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REPRESENTATIONS NAIMARK-RELATED TO *-REPRESENTATIONS; A CORRECTION

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Let A be a Banach *-algebra. A theorem is proved concerning a sufficient condition for a continuous representation of A on a Hilbert space H to be Naimark-related to a *-representation of A on H. One corollary of this result is that a continuous (topologically) irreducible representation of A on H is Naimark-related to a *-representation of A on H if and only if some coefficient of the representation is a nonzero positive functional of A.

One purpose of the paper is to correct in part a previously published result the proof of which contains a serious gap.

1. **Introduction.** Professor John Bunce has brought to my attention a gap in the proof of Theorem 3 in my paper [2]. Briefly, the problem is as follows: Let A be a Banach *-algebra, and let π be a continuous essential representation of A (not in general a *-representation) on a Hilbert space H. What is established at the beginning of the proof of [2, Theorem 3] is the existence of a π-invariant subspace H_0 of H and an inner-product on H_0 with the property that

$$\langle \pi(f)\xi, \eta\rangle = \langle \xi, \pi(f^*)\eta\rangle \qquad (\xi, \eta \in H, f \in A) .$$

At this point in the proof results are applied to this inner-product which may be applied only when this form is closable on H; see [4, pp. 313-315]. It is not proved in [2] that this form is closable, hence the gap in the proof. Although Theorem 3 and its corollaries have not been established in [2], we know of no counter-examples to these statements. All of the results of [2] outside of § 4 (including Theorem 1 and Theorem 7) are to our knowledge correct.

The aim of this note is to partially correct the error. Here we prove a result similar to [2, Theorem 3], and derive from it several corollaries. In particular, it is shown that a continuous, essential, (topologically) cyclic, separable representation of a C^*-algebra is Naimark-related to a *-representation of the algebra; and that a continuous (topologically) irreducible representation of a Banach *-algebra A is Naimark-related to a *-representation of A if and only if some nonzero coefficient of the representation is a positive functional on A.

2. **The results.** We use the same notation as in [2]. In parti-

cular, we use the terminology cyclic and irreducible as synonymous with what some authors call topologically cyclic and topologically irreducible. Given a representation (π, H) of the Banach *-algebra A, a coefficient of π is a functional on A of the form $f \rightarrow (\pi(f)\xi, \eta)$ for some $\xi, \eta \in H$.

The representation π is essential if for any $\xi \in H$ the condition $\pi(f)\xi = 0$ for all $f \in A$ implies $\xi = 0$. If α is a positive functional on A, then

$$K_\alpha = \{f \in A: \alpha(f^*f) = 0\} .$$

THEOREM. *Assume that π is a continuous representation of A on a Hilbert space H and that π is cyclic with cyclic vector $\xi_0 \in H$. Furthermore, assume that $\{\eta_m: m \in S\}$ is a collection of vectors in H, S being a finite or countably infinite index set, such that*

(i) $\alpha_m(f) = (\pi(f)\xi_0, \eta_m)$ *is a positive functional for all $m \in S$; and*

(ii) $\mathrm{span}\,\{\bigcup_{m \in S} (\pi(A)^*\eta_m)\}$ *is dense in H. Then π is Naimark-related to a *-representation of A on H.*

Proof. Choose positive numbers $\{\lambda_m: m \in S\}$ so that $\alpha = \sum_{m \in S}\lambda_m\alpha_m$ converges in the norm of the dual space of A, and

$$\eta_0 = \sum_{m \in S} \lambda_m\eta_m \text{ converges in } H .$$

Then by (i) α is a positive functional on A, and

$$\alpha(f) = (\pi(f)\xi_0, \eta_0) \quad (f \in A) .$$

First we verify that

$$K_\alpha = \{f \in A: \pi(f)\xi_0 = 0\} .$$

That K_α is the larger of these two sets is immediate. Now assume that $f \in K_\alpha$. Then $\alpha(f^*f) = 0$ implies $\alpha_m(f^*f) = 0$ for all $m \in S$. Thus

$$0 = \alpha_m(gf) = (\pi(f)\xi_0, \pi(g)^*\eta_m)$$

for all $g \in A$ and all $m \in S$. Thus by (ii) we have $\pi(f)\xi_0 = 0$.

