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PADÉ APPROXIMANTS ON BANACH SPACE OPERATOR EQUATIONS

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We examine an operator equation with a linear compact kernel in a Banach space and the Padé Approximant of its solution under a functional. We give a sufficient condition for convergence of a subsequence of Padé Approximants to the solution.

1. Introduction. If one is handling the Padé Approximation technique in multi-particle scattering theory one is interested in convergence. For the two-body case the scattering problem can be formulated as an operator equation with a compact kernel in Hilbert space. Baker [1] proved a result on the convergence of Padé Approximants derived from an operator equation in Hilbert space. Pointwise convergence of a series of Padé Approximants is established for solutions of operator equations as for the two-body scattering partial wave decomposed Lippmann Schwinger kernel and for trace class and compact operators under assumptions on subspace projection sequences. Going over to more particles one usually works in Banach spaces. In the three-particle case Faddeev [4] established an operator equation with compact kernels in a certain Banach space.

Baker's investigation is not easy to generalize onto Banach spaces because of the use of orthogonal projections. Nevertheless we prove a similar result which is mainly based on the properties of cyclic subspaces generated by the inhomogeneity g appearing in the operator equation and the kernel A, which is performed in §2.

In §3 we go over to Hilbert space. Our proposition can be formulated by means of the aperture of two subspaces, as defined by Nagy [8], Krein [6], Krasnoselskii [7]. Finally we discuss cases of validity.

2. Convergence theorem. Let us first present definitions. For standard notation used here see [3], [10], [12].

B is a Banach space, B^* its continuous dual space, A a linear compact operator mapping B into B, A^* the adjoint, g is an element of B, h^* an element of B^*. Let λ be a complex number and for $\lambda \neq 0$ let λ^{-1} be an element of $\rho(A)$, the resolvent set of A. The operator equation is

$$f = g + \lambda Af . \tag{2.1}$$

A unique solution exists.

$S_g^{(n)}$ is the linear span of the $A^i g$, $i = 0, 1, \cdots, n$.
S_g the closed hull of the union of all $S_g^{(n)}$.
$T_{h^*}^{(n)}$ the linear span of the $A^{*i}h^*$, $i = 0, 1, \cdots, n$
T_{h^*} the closed hull of the union of all $T_{h^*}^{(n)}$.

We assume S_g, T_{h^*} to be infinite dimensional. We use the definition and properties of the Padé Approximant given by Zinn-Justin [13]. Let $f(z)$ be an analytic function defined by its Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \ . \tag{2.2}$$

The Padé Approximant $f^{[n,m]}(z)$ of $f(z)$ is the following rational fraction

$$f^{[n,m]}(z) = \frac{P_n(z)}{Q_m(z)} = f(z) + O(z^{n+m+1}) \ . \tag{2.3}$$

The solution f of (2.1) can be expanded in a formal power series

$$f(\lambda) = g + \lambda A g + \lambda^2 A^2 g + \cdots \ .$$

It is well known [12] that the Neumann series converges to f for $|\lambda| < 1/\|A\|$.

Analoguously $(h^*)(f)$ can be formally expanded

$$(h^*)(f(\lambda)) = (h^*)(g) + \lambda(h^*)(Ag) + \lambda^2(h^*)(A^2 g) + \cdots \ .$$

Now $(h^*)(f)^{[n,m]}$ is the Padé Approximant of the above formal power series.

Here we give our basic assumption:

(2.4) There is a positive number M, such that for every sequence $x^{(n)} \in S_g^{(n)}$, $\|x^{(n)}\| = 1$ there exists a sequence $y^{*(n)} \in T_{h^*}^{(n)}$, $\|y^{*(n)}\| = 1$ such that $|(y^{*(n)})(x^{(n)})| \geqq M$.

THEOREM. *Let* $\{n'\} \subset \{n\}$ *be a subsequence such that* $(h^*)(f)^{[n',n'+1]}$ *exists. If* (2.4) *is fulfilled*

$$\lim_{n' \to \infty} (h^*)(f)^{[n',n'+1]} = (h^*)(f) \ . \tag{2.5}$$

The proof is cut in four parts. In (i) we give a projected equation proposed by Tani [11], Nuttal [9], prove the uniform boundedness of its solutions (ii) and the convergence under the functional h^* in (iii). In (iv) is shown, that the last expression is identical to the Padé Approximant of the original equation (2.1).

