PACIFIC JOURNAL OF MATHEMATICS
Vol. 85, No. 2, 1979

COMPACT AND WEAKLY COMPACT DERIVATIONS OF C^*-ALGEBRAS

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In a forthcoming paper, the second-named author asks if every compact derivation of a C^*-algebra $\mathscr{A}$ into a Banach $\mathscr{A}$-module X is the uniform limit of finite-rank derivations. We answer this question affirmatively in the present paper when $X = \mathscr{A}$ by characterizing the structure of compact derivations of C^*-algebras. In addition, the structure of weakly compact derivations of C^*-algebras is determined, and as immediate corollaries of these results, necessary and sufficient conditions are given for a C^*-algebra to admit a nonzero compact or weakly compact derivation.

To fix our notation, we we recall some basic definitions. A derivation of a C^*-algebra $\mathscr{A}$ is a linear map $\delta: \mathscr{A} \to \mathscr{A}$ for which $\delta(ab) = a\delta(b) + \delta(a)b$, $a, b \in \mathscr{A}$. If $x \in \mathscr{A}$, the map $a \to ax - xa$, $a \in \mathscr{A}$, defines a derivation of $\mathscr{A}$ which we denote by adx.

By an ideal of a C^*-algebra, we always mean a uniformly closed, two-sided ideal.

A C^*-algebra $\mathscr{A}$ is said to *act atomically* on a Hilbert space H if there exists an orthogonal family $\{P_\alpha\}$ of projections in $B(H)$, each commuting with $\mathscr{A}$, such that $\bigoplus_\alpha P_\alpha$ is the identity operator on H, $\mathscr{A}P_\alpha$ acts irreducibly on $P_\alpha(H)$, and $\mathscr{A}P_\alpha$ is not unitarily equivalent to $\mathscr{A}P_\beta$ for $\alpha \neq \beta$.

If $\{\mathscr{A}_n\}$ is a sequence of C^*-algebras, $\bigoplus_n \mathscr{A}_n$ denotes the C^*-direct sum of the $\mathscr{A}_n$'s, i.e., $\bigoplus_n \mathscr{A}_n$ is the C^*-algebra of all uniformly bounded sequences $\{a_n\}$, $a_n \in \mathscr{A}_n$, equipped with pointwise operations and the norm $\|\{a_n\}\| = \sup_n \|a_n\|$. $\hat{\bigoplus_n} \mathscr{A}_n$ denotes the C^*-subalgebra of $\bigoplus_n \mathscr{A}_n$ formed by all sequences $\{a_n\}$ with $\|a_n\| \to 0$.

Acknowledgment. The second-named author wishes to express his deep gratitude to V. S. Sunder for the warm hospitality he extended to that author during a stay at the University of California at Santa Barbara, which resulted in the initiation of the present work.

2. Compact derivations. The following lemma is due to Ho ([3], Corollary 1):

LEMMA 2.1. *Let H denote an infinite dimensional Hilbert space, $B(H)$ the algebra of all bounded linear operators on H. If δ is a compact derivations of $B(H)$, then $\delta \equiv 0$.*

Let $M_n = n \times n$ complex matrices, and let $\mathscr{A} = \bigoplus_n^{\wedge} M_n$ denote the restricted C^*-direct sum of $\{M_n\}_{n=1}^{\infty}$. If $x = (x_n) \in \mathscr{A}$, then adx is a compact derivation of $\mathscr{A}$ and is the uniform limit of the finite-rank derivations $\delta_n = ad(x_1, \cdots, x_n, 0, 0, \cdots)$. The following theorem, which determines the structure of compact derivations of C^*-algebras, shows that this seemingly very special example actually typifies the behavior of an arbitrary compact derivation.

Recall that a projection p of a C^*-algebra $\mathscr{A}$ is said to be *finite-dimensional* if $p\mathscr{A}p$ is finite-dimensional, and *has dimension* n if $p\mathscr{A}p$ has dimension n.