Now proceeding as in [2, (I)], define an inner-product on $\pi(A)\xi_0$ by

$$\langle\pi(f)\xi_0, \pi(g)\xi_0\rangle = \alpha(g^*f) \quad (f, g \in A) .$$

For $\xi, \eta \in \pi(A)\xi_0$, we use the notation $\tau(\xi, \eta) = \langle\xi, \eta\rangle$ and $\tau[\xi] = \tau(\xi, \xi)$. We prove that τ is a closable form [4, p. 315]. Assume $\{\xi_n\} \subset$

domain of $\tau = \pi(A)\xi_0$, and

$$\xi_n \longrightarrow 0(\text{in } H),\ \tau[\xi_n - \xi_m] \longrightarrow 0$$

as $n, m \to \infty$. We must show that $\tau[\xi_n] \to 0$ as $n \to \infty$. Now

$$\begin{aligned}|\tau[\xi_n]| &\leqq |\tau(\xi_n - \xi_m, \xi_n)| + |\tau(\xi_m, \xi_n)| \\ &\leqq \tau[\xi_n - \xi_m]^{1/2}\tau[\xi_n]^{1/2} + |\tau(\xi_m, \xi_n)| .\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Since $\tau[\xi_n - \xi_m] \to 0$, $\tau[\xi_n]$ is a Cauchy sequence, we may choose $M > 0$ such that $\tau[\xi_n] \leqq M$ for $n \geqq 1$. Choose N a positive integer such that $\tau[\xi_n - \xi_m] < \varepsilon$ whenever $n, m \geqq N$. For $n, m \geqq N$,

$$\tau[\xi_n] \leqq (M\varepsilon)^{1/2} + |\tau(\xi_m, \xi_n)| . \tag{1}$$

For each n choose $f_n \in A$ such that $\xi_n = \pi(f_n)\xi_0$. Then $\tau(\xi_m, \xi_n) = \alpha(f_n^* f_m) = (\pi(f_n^* f_m)\xi_0, \eta_0) = (\pi(f_m)\xi_0, \pi(f_n^*)^*\eta_0) \to 0$ as $m \to \infty$ for each fixed n. This fact together with (1) shows that $\tau[\xi_n] \to 0$ as $n \to \infty$.

Let $\bar{\tau}$ denote the closure of the form τ. By [4, Theorem 2.23, p. 331] there exists a self-adjoint operator U with $\mathscr{D}(U) = \mathscr{D}(\bar{\tau})$ such that

$$\bar{\tau}(\xi, \eta) = (U\xi, U\eta) \quad (\xi, \eta \in \mathscr{D}(U)) .$$

Next we prove that $\mathscr{N}(U) = \{0\}$. By [4, Corollary 2.27, p. 332] $\pi(A)\xi_0$ is a core of U (this means that the set $\{(\pi(f)\xi_0, U\pi(f)\xi_0): f \in A\}$ is dense in the graph of U [4, p. 166]). Suppose $\xi \in \mathscr{N}(U)$. Choose $\{f_n\} \subset A$ such that

$$\pi(f_n)\xi_0 \longrightarrow \xi \text{ and } U\pi(f_n)\xi_0 \longrightarrow 0 .$$

Then $\alpha(f_n^* f_n) = \|U\pi(f_n)\xi_0\|^2 \to 0$. Thus $\alpha_m(f_n^* f_n) \to 0$ for each m. Therefore for each $m \in S$ and all $g \in A$

$$|\alpha_m(gf_n)| \leqq \alpha_m(gg^*)^{1/2}\alpha_m(f_n^* f_n)^{1/2} \longrightarrow 0$$

as $n \to \infty$. It follows that for each m

$$(\pi(f_n)\xi_0, \pi(g)^*\eta_m) = (\pi(gf_n)\xi_0, \eta_m) = \alpha_m(gf_n) \longrightarrow 0$$

as $n \to \infty$. Therefore $(\xi, \pi(g)^*\eta_m) = 0$ for all $g \in A$ and all $m \in S$, so by (ii), $\xi = 0$.

Since $\mathscr{N}(U) = \{0\}$, U has dense range. Then again using the fact that $\pi(A)\xi_0$ is a core of U, we have that $U\pi(A)\xi_0$ is dense in H.

Now we complete the proof that π is Naimark-related to a *-representation of A. By [2, (I)] we have

$$\tau(\pi(f)\xi, \eta) = \tau(\xi, \pi(f^*)\eta) \quad (\xi, \eta \in \pi(A)g_0,\ f \in A) .$$

For $f \in A$ define $\varphi_0(f)$ on $U\pi(A)\xi_0$ by

$$\varphi_0(f)U\xi = U\pi(f)\xi \quad (\xi \in \pi(A)\xi_0) .$$

A routine calculation using the observation above shows that for all $f \in A$ and all $\xi, \eta \in U\pi(A)\xi_0$

$$(\varphi_0(f)\xi, \eta) = (\xi, \varphi_0(f^*)\eta) .$$

By [5, Prop. 5] there exists a unique extension of φ_0 to a *-representation φ of A on H (=the closure of $U\pi(A)\xi_0$). Let $\xi \in \mathscr{D}(U)$. Again using that $\pi(A)\xi_0$ is a core of U, choose $\{f_n\} \subset A$ such that

$$\pi(f_n)\xi_0 \longrightarrow \xi \text{ and } U\pi(f_n)\xi_0 \longrightarrow U\xi .$$

Then for any $f \in A$

$$\pi(f)\pi(f_n)\xi_0 \longrightarrow \pi(f)\xi ,$$

and

$$U\pi(f)\pi(f_n)\xi_0 = \varphi_0(f)U\pi(f_n)\xi_0 \longrightarrow \varphi(f)U\xi .$$

Since U is closed, $\pi(f)\xi \in \mathscr{D}(U)$, and $U\pi(f)\xi = \varphi(f)U\xi$. Thus π is Naimark-related to φ.