(i) One defines a series of operators $P^{(n)}: B \to B$, $n = 0, 1, 2, \cdots$

requiring $P^{(n)}$ and its dual adjoint $P^{(n)*}$ to be projections with

$$\text{image } P^{(n)} = S_g^{(n)} , \quad \text{image } P^{(n)*} = T_{h^*}^{(n)} . \tag{2.6}$$

Such an operator $P^{(n)}$ exists if the matrix $c_{ij}^{(n)} = (A^{*i}h^*)(A^j g)$ is invertible; $P^{(n)}$ is unique and can be explicitly written

$$\forall_{x \in B}, \; P^{(n)}x = \sum_{i,j=0}^{n} A^i g (c^{(n)-1})_{ij} (A^{*j}h^*)(x) . \tag{2.7}$$

The projected equation is defined as

$$f^{(n)} = g + \lambda P^{(n)} A f^{(n)} . \tag{2.8}$$

One can formally expand $f^{(n)}$ in a power series

$$f^n(\lambda) = g + \lambda P^{(n)} A g + \lambda^2 (P^{(n)} A)^2 g + \cdots .$$

This Neumann series converges for $|\lambda| < 1/\|P^{(n)}A\|$. The properties of $P^{(n)}$ imply that the first $n+1$ terms of the expansions on $f(\lambda)$ and $f^{(n)}(\lambda)$ are identical, that $f^{(n)}$ can be written as

$$f^{(n)} = \sum_{k=0}^{n} f_k^{(n)} A^k g ,$$

and that an equivalent to definition (2.8) is the set of equations

$$(A^{*k}h^*)(f^{(n)} - g - \lambda A f^{(n)}) = 0, \; k = 0, 1, \cdots, n$$

which gives an algebraic set of equations in order to determine the coefficients $f_k^{(n)}$.

Now let $\{n'\}$ be a subsequence such that $P^{(n')}$ and $f^{(n')}$ exist. The existance of $P^{(n')}$ and $f^{(n')}$ is guaranteed if the determinant of $c_{ij}^{(n)}$ and the determinant of the algebraique set for $f_k^{(n)}$ do not vanish. Baker shows in [1], [2] that either the Padé Approximant of finite order is equal to the exact solution or there exists at least an infinite subsequence $\{n'\}$, such that the determinants for $P^{(n')}$ and $f^{(n')}$ do not vanish and thus $P^{(n')}$ and $f^{(n')}$ exist. To that subsequence we will confine our attention.

In the following we need the transformation properties of S_g and T_{h^*} under $(1 - \lambda A)^{-1}$ and $(1 - \lambda A^*)^{-1}$ respectively,

$$\begin{aligned} &(1 - \lambda A)^{-1} S_g \subset S_g , \\ &(1 - \lambda A^*)^{-1} T_{h^*} \subset T_{h^*} . \end{aligned} \tag{2.9}$$

This can be seen as follows. First look upon $(1 - \lambda A)^{-1} g$. From the Neumann series one knows $(1 - \delta A)^{-1} g = \sum_{k=0}^{\infty} (\delta A)^k g \in S_g$ for $|\delta| < 1/\|A\|$. In the domain $D = \{\delta \mid \delta \in C, |\delta| < 1/\|A\|\}$ is $(1 - \delta A)^{-1}$ an analytic operator in δ. Thus $(1 - \delta A)^{-1} g / S_g$ is an analytic vector in the quotient space B/S_g. But $(1 - \delta A)^{-1} g / S_g = 0$ for all δ in D. Then analytic continuation gives $(1-\delta A)^{-1}g/S_g = 0$ for all $\delta \in C$ and thus $(1-\lambda A)^{-1}g \in S_g$. Analogously $(1-\lambda A)^{-1}A^k g \in S_g$ for each k and as $(1-\lambda A)^{-1}$ is continuous in B and S_g is closed, the first invariance property follows. Similarly goes the second property.

2.1 reads $f = (1-\lambda A)^{-1}g \in S_g$.

(ii) In this section starting from (2.4) we prove the uniform boundedness of solutions of (2.8),

$$\exists_{K>0},\ \|f^{(n')}\| \leqq K .$$

If we assume $\{f^{(n')}\}_{n'}$ solutions of (2.8) not uniformly bounded, then we have a subsequence $\{n''\} \subset \{n'\}$ such that

$$u_{n''} = 1/\|f^{(n'')}\|,\ \lim_{n''\to\infty} u_{n''} = 0 . \tag{2.10}$$

With the definition $e^{(n'')} = f^{(n'')}/\|f^{(n'')}\|$ (2.8) reads

$$\begin{aligned} e^{(n'')} &= u_{n''}g + \lambda P^{(n'')}Ae^{(n'')} \\ &= u_{n''}g + \lambda(P^{(n'')} - 1)Ae^{(n'')} + \lambda Ae^{(n'')} , \end{aligned} \tag{2.11}$$

which can be rewritten

$$e^{(n'')} = (1-\lambda A)^{-1}(u_{n''}g + \lambda(P^{(n'')} - 1)Ae^{(n'')}) . \tag{2.12}$$

