams of the form (7.3).

$$\begin{array}{ccccc} \varnothing_i & \longrightarrow & \eta_i & \longrightarrow & \beta_i \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ \bar{F}_i & \longrightarrow & \hat{E}_i & \longrightarrow & \hat{B}_i \end{array} \tag{7.3}$$

These diagrams satisfy

(a) each row and column is a fibration.

(b) each space is reduced and fibrant.

(c) $\{\bar{F}_i\}$, $\{\hat{E}_i\}$, and $\{\hat{B}_i\}$, $i\in\mathscr{C}$, are rigid pro-l spaces and represent $\bar{F}$, $\hat{E}$, and $\hat{B}$, respectively.

(d) E is fibred over $\hat{E}_i \times_{\hat{B}_i} B$.

(e) all maps in the diagram

$$\begin{array}{ccc} \pi_1 E & \longrightarrow & \pi_1 B \\ \downarrow & & \downarrow \\ \pi_1 \hat{E}_i & \longrightarrow & \pi_1 \hat{B}_i \end{array}$$

are epic.

(f) $\pi_1\eta_i \to \pi_1\beta_i$ is epic.

Let $\varnothing = \{\varnothing_i\}$, $\eta = \{\eta_i\}$, $\beta = \{\beta_i\}$, $i\in\mathscr{C}$. By [2; Prop. 5.1], $\hat{\eta}_l$ and $\hat{\beta}_l$ are weakly contractible. Our strategy in proving (4.3) is to first show that $\hat{\varnothing}_l$ is weakly contactible; then, we will use that fact to show that $\hat{F} \to \bar{F}$ is a weak equivalence. We begin by showing that

$$\tilde{H}^*(\varnothing; \boldsymbol{Z}/p) = 0$$

for all $p \in l$.

Let N be some large integer. Since $F \to E \to B$ is an l-virtually nilpotent fibration, for $i\in\mathscr{C}$ sufficiently large — i.e., for all i in a cofinal subcategory of $\mathscr{C}$ — we may assume that $\operatorname{Im}(\pi_1\eta_i \to \pi_1 E)$ acts nilpotently on $\pi_n F$ for $n \leq N$.

LEMMA 7.4. *For all* $i \in \mathscr{C}$, $\pi_1\eta_i$ *acts trivially on* $\operatorname{Im}(\partial: \pi_{n+1}\bar{F}_i \to \pi_n\varnothing_i)$, $n \geqq 2$.

Proof. Let $\bar{\alpha}: (S^n \times I, S^n \times \dot{I} \cup \{*\} \times I) \to (\bar{F}_i, *)$, where $I = \Delta[1]$ is the simplicial unit interval and S^n the standard n-sphere, represent $[\bar{\alpha}] \in \pi_{n+1}\bar{F}_i$, and let $\beta: (I, \dot{I}) \to (\eta_i, *)$ represent $[\beta] \in \pi_1\eta_i$. Lift $\bar{\alpha}$ to a map $\alpha: (S^n \times I, S^n \times \{0\} \cup \{*\} \times I) \to (F_i, *)$. Then $\alpha | S^n \times \{1\}$ represents $\partial([\bar{\alpha}])$. Let $\bar{\beta}$ denote β composed with $E \to B$. We will construct a map $\gamma: S^n \times I \times I \to E$ as follows. Let $\gamma | S^n \times I \times \dot{I}$ be α composed with projection onto $S^n \times I$; let $\gamma | (S^n \times \{0\} \cup \{*\} \times I) \times I$ be β composed with projection on I. By the covering homotopy extension theorem, γ may be extended to a map $S^n \times I \times I \to$ covering $\bar{\alpha} \times \bar{\beta}: S^n \times I \times I \to \hat{E}_i \times_{\hat{B}_i} B_i$. Then $\gamma(S^n \times \{1\} \times I) \subseteqq \eta_i$, and $\gamma|_{S^n \times \{1\} \times \{1\}}$ represents ${}^{[\beta]}\partial([\bar{\alpha}]) \in \pi_n\varnothing_i$. But $\gamma|_{S^n \times \{1\} \times \{1\}}$ also represents $\partial([\bar{\alpha}])$.

COROLLARY 7.5. *For* $i \in \mathscr{C}$ *sufficiently large,* $\pi_1\eta_i$ *acts nilpotently on* $\pi_n\varnothing_i$, $n \leqq N$.

Proof. For i sufficiently large,

$$0 \longrightarrow \operatorname{Im}\partial \longrightarrow \pi_n\varnothing_i \longrightarrow \pi_n F$$

is an exact sequence of $\pi_1\eta_i$ modules with $\operatorname{Im}\partial$ and $\pi_n F$ nilpotent.

COROLLARY 7.6. *For* $i \in \mathscr{C}$ *sufficiently large,* $\pi_1\beta_i$ *acts nilpotently on* $H^n(\varnothing_i; \boldsymbol{Z}/p)$, $n \leqq N$.