COROLLARY 1. *Let (π, H) be a continuous irreducible representation of A. Then π is Naimark-related to a *-representation of A on H if and only if some coefficient of π is a nonzero positive functional on A.*

Proof. Assume that $\xi_0, \eta_0 \in H$ with $f \to (\pi(f)\xi_0, \eta_0)$ a nonzero positive functional on A. Since π is irreducible, ξ_0 is a cyclic vector for π. Now $\pi(A)^*\eta_0$ is a nonzero subspace of H with $(\pi(A)^*\eta_0)^\perp$ a closed π − invariant subspace. Then since π is irreducible, $\pi(A)^*\eta_0$ is dense in H. By the theorem, π is Naimark-related to a *-representation of A.

Conversely, assume that U is a closed densely defined operator on H, φ is a *-representation of A on H, and

$$U\pi(f)\xi = \varphi(f)U\xi \quad (\xi \in \mathscr{D}(U), f \in A) .$$

Then U^*U is densely defined [4, Theorem 3.24, p. 275]. Choose a vector $\xi_0 \in \mathscr{D}(U)$, $\xi_0 \neq 0$, such that $U\xi_0 \in \mathscr{D}(U^*)$. Set $\eta_0 = U^*U\xi_0$. Then for $f \in A$,

$$\begin{aligned}(\pi(f)\xi_0, \eta_0) &= (\pi(f)\xi_0, U^*U\xi_0) \\ &= (U\pi(f)\xi_0, U\xi_0) \\ &= (\varphi(f)U\xi_0, U\xi_0) .\end{aligned}$$

Thus $f \to (\pi(f)\xi_0, \eta_0)$ is a nonzero positive functional on A.

For group representations the result analagous to Corollary 1 is the following.

COROLLARY 2. *Assume that G is a locally compact group. Let (π, H) be a bounded weakly continuous irreducible representation of G on a Hilbert space H. Then π is Naimark-related to a weakly continuous unitary representation of G (on H) if and only if some coefficient of π is a (nonzero) positive definite function.*

If A is a C^*-algebra, we use the notation $\bar{A}$ to denote the von-Neumann enveloping algebra of A [3, 12.1.5].

COROLLARY 3. *Let A be a C^*-algebra. Let π be an essential continuous cyclic representation of A on a separable Hilbert space H. Then π is Naimark-related to a *-representation of A on H.*

Proof. By [1, Theorem 1] π extends to an ultraweakly continuous representation $\bar{\pi}$ of $\bar{A}$ on H. Let ξ_0 be a cyclic vector for π. Assume $\xi \in H$ and $\xi \perp (\pi(A)^*H)$. Then $(\pi(A)\xi) \perp H$, so that $\xi = 0$. Therefore $\pi(A)^*H$ is dense in H. Choose a sequence $\{\zeta_m\} \subset H$ such that

$$\text{span}\left(\bigcup_{m=1}^{\infty} (\pi(A)^*\zeta_m)\right) \text{ is dense in } H .$$

For each m, set

$$\beta_m(f) = (\bar{\pi}(f)\xi_0, \zeta_m) \quad (f \in \bar{A}) .$$

Since β_m is a normal functional on $\bar{A}$, by the Polar Decomposition Theorem [3, p. 240] there exists a partial isometry $u_m \in \bar{A}$ and a positive functional α_m on $\bar{A}$ such that

$$\beta_m(f) = \alpha_m(u_m f) \text{ and } \alpha_m(f) = \beta_m(u_m^* f) \quad (f \in \bar{A}) .$$

Let $\eta_m = (\bar{\pi}(u_m^*))^*\zeta_m$ for each m. Then for each m and all $f \in \bar{A}$

$$\begin{aligned} \alpha_m(f) &= (\bar{\pi}(u_m^* f)\xi_0, \zeta_m) \\ &= (\bar{\pi}(f)\xi_0, (\bar{\pi}(u_m^*))^*\zeta_m) \\ &= (\bar{\pi}(f)\xi_0, \eta_m) . \end{aligned}$$

Also, $(\bar{\pi}(f)\xi_0, \zeta_m) = \beta_m(f) = \alpha_m(u_m f) = (\bar{\pi}(f)\xi_0, (\bar{\pi}(u_m))^*\eta_m)$ for all $f \in \bar{A}$. Therefore $(\bar{\pi}(u_m))^*\eta_m = \zeta_m$. It follows that

$$\text{span}\left(\bigcup_{m=1}^{+\infty} (\bar{\pi}(\bar{A})^*\eta_m)\right) \text{ is dense in } H .$$

By the theorem $\bar{\pi}$ is Naimark-related to a *-representation of $\bar{A}$ on H. This completes the proof.

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