DEHN'S LEMMA AND HANDLE DECOMPOSITIONS
OF SOME 4-MANIFOLDS

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We give two short proofs of a weak version of the theorem of Laudenbach, Poenaru [3]. Also we show that an embedded $S^1 \times S^2$ in $S^4$ bounds a copy of $B^2 \times S^2$. Finally we establish that if $W$ is a smooth 4-manifold with $\partial W = \#_n S^1 \times S^2$ and $W$ is built from $\#_{n-1} B^3 \times S^2$ by attaching a 2-handle, then $W$ is homeomorphic to $\#_n B^3 \times S^2$.

1. 4-Dimensional handlebodies. Let $X$, $Y$ be the following smooth 4-manifolds:

$$X = \#_n B^3 \times S^1 \quad \text{and} \quad Y = \#_n B^2 \times S^2.$$ 

In [3] it is proved that if $h : \partial X \rightarrow \partial Y$ is a diffeomorphism, then the smooth closed 4-manifold $X \cup_h Y$ which is obtained by gluing along $h$, is diffeomorphic to $S^4$.

We begin with two brief proofs, one using the Dehn's lemma in [5] and the other employing unknotting in codimension 3, of the following result:

**Theorem.** Let $X$, $Y$, $h$ be as above. Then $X \cup_h Y$ is homeomorphic to $S^4$.

**Proof.** (1) Let $\{x_i\} \times S^1$ be a circle in the boundary of the $i$th copy of $B^3 \times S^1$ in the connected sum $X = \#_n B^3 \times S^1$, for $1 \leq i \leq n$. Without loss of generality, all the loops $\{x_i\} \times S^1$ can be assumed to miss the cells which are used to construct $X$ as a connected sum. By the Dehn's lemma in [5], it follows that all of the circles $h(\{x_i\} \times S^1)$ bound disjoint smooth embedded disks $D_i$ in $Y$, for $1 \leq i \leq n$.

Let $N(D_i)$ denote a small tubular neighborhood of $D_i$ in $Y$. Clearly $X \cup_h (N(D_1) \cup \cdots \cup N(D_n))$ is diffeomorphic to $B^4$, since $N(D_i)$ can be thought of as a 2-handle which geometrically cancels a 1-handle of $X$. On the other hand, let $W$ denote the closure of $Y - N(D_1) - \cdots - N(D_n)$. Then $\partial W = S^3$ and $W$ is contained in $Y$ which can be embedded in $S^4$. By the topological Schoenflies theorem [1], $W$ is homeomorphic to $B^4$. Consequently $X \cup_h Y$ is homeomorphic to $B^4 \cup B^4 = S^4$.

(2) By Van Kampen's theorem, $\pi_1(X \cup_h Y) = \{1\}$. Let $Z$ be a bouquet of $n$ circles which is embedded in $X$ and is a deformation retract of $X$. By isotopic unknotting in codimension 3, $Z$ is con-
tained in the interior of a PL 4-cell $B$ in $X \cup_h Y$. Therefore, by an isotopy we can shrink $X$ down towards $Z$ until $X$ is included in $\text{int } B$. Exactly as in (1), by the topological Schoenflies theorem we obtain that $X \cup_h Y - \text{int } B$ is homeomorphic to $B^4$ and so the result follows.

**Remark.** Note that if the PL or smooth 4-dimensional Schoenflies theorem was known, then these arguments would establish that $X \cup_h Y$ is PL isomorphic or diffeomorphic to $S^4$.

2. Embeddings of $S^1 \times S^2$ in $S^4$. The following result was first proved by I. Aitchison (unpublished). We present a simplification of his method, which again uses the Dehn's lemma in [5].

**Theorem.** Let $h : S^1 \times S^2 \to S^4$ be a smooth embedding. Then $h$ extends to a topological embedding of $B^2 \times S^2$ in $S^4$.

**Proof.** Let $V$, $W$ be the closures of the components of $S^4 - h(S^1 \times S^2)$ (by Alexander duality there are two such components). By the Mayer-Vietoris sequence, without loss of generality the inclusion $h(S^1 \times S^2) \to V$ induces an isomorphism $H_i(h(S^1 \times S^2)) \to H_i(V)$ and $H_i(W) = 0$.