THEOREM 2.2. *Let $\mathscr{A}$ be a C^*-algebra, $\delta: \mathscr{A} \rightarrow \mathscr{A}$ a compact derivation. Then there is an orthogonal sequence $\{x_n\}$ of minimal, finite-dimensional, central projections of $\mathscr{A}$ and an element d of $\mathscr{A}$ such that $\delta = ad\,d$ and $\sum_n x_n d$ converges uniformly to d.*

Proof. Let π denote the reduced atomic representation of $\mathscr{A}$ ([5], p. 35). π is constructed as follows: partition the class of irreducible representations of $\mathscr{A}$ according to unitary equivalence, and from each equivalence class, choose a representation π_α, acting on a Hilbert space H_α. Then $\pi = \bigoplus_\alpha \pi_\alpha$, with π acting on $H = \bigoplus_\alpha H_\alpha$. Since π is a faithful *-representation of $\mathscr{A}$, we may hence assume with no loss of generality that $\mathscr{A}$ acts atomically on a Hilbert space $H = \bigoplus_\alpha H_\alpha$.

Letting $\mathscr{A}^-$ denote the closure of $\mathscr{A}$ in the weak operator topology, we assert that δ extends to a compact derivation $\tilde{\delta}$ of $\mathscr{A}^-$. Identifying $\mathscr{A}$ in the usual way with a subalgebra of $\mathscr{A}^{**}$, the enveloping von Neumann algebra of $\mathscr{A}$, we may extend the inclusion $\mathscr{A} \hookrightarrow \mathscr{A}^{**}$ to a representation π_w of $\mathscr{A}^{**}$ onto $\mathscr{A}^-$ which is $\sigma(\mathscr{A}^{**}, \mathscr{A}^*)$-ultraweakly continuous ([6], p. 53). Let $1 - z$ be the support projection of $\ker \pi_w$. Then z is central in $\mathscr{A}^{**}$ and $\mathscr{A}^{**}z$ is isomorphic to $\mathscr{A}^-$ via the isomorphism $az \rightarrow \pi_w(a)$, $a \in \mathscr{A}^{**}$. Now $\delta^{**}|_{\mathscr{A}^{**}z}$ is a compact derivation of $\mathscr{A}^{**}z$. Thus, if we define $\tilde{\delta}: \mathscr{A}^- \rightarrow \mathscr{A}^-$ by

$$\tilde{\delta}: \pi_w(az) \longrightarrow \pi_w(z\delta^{**}(a)),\ a \in \mathscr{A}^{**} ,$$

it follows that $\tilde{\delta}$ is a compact derivation of $\mathscr{A}^-$ which extends δ.

Since $\mathscr{A}$ acts atomically on $H = \bigoplus_\alpha H_\alpha$, by Corollary 4 of [2], $\mathscr{A}^- = \bigoplus_\alpha B(H_\alpha)$. Let $q_\alpha =$ the orthogonal projection of H onto H_α. Since $\tilde{\delta}$ is ultraweakly continuous and q_α commutes with $\delta(\mathscr{A}) = \tilde{\delta}(\mathscr{A})$, q_α commutes with $\tilde{\delta}(\mathscr{A}^-)$, so that if $\tilde{\delta}_\alpha$ denotes the restriction of $\tilde{\delta}$ to $B(H_\alpha)$, then $\tilde{\delta} = \bigoplus_\alpha \tilde{\delta}_\alpha$.