Next we have

$$\forall_{y^*\in T_{h^*}},\ \lim_{n''\to\infty} (y^*)(\lambda(P^{(n'')} - 1)Ae^{(n'')}) = 0 , \tag{2.13}$$

which can be seen as follows. From (2.11) we know that

$$\{\lambda(P^{(n'')} - 1)\, \mathrm{A}e^{(n'')}\}$$

is uniformly bounded. From the definition *of* T_{h^*} follows the existence of a sequence $y^{*(n'')} \in T_{h^*}^{(n'')}$ approximating y^* in the norm,

$$\begin{aligned} |(y^*)(\lambda(P^{(n'')} - 1)Ae^{(n'')})| \leqq \|y^* - y^{*(n'')}\| \quad \|\lambda(P^{(n)} - 1)Ae^{(n'')}\| \\ + |(y^{*(n'')})(\lambda(P^{(n'')} - 1)Ae^{(n'')})| . \end{aligned}$$

From (2.6) follows $\forall_{y^{*(n)}\in T_{h^*}^{(n)}} P^{(n)*}y^{*(n)} = y^{*(n)}$. Thus the second term is identical 0.

We apply $y^* \in T_{h^*}$ to (2.12):

$$\begin{aligned} \forall_{y^*\in T_{h^*}^{(n)}},\ (y^*)(e^{(n'')}) &= (y^*)((1-\lambda A)^{-1}(u_{n''}g + \lambda(P^{(n'')} - 1)Ae^{(n'')})) \\ &= ((1-\lambda A)^{-1*}y^*)(u_{n''}g + \lambda(P^{(n'')} - 1)Ae^{(n'')}) \\ &= (z^*)(u_{n''}g) + (z^*)(\lambda(P^{(n'')} - 1)Ae^{(n'')}) . \end{aligned}$$

(2.9) guarantees $(1-\lambda A)^{-1*}y^* = z^* \in T_{h^*}$ and the second term tends to 0 because of (2.13), thus leading to

$$\forall_{y^* \in T_{h^*}}, \lim_{n'' \to \infty} (y^*)(e^{(n'')}) = 0 . \tag{2.14}$$

At this point enters the compactness of A. Because B is a Banach space and $\|e^{(n'')}\|=1$ there is a subsequence $\{n'''\}\subset\{n''\}$ and an element $s\in B$ with

$$\lim_{n''' \to \infty} \| Ae^{(n''')} - s \| = 0 . \tag{2.15}$$

As S_g is a closed subspace of B, the limit point s is contained in S_g. Then we find that $(y^*)(Ae^{(n''')})$ goes to 0 uniformly for $y^* \in T_{h^*}$, more explicit

$$\forall_{\varepsilon>0}, \exists_{n_0'''}, \forall_{n''' \geqq_0''', y^* \in T_{h^*}, \|y^*\|=1}, |(y^*)(Ae^{(n''')})| < \varepsilon . \tag{2.16}$$

This can be shown in three steps:

$$\forall_{y^* \in T_{h^*}} (y^*)(Ae^{(n''')}) = (A^* y^*)(e^{(n''')}) = (z^*)(e^{(n''')})$$
$$\text{and } \lim_{n''' \to \infty} (z^*)(e^{(n''')}) = 0 \quad \text{by 2.14} ,$$
$$\forall_{y^* \in T_{h^*}} (y^*)(s) = 0\colon \forall_{\varepsilon>0}, \exists_{n_0'''}, \forall_{n''' \geqq n_0'''}, \|| Ae^{(n''')} - s\| < \varepsilon ,$$
$$|(y^*)(s)| \leqq \|y^*\| \, \|s - Ae^{(n''')}\| + |(y^*)(Ae^{(n''')})| < \|y^*\|\varepsilon + \varepsilon ,$$
$$\forall_{\varepsilon>0}, \exists_{n_0'''}, \forall_{n''' \geqq n_0'''}, \|Ae^{(n''')} - s\| < \varepsilon ,$$
$$\forall_{n''' \geqq n_0'''}, y^* \in T_{h^*}, \|y^*\| = 1$$
$$|(y^*)(Ae^{(n''')})| = |(y^*)(Ae^{(n''')} - s) + (y^*)(s)| \leqq \|y^*\| \, \|Ae^{(n''')}\| < \varepsilon .$$