Proof. [5; I 5.4].

LEMMA 7.7. *Let* $X \to Y \to Z$ *be a fibration of connected spaces such that*

(i) $\pi_1 Z$ *is a finite l-group,*

(ii) Z *is of finite type, and*

(iii) $H^n(Y; M)$ *is finite for all* $n \geqq 0$ *and finite l-coefficient systems* M.

Then $H^n(X; Z/p)$ *is finite for all* $n \geqq 0$ *and all* $p \in l$.

Proof. Let $* \to \pi \to \pi_1 Y \to \pi_1 Z \to *$ be exact, Y_π the covering space of Y corresponding to π, and $\tilde{Z}$ the universal cover of Z. Then

$$X \longrightarrow Y_\pi \longrightarrow \tilde{Z}$$

is a fibration. By (iii), $H^n(Y_\pi; \boldsymbol{Z}/p)$ is finite for all $n \geqq 0$ and $p \in l$.

Furthermore, $\tilde{Z}$ is of finite type. A simple Serre spectral sequence argument implies that $H^n(X; \boldsymbol{Z}/p)$ is finite for all $n \geqq 0$.

COROLLARY 7.8. *If B (respectively F) is locally of finite type with respect to l, then $H^n(\beta_i; \boldsymbol{Z}/p)$ (respectively $H^n(\varnothing_i; \boldsymbol{Z}/p)$) is finite for all $n \geqq 0$ and $p \in l$.*

LEMMA 7.9. *Let M be a $\boldsymbol{Z}/p$ vector space, $p \in l$ and suppose either*

(i) M is finite dimensional, or

(ii) B is locally of finite type.

Then

$$\tilde{H}^*(\beta; M) = 0 .$$

Proof. $\tilde{H}^*(\beta; \boldsymbol{Z}/p) = 0$ [2; Prop. 5.1]. If M has finite dimension, the result follows immediately. Suppose B is locally of finite type. Then $H^n(\beta_i; \boldsymbol{Z}/p)$ is finite for all i, n. Write

$$M = \varinjlim M_j$$

where $\{M_j\}_{j \in J}$ is a direct system of finite $\boldsymbol{Z}/p$-vector spaces. Then

$$\begin{aligned} H^n(\beta_i; M) &= \operatorname{Hom}(H_n(\beta_i; \boldsymbol{Z}/p), M) \\ &= \operatorname{Hom}(H_n(\beta_i; \boldsymbol{Z}/p), \varinjlim M_j) \\ &= \varinjlim (\operatorname{Hom}(H_n(\beta_i; \boldsymbol{Z}/p), M_j)) \end{aligned}$$

since $H_n(\beta_i; \boldsymbol{Z}/p)$ is finite dimensional. But then

$$\begin{aligned} \tilde{H}^n(\beta; M) &= \varinjlim_i \tilde{H}^n(\beta_i; M) \\ &= \varinjlim_i \varinjlim_j \tilde{H}^n(\beta_i; M_j) \\ &= \varinjlim_j \tilde{H}^n(\beta; M_j) \\ &= 0 . \end{aligned}$$

LEMMA 7.10. $\tilde{H}^*(\varnothing; \boldsymbol{Z}/p) = 0$, $p \in l$.

Proof. Consider the directed system of Serre spectral sequences

$$E_2^{r,s} = H^r(\beta_i; H^s(\varnothing_i; \boldsymbol{Z}/p)) \Longrightarrow H^{r+s}(\eta_i; \boldsymbol{Z}/p) .$$

Taking direct limits, we obtain a spectral sequence

$$E_2^{r,s} = \varinjlim_i H^r(\beta_i; H^s(\varnothing_i; \boldsymbol{Z}/p)) \Longrightarrow H^{r+s}(\eta; \boldsymbol{Z}/p) .$$

By [2; Prop. 1.10, p. 152],

$$\begin{aligned} E_2^{r,s} &= \varinjlim_i H^r(\beta_i; H^s(\varnothing_i; \boldsymbol{Z}/p)) \\ &= \varinjlim_i \varinjlim_{j\to i} H^2(\beta_j; H^s(\varnothing_i; \boldsymbol{Z}/p)) \\ &= \varinjlim_i H^r(\beta; H^s(\varnothing_i; \boldsymbol{Z}/p)) . \end{aligned}$$