Since $\tilde{\delta}$ is compact, its restriction $\tilde{\delta}_\alpha$ is a compact derivation of $B(H_\alpha)$. Furthermore, the compactness of $\tilde{\delta}$ implies that for each $\varepsilon > 0$, $\{\alpha: \|\tilde{\delta}_\alpha\| > \varepsilon\}$ is finite (see Lemma 3.2 in the next section). In particular, $\{\alpha: \|\tilde{\delta}_\alpha\| > 0\}$ is countable, say $\{\alpha_n\}_{n=1}^\infty$, and setting $\tilde{\delta}_n = \tilde{\delta}_{\alpha_n}$, we have $\lim_n \|\tilde{\delta}_n\| = 0$. Since $\tilde{\delta}_n \not\equiv 0$, we conclude by Lemma 2.1 that H_{α_n} is finite-dimensional for each n.

We assert next that $\tilde{\delta}(\mathscr{A}^-) \subseteqq \mathscr{A}$. This will follow from the identification of $\mathscr{A}^-$ with $\mathscr{A}^{**}z$ via π_w as defined above, provided $\delta^{**}(\mathscr{A}^{**}) \subseteqq \mathscr{A}$. But by the Kaplansky density theorem, the unit ball $\mathscr{A}_1^{**}$ of $\mathscr{A}^{**}$ is the $\sigma(\mathscr{A}^{**}, \mathscr{A}^*)$-closure of the unit ball $\mathscr{A}_1$ of $\mathscr{A}$, and so by [1], Theorem 6, p. 486, and the compactness of δ^*,

$$\begin{aligned}\delta^{**}(\mathscr{A}_1^{**}) &= \delta^{**}(\sigma(\mathscr{A}^{**}, \mathscr{A}^*)\text{-closure of } \mathscr{A}_1) \\ &\subseteqq \text{uniform closure of } \delta^{**}(\mathscr{A}_1) \\ &= \text{uniform closure of } \delta(\mathscr{A}_1)\end{aligned}$$

so that $\tilde{\delta}(\mathscr{A}^-) \subseteqq \mathscr{A}$.

Let $q_n = q_{\alpha_n}$. We claim that $q_n \in \mathscr{A}$. This is true since $\mathscr{A} \cap B(H_{\alpha_n})$ is a nonzero ideal of $B(H_{\alpha_n})$ (it contains the range of $\tilde{\delta}_n \not\equiv 0$, since $\tilde{\delta}(\mathscr{A}^-) \subseteqq \mathscr{A}$), whence $B(H_{\alpha_n}) \subseteqq \mathscr{A}$. Set $x_n = q_n$.

It follows that $\{x_n\}$ is an orthogonal sequence of minimal, finite-dimensional, central projections in $\mathscr{A}$. Choose $d_n \in B(H_{\alpha_n}) = \mathscr{A}x_n$ such that $\tilde{\delta}_n = ad\, d_n$ and $\|d_n\| \leqq \|\tilde{\delta}_n\|$. Since d_n is in $B(H_{\alpha_n})$, $\{d_n\}$ is an orthogonal sequence, and since $\|\tilde{\delta}_n\| \to 0$, $\sum_n d_n$ converges uniformly to an element $d \in \mathscr{A}$. But then

$$\begin{aligned}\delta = \tilde{\delta}|_a &= \bigoplus_n \tilde{\delta}_n|_a = \bigoplus ad\, d_n|_a \\ &= ad\Big(\bigoplus_n d_n\Big)|_a = ad\, d|_a\,.\end{aligned}$$

COROLLARY 2.3. *Every compact derivation of a C^*-algebra is the uniform limit of finite-rank derivations of that algebra.*

COROLLARY 2.4. *A C^*-algebra admits nonzero compact derivations if and only if it contains nonzero finite-dimensional central projections.*

Motivated by the concept of strong amenability of C^*-algebras, in [7] a derivation δ of a unital C^*-algebra $\mathscr{A}$ was called *strongly inner* if $\delta = ad\, x$ for x in the uniformly closed convex hull of $\{\delta(u)u^*: u \text{ a unitary element of } \mathscr{A}\}$. Thus by Corollary 2.3 above and Corollary 2.5 of [7], we deduce

COROLLARY 2.5. *Every compact derivation of a unital C^*-algebra is strongly inner.*

3. Weakly compact derivations. In this section, the structure of weakly compact derivations of C^*-algebras is determined.