Now we conclude that $(y^{*(n''')})(e^{(n''')})$ tends to 0 uniformly in $y^{*(n''')} \in T_{h^*}^{(n''')}$,

$$\forall_{\varepsilon>0}, \exists_{n_0'''}, \forall_{n''' \geqq_0''', y^{*(n''')} \in T_{h^*}^{(n''')}, \|y^{*(n''')}\|=1}, |(y^{*(n''')})(e^{(n''')})| < \varepsilon . \tag{2.17}$$

To see this use (2.11). From (2.10) and (2.16) we have

$$\forall_{\varepsilon>0}, \exists_{n_0'''}, \forall_{n''' \geqq n_0'''}, \|u_{n'''} g\| < \varepsilon ,$$
$$\forall_{n''' \geqq n_0''', y^* \in T_{h^*}, \|y^*\|=1}, |(y^*)(\lambda Ae^{(n''')})| < \varepsilon |\lambda| .$$

Remembering, that $P^{(n''')*}$ is a projection on $T_{h^*}^{(n''')}$, one has

$$\begin{aligned} \forall_{\varepsilon>0}, \exists_{n_0'''}, \forall_{n''' \geqq n_0''', y^{*(n''')} \in T_{h^*}^{(n''')}, \|y^{*(n''')}\|=1}, & |(y^{*(n''')})(e^{(n''')})| \\ &= |(y^{*(n''')})(u_{n'''} g) + (y^{*(n''')})(\lambda P^{(n''')} Ae^{(n''')})| \\ &\leqq \|y^{*(n''')}\| \, \|u_{n'''} g\| + |(y^{*(n''')})(\lambda Ae^{(n''')})| < \varepsilon(1 + |\lambda|) . \end{aligned}$$

But it is clear that (2.17) contradicts our assumption (2.4).

(iii) Here we will ensure us of the convergence of $(h^*)(f^{(n')})$. We regard (2.8).

$$f^{(n')} = g + \lambda P^{(n')} Af^{(n')} = g + \lambda(P^{(n')} - 1)Af^{(n')} + \lambda Af^{(n')} .$$

From the uniform boundedness of $\{f^{(n)}\}$ we derive the uniform boundedness of $\{\lambda(P^{(n')}-1)Af^{(n')}\}$ and one has

$$\forall_{y^*\in T_{h^*}}, \lim_{n'\to\infty}(y^*)(\lambda(P^{(n')}-1)Af^{(n')})=0\ . \tag{2.18}$$

The reason is the same as for (2.13). Investigating the difference of (2.1), (2.8) gives

$$\begin{aligned} d^{(n')} &= f - f^{(n')} = \lambda(Af - P^{(n')}Af^{(n')}) \\ &= \lambda Ad^{(n')} + \lambda(1-P^{(n')})Af^{(n')} \\ &= (1-\lambda A)^{-1}\lambda(1-P^{(n')})Af^{(n')}\ . \end{aligned}$$

We apply $h^*\in T_{h^*}$

$$\begin{aligned} (h^*)(d^{(n')}) &= (h^*)((1-\lambda A)^{-1}\lambda(1-P^{(n')})Af^{(n')}) \\ &= ((1-\lambda A)^{-1*}h^*)(\lambda(1-P^{(n')})Af^{(n')}) \\ &= (i^*)(\lambda(1-P^{(n')})Af^{(n')})\ , \\ \lim_{n'\to\infty}(h^*)(d^{(n')}) &= \lim_{n'\to\infty}(i^*)(\lambda(1-P^{(n')})Af^{(n')}) = 0\ , \end{aligned}$$

which follows from $(1-\lambda A)^{-1*}h^* = i^*\in T_{h^*}$ guaranteed by (2.9) and application of (2.18), hence

$$\lim_{n'\to\infty}(h^*)(f^{(n')}) = (h^*)(f)\ . \tag{2.19}$$

(iv) Finally the relation between $(h^*)(f^{(n')}$ (and the Padé Approximant $(h^*)(f)^{[n',n'+1]}$ has to be established. Some linear algebra shows that $(h^*)(f^{(n')})$ regarded as a function in λ is a rational fraction of degree $[n', n'+1]$.