We assert that $H^r(\beta; H^s(\varnothing_i; \boldsymbol{Z}/p)) = 0$ for $r > 0$. For when i sufficiently large $H^s(\varnothing_i; \boldsymbol{Z}/p)$ has a filtration

$$0 \subseteqq A_\nu \subseteqq \cdots \subseteqq A_1 = H^s(\varnothing_i; \boldsymbol{Z}/p)$$

of $\pi_1\beta_i$ modules such that each $A_\tau/A_{\tau+1}$ is a trivial $\pi_1\beta_i$ module. Since $\pi_1\beta$ acts on $H^s(\varnothing_i; \boldsymbol{Z}/p)$ through $\pi_1\beta_i$, (7.9) implies

$$H^r(\beta; A_\tau/A_{\tau+1}) = 0 ,$$

$r > 0$. From the long exact sequence induced by

$$0 \longrightarrow A_{\tau+1} \longrightarrow A_\tau \longrightarrow A_\tau/A_{\tau+1} \longrightarrow 0 ,$$

we see that

$$H^r(\beta; H^s(\varnothing_i; \boldsymbol{Z}/p)) = 0 ,$$

$r > 0$. By the same argument,

$$H^0(\beta; H^s(\varnothing_i; \boldsymbol{Z}/p)) = H^s(\varnothing_i; \boldsymbol{Z}/p) ,$$

since $H^0(\beta; A_\tau/A_{\tau+1}) = A_\tau/A_{\tau+1}$. Thus,

$$E_2^{r,s} = \begin{cases} H^s(\varnothing; \boldsymbol{Z}/p), & r = 0 \\ 0, & r > 0 . \end{cases}$$

But then

$$\tilde{H}^s(\varnothing; \boldsymbol{Z}/p) = \tilde{H}^s(\eta; \boldsymbol{Z}/p) = 0$$

[2; Prop. 5.1].

LEMMA 7.11. *If M is an l-coefficient system on $\bar{F}$, then $H^r(\bar{F}; \boldsymbol{H}^s(\varnothing; M)) = 0$ for $s > 0$. Therefore,*

$$H^r(\bar{F}; M) \xrightarrow{\sim} H^r(F; M)$$

for $r \geqq 0$.

Proof. See [2; p. 67].

LEMMA 7.12. $(\pi_1\varnothing)\hat{}_l = 0$.

Proof. Let G be a finite l-group, $\alpha_i: \pi_1\varnothing_i \to G$. Since $\pi_i\varnothing$ is nilpotent, we may suppose G is nilpotent. Therefore, we may suppose G is abelian. But then α_i represents an element of $H^1(\varnothing; G)$ and is thus zero on some $\pi_1\varnothing_j$, $j \to i$.

The proof of Theorem 4.3 is now completed by the argument of [2; p. 67-68].

REFERENCES

1. M. P. Anderson, *Exactness properties of profinite completion functors*, Topology, **13** (1974), 229-240.
2. M. Artin and B. Mazur, *Étale Homotopy*, Lecture notes in Mathematics No. 100, Springer-Verlag, 1969.
3. N. Bourbaki, *Topologie Général*, 3 è meédit, Hermann, Paris.
4. A. K. Bousfield, *Homological localization towers for groups and π-modules*, Memoirs Amer. Math. Soc., **10** (1977), No. 186.
5. A. K. Bousfield and D. M. Kan, *Homotopy limits completions and localizations*, Lecture Notes in Mathematics No. 304, Springer-Verlag, 1972.
6. H. Cartan, *Algèbre d'Eilenberg-MacLane et homotopie*, Seminaire Henri Cartan, 2e année, 1956.
7. A. Dold and D. Puppe, *Homologie nicht-additiver functoren. Anwendungen*, Ann. Inst. Fourier, Grenoble, **11** (1961), 201-312.
8. E. M. Friedlander, *Computations of K-theories of finite fields*, Topology, **15** (1976), 87-109.
9. M. Hall, Jr., *The Theory of Groups*, Macmillan, New York, 1959.
10. R. Hoobler and D. L. Rector, *Arithmetic K-theory*, Localization in Group Theory and Homotopy Theory, Lecture Notes in Mathematics, No. 418, Springer-Verlag, 1974, 78-95.
11. P. May, *Simplicial Objects in Algebraic Topology*, Van Nostrand, 1967.
12. S. Priddy, *On $\Omega^\infty S^\infty$ and the infinite symmetric group*, Proc. Symp. Pure Math. Amer. Math. Soc., **22** (1971), 217-220.
13. D. G. Quillen, *An application of simplicial profinite groups*, Comm. Math. Helv., **44** (1969), 45-60.
14. S. Shatz, *Profinite Groups, Arithmetic, and Geometry*, Ann. of Math. Studies, No. 67, Princeton University Press, 1972.
15. J. P. Serre, *Cohomologie Galoisienne*, Lecture Notes in Mathematics, No. 5, Springer-Verlag, 1965.
16. ———, *Représentations Linéaires des Groupes Finis*, Hermann, Paris, 1967.
17. D. Sullivan, *Genetics of homotopy theory and the Adams conjecture*, Ann. of Math., **100** (1974), 1-79.
18. ———, *Geometric Topology, part I, Localization, Periodicity and Galois Symmetry*, notes, M.I.T., 1970.

Received November 27, 1978.

UNIVERSITY OF CALIFORNIA
IRVINE, CA 92717