Let H be a Hilbert space, $B(H)$ the algebra of bounded linear operators on H, and let $\mathscr{K}$ denote the ideal of compact operators in $B(H)$. The first theorem gives an analog of Ho's theorem for weakly compact derivations of $B(H)$.

THEOREM 3.1. *Let δ be a derivation of $B(H)$. The following are equivalent:*

(1) *δ is weakly compact.*

(2) *The range of δ is contained in $\mathscr{K}$.*

(3) *$\delta = ad\,T$ with $T \in \mathscr{K}$ (and $\|T\| \leqq \|\delta\|$).*

Proof. (1) $\Rightarrow$ (2). Since δ is inner, $\delta(\mathscr{K}) \subseteq \mathscr{K}$, and $\mathscr{K}^{**} = B(H)$, whence $\delta = (\delta|_{\mathscr{K}})^{**}$. Now $\delta|_{\mathscr{K}}$ is weakly compact, so by Theorem 2, p. 482 of [1], $\delta = (\delta|_{\mathscr{K}})^{**}$ maps $B(H)$ into $\mathscr{K}$.

(2) $\Rightarrow$ (3). This is an immediate consequence of Lemma 3.2 of [4].

(3) $\Rightarrow$ (1). By considering real and imaginary parts of T, we may assume that T is self-adjoint. Since T is compact, the spectral theorem allows us to approximate T uniformly by linear combinations of finite-rank projections, and so we may approximate $\delta = ad\,T$ uniformly by linear combinations of derivations of the form $ad\,p$, p a finite-rank projection. By [1], Corollary 4, p. 483, we may hence assume that T is a finite-rank projection. But then $\delta = ad\,T$ is a sum of derivations of the form $ad\,p$, p a rank-one projection, so we assume that $T = p$ is a rank-one projection.

Let X denote the Banach space $H \oplus H$ endowed with the norm $\|(x, y)\| = \max\{\|x\|, \|y\|\}$. Simple matricial computations show the existence of a one-dimensional subspace S of $pB(H) + B(H)p$ such that $(pB(H) + B(H)p)/S$ is isometrically Banach space isomorphic to X, and is hence reflexive. It follows that $pB(H) + B(H)p$ is reflexive. Since $\delta = ad\,p$ maps $B(H)$ into $pB(H) + B(H)p$, we conclude by [1], Corollary 3, p. 483 that δ is weakly compact.

LEMMA 3.2. *Let $\{H_\alpha\}$ be a family of Hilbert spaces, and let $\delta: \bigoplus_\alpha B(H_\alpha) \to \bigoplus_\alpha B(H_\alpha)$ be a weakly compact derivation. Then for all $\varepsilon > 0$, $\{\alpha: \|\delta|_{B(H_\alpha)}\| > \varepsilon\}$ is finite.*

Proof. Suppose the lemma is false. Then there exists a sequence $\{\alpha_n\}_{n=2}^{\infty}$ of indices and $a = (a_\alpha) \in \bigoplus_\alpha B(H_\alpha)$ such that $\|\delta(a)_{\alpha_n}\| > 1$, for all n.

Since any compression of δ is weakly compact, we can assume that δ acts on $\bigoplus_n B(H_{\alpha_n})$. Let $\bigoplus_n^{\wedge} B(H_{\alpha_n})$ denote the restricted direct sum of $\{B(H_{\alpha_n})\}$. Then since $\|\delta(a)_{\alpha_n}\| > 1$, for all n, there is a linear functional f such that $f(\delta(a)) = 1$ and f vanishes on $\hat{\bigoplus}_n B(H_{\alpha_n})$.