Let $G$ denote the group which is the pushout of the homomorphisms $\pi_i(h(S^1 \times S^2)) \to \pi_i(V)$ and $\pi_i(h(S^1 \times S^2)) \to \pi_i(W)$. By Van Kampen's theorem, $G = \{1\}$. On the other hand there is a homomorphism of $G$ onto $\pi_i(W)$ induced by the epimorphism $\pi_i(V) \to H_i(V) \cong H_i(h(S^1 \times S^2)) \cong \pi_i(h(S^1 \times S^2))$. Consequently $\pi_i(W) = \{1\}$ follows.

Now we can apply the Dehn's lemma in [5] to obtain that $h(S^1 \times *)$ bounds a smooth embedded disk $D$ in $W$. Let $N(D)$ be a small tubular neighborhood of $D$ in $W$. Then the closure of $W - N(D)$ is a topological 4-cell, by the topological Schoenflies theorem [1]. Therefore $W$ is homeomorphic to $B^2 \times S^2$ and $h$ extends to a topological embedding of $B^2 \times S^2$ as desired.

**Remark.** This result is analogous to the classical theorem of Alexander that any smooth embedded $S^1 \times S^1$ in $S^3$ bounds a smooth solid torus $B^2 \times S^1$.

3. Handle decompositions and slice links. In [2], Kirby, Melvin proved that if a smooth 4-manifold $M$ has boundary $S^1 \times S^2$ and is constructed by attaching a 2-handle to $B^4$ along a curve $C$ with the 0-framing, then $M$ is homeomorphic to $B^2 \times S^2$ and $C$ is a slice knot. We prove the following generalization of their result:
THEOREM. Let $W$ be a smooth 4-manifold which is obtained by adding $n$ 2-handles to $B^4$ along the curves $C_1, \ldots, C_n$. The 2-handles induce a framing of the link $C_1 \cup \cdots \cup C_n$. Assume that framed surgery on the sublink $C_1 \cup \cdots \cup C_i$ in $S^2$ yields $\#_i S^1 \times S^2$, for all $i$ with $1 \leq i \leq n$. Then $W$ is homeomorphic to $\#_n B^2 \times S^2$ and $C_1 \cup \cdots \cup C_n$ is a slice link.

COROLLARY. Let $W$ be a smooth 4-manifold such that $\partial W$ is diffeomorphic to $\#_n S^1 \times S^2$ and $W$ is built by attaching a 2-handle to $\#_n S^1 \times S^2 \times I$ along $C_1 \times \{t\}, C_2 \times \{t\}, \ldots, C_n \times \{t\}$ and then adding a 4-handle. Then $W$ is homeomorphic to $\#_n B^2 \times S^2$. 

Proof of theorem. By the assumption that surgery on the link $C_1 \cup \cdots \cup C_n$ gives $\#_n S^1 \times S^2$, it immediately follows that $\partial W$ is diffeomorphic to $\#_n S^1 \times S^2$. If the handle decomposition of $W$ is turned upside down, then $W$ is constructed by attaching $n$ 2-handles to $(\#_n S^1 \times S^2) \times I$ along some curves $C_1 \times \{1\}, C_2 \times \{1\}, \ldots, C_n \times \{1\}$ and then adding a 4-handle. We will assume that the 2-handle glued along $C_i \times \{1\}$ is dual to the 2-handle added along $C_i$ to $B^4$. 

Let $W_i$ or $W_i'$ denote the 4-manifold which is obtained by adjoining $i$ 2-handles to $B^4$ or $(\#_n S^1 \times S^2) \times I$ respectively along the curves $C_i, \ldots, C_n$ or $C_{n-i+1} \times \{1\}, \ldots, C_n \times \{1\}$ respectively. Then $\partial W_i$ is diffeomorphic to $(\#_n S^1 \times S^2) \times I$ since surgery on $C_1 \cup \cdots \cup C_i$ gives $\#_i S^1 \times S^2$. Also $W - \text{int} W_i'$ is diffeomorphic to $W_{n-i}$ and therefore $W_i'$ is a cobordism between $\#_i S^1 \times S^2$ and $\#_{n-i} S^1 \times S^2$. Note that $W_i'$ can also be constructed by adding $n-i$ 2-handles to $(\#_n S^1 \times S^2) \times I$.