$(h^*)(f^{(n')})^{[n',n'+1]}$ is the Padé Approximant of the formal power series (2.8) under the functional h^*

$$(h^*)(g) + \lambda(h^*)(P^{(n')}Ag) + \lambda^2(h^*)((P^{(n')}A)^2g) + \cdots\ .$$

If we confer this power series with that of (2.1)

$$(h^*)(g) + \lambda(h^*)(Ag) + \lambda^2(h^*)(A^2g) + \cdots\ .$$

We find with help of (2.6) the first $2n'+2$ terms to be identical. From the definition (2.3) follows

$$(h^*)(f)^{[n',n'+1]} = (h^*)(f^{(n')})^{[n',n'+1]}\ . \tag{2.20}$$

Baker [1] shows, that in Hilbert space the coefficients $f_k^{(n)}$ in $f^{(n)} = \sum_{k=0}^{n} f_k^{(n)}A^kg$ constructed from matrix elements (h, A^mg) are rational functions in λ of degree $[n, n+1]$. This also applies to the Banach space case, where the coefficients are constructed from $(h^*)(A^mg)$. Then clearly $(h^*)(f^{(n)})$ is a rational function in λ of degree $[n, n+1]$.

Zinn-Justin [13] proves that a Padé Approximant $[n, m]$ of a rational fraction of degree $[n, m]$ is identical to the rational fraction. That means

$$(h^*)(f^{(n')})^{[n',n'+1]} = (h^*)(f^{(n')}) . \tag{2.21}$$

Thus (2.19), (2.20), (2.1) together give (2.5).

3. Hilbert space formulation. Here we turn to Hilbert space $H = B$, we substitute the adjoint A^* by the Hilbert adjoint A^+ of A and every $y^* \in H^*$ by its isomorphic $y \in H$. Then (2.4) reads

$$\exists_{M>0}, \forall_{x^{(n)} \in S_g^{(n)}, ||x^{(n)}||=1}, \exists_{y^{(n)} \in T_h^{(n)}, ||y^{(n)}||=1}, |(y^{(n)}, x^{(n)}| \geqq M . \tag{3.1}$$

It can be expressed in terms of the aperture of Hilbert spaces [8], [5]. Let H_1, H_2 be closed subspaces of a Hilbert space, P_1, P_2 the related orthogonal projections, θ is defined as

$$\theta(H_1, H_2) = ||P_1 - P_2|| . \tag{3.2}$$

In our case $H_1 = T_h^{(n)}$, $H_2 = S_g^{(n)}$, $P_1 = P_{T_h^{(n)}}$, $P_2 = P_{S_g^{(n)}}$.

$$\theta_n = \theta(T_h^{(n)}, S_g^{(n)}) . \tag{3.3}$$

Krasnoselskii [5] defines

$$\tau_n = \inf_{t^{(n)} \in T_h^{(n)}, ||t^{(n)}||=1} ||P_{S_g^{(n)}} t^{(n)}|| , \tag{3.4}$$

and shows

$$\theta_n^2 + \tau_n^2 = 1 . \tag{3.5}$$

We define

$$\mu_n = \inf_{t^{(n)} \in T_h^{(n)}, ||t^{n)}||=1} \ \sup_{s^{(n)} \in S_g^{(n)}, ||s^{(n)}||=1} |(t^{(n)}, s^{(n)})| . \tag{3.6}$$

Between τ_n and μ_n the following relation is valid

$$\tau_n^2 \leqq \mu_n < \tau_n , \tag{3.7}$$

which is easy to check. Obviously $0 \leqq \theta_n, \tau_n, \mu_n \leqq 1$ holds, such that we can formulate (3.1) as

$$\begin{aligned} &\exists_{M>0}, \forall_{\mu_n}, \mu_n > M \Longleftrightarrow \inf_n \mu_n > 0 \Longleftrightarrow \inf_n \tau_n^2 > 0 \\ &\Longleftrightarrow \sup_n \theta_n^2 < 1 \Longleftrightarrow \sup_n \theta_n < 1 . \end{aligned} \tag{3.8}$$

To finish up we illustrate the validity of (3.1). For example take A self adjoint, C is a linear bounded operator commuting with A.

$$h = g + Cg,\ \|C\| < 1\,,$$

$$\forall_{x^{(n)} \in S_g^{(n)}, \|x^{(n)}\|=1},\ x^{(n)} = \sum_{i=0}^{n} x_i^{(n)} A^i g\,,$$

$$v^{(n)} = \sum_{i=0}^{n} x_i^{(n)} A^i h = \sum_{i=0}^{n} x_i^{(n)} A^i (g + Cg) = x^{(n)} + Cx^{(n)}\,,$$

$$y^{(n)} = v^{(n)} / \|v^{(n)}\|\,,$$

$$|(y^{(n)}, x^{(n)})| = \left|\left(\frac{(1 + C)x^{(n)}}{\|(1 + C)x^{(n)}\|}, x^{(n)}\right)\right| \geqq \frac{1 - \|C\|}{1 + \|C\|}\,.$$

Thus (3.1) is fulfilled.

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