Define $\{b_k\} \subseteqq \bigoplus_n B(H_{\alpha_n})$ by

$$(b_k)_{\alpha_n} = \begin{cases} 0\,, & \text{if} \quad n < k\,, \\ a_{\alpha_n}\,, & \text{if} \quad n \geqq k\,. \end{cases}$$

Then $b_k \to 0$ in the weak operator topology (WOT), and so $\delta(b_k) \to 0$(WOT) (since δ is inner, it is WOT-continuous), whence by weak compactness of δ, $\delta(b_k) \to 0$ weakly. But $\delta(b_k) - \delta(a) \in \hat{\bigoplus}_n B(H_{\alpha_n})$, for all k, and so by the choice of f, $f(\delta(b_k)) = f(\delta(a)) = 1$, for all k, a contradiction.

The next theorem determines the structure of weakly compact derivations.

THEOREM 3.3. *Let $\delta: \mathscr{A} \to \mathscr{A}$ be a derivation of a C^*-algebra $\mathscr{A}$. The following are equivalent:*

(1) *δ is weakly compact.*

(2) *There exists a sequence $\{I_n\}$ of orthogonal ideals of $\mathscr{A}$ such that each I_n is isomorphic to the C^*-algebra $\mathscr{K}_n$ of compact operators on a Hilbert space H_n, and an element $d \in \hat{\bigoplus}_n I_n \subseteqq \mathscr{A}$ with $\delta = ad\, d$.*

Proof. (1) $\Rightarrow$ (2). We use Theorem 3.1 and Lemma 3.2 to adopt the proof of Theorem 2.2 to the present situation. As before, we may assume that $\mathscr{A}$ acts atomically on $H = \bigoplus_\alpha H_\alpha$. As in the proof of Theorem 2.2, we extend δ to a weakly compact derivation $\tilde{\delta}$ of $\mathscr{A}^-$ and use Lemma 3.2 to deduce the existence of a countable set $\{\alpha_n\}$ of indices such that $\tilde{\delta}_\alpha \equiv 0$ except when $\alpha = \alpha_n$, $\tilde{\delta}_n = \tilde{\delta}_{\alpha_n}$ is a weakly compact derivation of $B(H_{\alpha_n})$, and $\|\tilde{\delta}_n\| \to 0$. By the weak compactness of $\tilde{\delta}$ and [1], Theorem 2, p. 482, we also deduce as before that $\tilde{\delta}(\mathscr{A}^-) \subseteqq \mathscr{A}$. By Theorem 3.1, $\tilde{\delta}_n$ has range in $\mathscr{K}_n$ = compact operators in $B(H_{\alpha_n})$ and $\tilde{\delta}_n = ad\, c_n$ for $c_n \in \mathscr{K}_n$ with $\|c_n\| \leqq \|\tilde{\delta}_n\|$. Since $\mathscr{A} \cap \mathscr{K}_n$ is a nonzero ideal of $\mathscr{A} q_{\alpha_n} \supseteqq \mathscr{K}_n$ (it contains the range of $\tilde{\delta}_n \not\equiv 0$), $\mathscr{A} \cap \mathscr{K}_n$ is a nonzero ideal of $\mathscr{K}_n$, whence $\mathscr{K}_n \subseteqq \mathscr{A}$. Thus $I_n = \mathscr{K}_n$ and $d = \sum_n c_n$ satisfy the conditions of (2) for δ.

(2) $\Rightarrow$ (1). Since $d = \sum_n d_n \in \hat{\bigoplus}_n I_n$, $\delta = ad\, d$ is the uniform limit of the derivations $\delta_n = ad(\sum_1^n d_k)$, and so it suffices to show that each $ad\, d_k$ is weakly compact.