Let $\langle \cdot \rangle$ denote the homotopy class of a loop $C$ relative to some base point and let $\langle * \rangle$ denote the normal closure of the set of elements $\ast$ in some group. By Van Kampen's theorem applied to the two handle decompositions of $W_i'$, we conclude that 

$$\pi_i(W_i') \cong \pi_i(\#_i S^1 \times S^2)/\langle \{C_{n-i+1}\}, \ldots, \{C_n\} \rangle$$

and $\pi_i(W_i')$ has rank $\leq n - i$. Consider the case when $i = 1$. By a classical theorem of Whitehead (see Exercise 20 on p. 283 of [4]) and by Corollary 5.14.2 on p. 354 of [4], it follows that $\pi_i(W_1')$ is free and $\{C_n\}$ is primitive, i.e., is contained in a free basis of the free group $\pi_i(\#_n S^1 \times S^2)$.

Next, $\pi_i(W_2')$ has a presentation consisting of a set of free generators of $\pi_i(W_2') \cong \pi_i(\#_n S^1 \times S^2)/\langle \{C_n\} \rangle$ and the one relation $\{C_{n-1}\}$. Hence by the results on p. 283 and p. 354 of [4] again, $\pi_i(W_2')$ is free and $\{C_{n-1}\}$ is primitive. Therefore we obtain that $\{\{C_{n-1}\}, \{C_n\}\}$ is contained in a free basis for $\pi_i(\#_n S^1 \times S^2)$. Continuing on with this argument, we conclude that $\{\{C_{n-1}\}, \ldots, \{C_n\}\}$ is a free basis of $\pi_i(\#_n S^1 \times S^2)$. So by Lemma 2 of [3], there is a diffeomorphism $h: \#_n S^1 \times S^2 \to \#_n S^1 \times S^2$ such that $h(S^1 \times \{x_i\})$ is homotopic to $C_i$ for
all $i$, $1 \leq i \leq n$, where $S^1 \times \{x_i\}$ is contained in the $i$th copy of $S^1 \times S^2$ used to form $\#_n S^1 \times S^2$ and is disjoint from the 3-cells employed for the connected sum.

Let $M$ be the smooth 4-manifold with $\partial M = S^3$ which is built by adding $n$ 3-handles and 4-handles to $W_n$, using the component $(\#_n S^1 \times S^2) \times \{0\}$ of $\partial W_n$. The 3-handles can be attached along the 2-spheres $h_i(\{y_i\} \times S^2) \times \{0\}$, for $1 \leq i \leq n$, where $\{y_i\} \times S^2$ is in the $i$th copy of $S^1 \times S^2$ used to obtain $\#_n S^1 \times S^2$ and $\{y_i\} \times S^2$ misses the 3-cells utilized for the connected sum. Turning the 3- and 4-handles of $M$ upside down, we find that $M$ can be constructed with a 0-handle, $n$ 1-handles and $n$ 2-handles. Note that each 2-handle of $M$ algebraically cancels one of the 1-handles, since $C_i$ is homotopic to $h(S^1 \times \{x_i\})$.

The Mazur trick can now be applied. $M \times I$ is a 5-manifold composed of a 0-handle, $n$ 1-handles and $n$ 2-handles. By the Whitney trick, the 2-handles geometrically cancel the 1-handles. Consequently $M \times I$ is diffeomorphic to $B^5$ and $2M = \partial (M \times I)$ is diffeomorphic to $S^4$. By the topological Schoenflies theorem [1], $M$ is homeomorphic to $B^4$.

Let $N$ denote the smooth closed 4-manifold which is obtained by gluing a 4-cell to $M$ along $\partial M = S^3$. Then $N$ is homeomorphic to $S^4$. Since $N = W \cup \#_n B^2 \times S^1$ it follows that $W$ is homeomorphic to $\#_n B^2 \times S^2$, either by isotopic unknotting in codimension 3 or by using the Dehn’s lemma in [5] plus the topological Schoenflies theorem as in §2. This proves the first part of the theorem.

Finally, exactly the same argument as in [2] applies to show that $C_1 \cup \cdots \cup C_n$ is a slice link.

Proof of corollary. If $W$ satisfies the conditions of the corollary, then $W$ can be constructed by adding $n$ 2-handles to $B^4$ along the curves $C_1, \ldots, C_n$ where $C_1 \cup \cdots \cup C_{n-1}$ is a trivial link of $n-1$ components in $S^3$. Hence $W$ satisfies the hypotheses of the theorem and so $W$ is homeomorphic to $\#_n B^2 \times S^2$.

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References


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