We suppress the k's and assume with no loss of generality that $d \geqq 0$. Theorem 3.1 implies that every inner derivation of $\mathscr{K}$ is weakly compact, and so $ad\, d|_I$ is weakly compact. It hence follows by induction and the formula

$$ad\, d^{n+1} = d^n ad\, d + (ad\, d^n)d$$

that $ad\,d^n|_I$ is weakly compact for all positive integers n. We conclude that $ad\,d^{1/2}|_I$ is weakly compact; but since

$$ad\,d(a) = ad\,d^{1/2}(ad^{1/2}) + ad\,d^{1/2}(d^{1/2}a)$$

and $d^{1/2}a$, $ad^{1/2} \in I$, for all $a \in \mathscr{A}$, it follows that $ad\,d$ is weakly compact on $\mathscr{A}$.

COROLLARY 3.4. *A C^*-algebra admits nonzero weakly compact derivations if and only if it contains a nonzero ideal isomorphic to the C^*-algebra of compact operators on a Hilbert space.*

The next two corollaries determine the von Neumann algebras which admit nonzero compact or weakly compact derivations. We first preface them with the following remarks.

Let R be a von Neumann algebra, and let $R = R_I \oplus R_{II} \oplus R_{III}$ be the decomposition of R into its type I, II, and III parts. Since R_I is type I, there exists a family $\{p_\alpha\}$ of pairwise orthogonal central projections in R_I such that $\bigoplus_\alpha p_\alpha =$ identity of R_I and $p_\alpha R_I \cong p_\alpha Z_I \otimes B(H_\alpha)$ ($\cong$ denotes isomorphism), where H_α is a Hilbert space and $Z_I =$ center of R_I ([6], §2.3). We set $R_\alpha = p_\alpha R_I (Z_\alpha = p_\alpha Z_I)$ and call $\{R_\alpha\}(\{Z_\alpha\})$ the *discrete components* of $R_I(Z_I)$.

COROLLARY 3.5. *A von Neumann algebra R admits a nonzero weakly compact derivation if and only if a discrete component of the center of its type* I *part contains a one-dimensional projection.*

Proof. ($\Rightarrow$). Let $\delta: R \to R$ be a nonzero weakly compact derivation. We show first that $\delta \equiv 0$ on R_{II} and R_{III}. Suppose $\delta \not\equiv 0$ on R_{II}. δ maps R_{II} into R_{II}, so $\delta|_{R_{II}}$ is a nonzero weakly compact derivation of R_{II}. Hence by Corollary 3.4, R_{II} contains an ideal $\mathscr{I}$ isomorphic to the C^*-algebra of compact operators on some Hilbert space H. If $\mathscr{I}^{-\sigma}$ denotes the ultraweak closure of $\mathscr{I}$, then $\mathscr{I}^{-\sigma}$ is an ultraweakly closed ideal of R_{II} such that $\mathscr{I}^{-\sigma} \cong \mathscr{I}^{**} \cong B(H)$, and so $\mathscr{I}^{-\sigma}$ is a type I direct summand of R_{II}, which is impossible. The same argument shows that $\delta \equiv 0$ on R_{III}.

We conclude that $\delta|_{R_I}$ is a nonzero weakly compact derivation of R_I, so there is no loss of generality by assuming that $R = Z \otimes B(H)$ for an abelian von Neumann algebra Z and a Hilbert space H. Applying Corollary 3.4 and reasoning as before, we find a projection $p \in Z$ such that $pZ \otimes B(H) \cong B(K)$ for some Hilbert space K. Thus $pZ \otimes B(H)$ is a factor, whence pZ is one-dimensional.

($\Leftarrow$). If p is a one-dimensional projection of the discrete component Z_α corresponding to $Z_\alpha \otimes B(H_\alpha)$ and T_α is a nonzero compact operator on H_α, it is immediate from Theorem 3.1 that $ad(p \otimes T_\alpha)$

is a nonzero weakly compact derivation of R.

COROLLARY 3.6. *A von Neumann algebra admits a nonzero compact derivation if and only if its type* I *part has a nonzero finite-dimensional discrete component.*

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Received November 17, 1978. The first author was partially supported by NSF grant MCS 78-01870. The second author was partially supported by an Oakland University faculty research fellowship.

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