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THE RADON-NIKODYM-PROPERTY, σ-DENTABILITY AND MARTINGALES IN LOCALLY CONVEX SPACES

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In this paper we give relations between the Radon-Nikodym-Property (RNP), in sequentially complete locally convex spaces, mean convergence of martingales, and σ-dentability. (RNP) is equivalent with the property that a certain class of martingales is mean convergent, while σ-dentability is equivalent with the property that the same class of martingales is mean Cauchy. We give an example of a σ-dentable space not having the (RNP). It is also an example of a sequentially incomplete space of integrable functions, the range space being sequentially complete.

1. Introduction, terminology and notation. A nonempty subset B of a locally convex space (l.c.s.) (over the reals) is called dentable, if for every neighborhood (nbhd) V of o, there exists a point x in B such that

$$x \notin \overline{\mathrm{con}}\,(B\backslash(x+V))$$

($\overline{\mathrm{con}}$ denotes the closed convex hull). X is called dentable if every bounded subset of X is dentable. When we replace $\overline{\mathrm{con}}$ by σ, where

$$\sigma(A)=\left\{\sum_{n=1}^{\infty}\lambda_n x_n \,\|\, x_n\in A,\ \forall n\in N,\ \sum_{n=1}^{\infty}\lambda_n=1,\ \sum_{n=1}^{\infty}\lambda_n x_n \text{ convergent},\ \lambda_n\geqq 0\right\},$$

we get the corresponding definitions for σ-dentability.

We use the following integral:

Let X be a sequentially complete l.c.s., and (Ω, Σ, μ) a finite complete positive measure space.

A function $f\colon \Omega\to X$ is said to be μ-integrable, if there exists a sequence $(f_n)_{n=1}^{\infty}$ of simple functions such that:

(i) $\lim_n f_n(\omega)=f(\omega)$, μ – a.e..

(ii) For every continuous seminorm p on X:

$$\lim_n \int_\Omega p(f_n(\omega)-f(\omega))d\mu(\omega)=0\ .$$

Put $\int_A f d\mu=\lim_n\int_A f_n d\mu$, $\forall A\in\Sigma$. This limit exists and is in X. Denote $L^1_X(\mu,\Sigma)$ as the space of classes $[f]$ of μ-integrable functions, where $[f]=[g]$ iff $f=g$, μ – a.e..

Put $q(f)=\int_\Omega p(f)d\mu$, where p is any continuous seminorm on X.

The topology on L_X^1 considered is these, generated by all the q.

Note. It is easily seen by Lebesgue's convergence theorem and (i), that we can replace (ii) by:

(ii)′ $\lim_{m,n} \int_\Omega p(f_n(\omega) - f_m(\omega))\, d\mu(\omega) = 0$ for every continuous seminorm p on X.

Let B be a closed bounded subset of X. We say that B has the Radon-Nikodym-Property, (RNP), if, for every positive finite separable measure space $(\Omega, \sum, \mu)$, and every vector measure $m: \sum \to X$, with

$$A_m(\sum, \mu) = \left\{\frac{m(A)}{\mu(A)} - \middle\| A \in \sum, \mu(A) > 0\right\}$$

contained in B, there is a μ-integrable function $f: \Omega \to X$, such that

$$m(A) = \int_A f d\mu , \quad \forall A \in \sum .$$

We say that X has the (RNP) if each closed bounded convex subset of X has the (RNP).

A sequence $(x_n, \sum_n)_{n=1}^\infty$ is called an X-valued martingale, if every x_n is in $L_X^1(\mu, \sum_n)$, where $(\Omega, \sum, \mu)$ is a measure space and the $\sum_n$ are σ-algebras such that $\sum_n \subset \sum_{n+1} \subset \sum$, $\forall n \in N$, and if, for every A in $\sum_n$:

$$\int_A x_n d\mu = \int_A x_{n+1} d\mu, \ \forall n \in N .$$

We call a l.c.s. in which every bounded set is metrizable, a (BM)-space. In this case our definition of (RNP) corresponds to this given in [10]. (This is a consequence of Theorems 1 and 2 below.)

2. The results.

The following theorem is well-known in Banach spaces (see [1] and [8]):

THEOREM. *The following assertions are equivalent in a Banach space X:*

(i) *X has* (RNP).

(ii) *Every uniformly bounded martingale $(x_n, \sum_n)_{n=1}^\infty$ is L_X^1-convergent.*

(iii) *X is dentable.*

(iv) *X is σ-dentable.*

In our case the space $L_X^1(\mu, \sum)$ is in general not complete, so that we might get some Cauchy-results, when (ii) is relied to (iii) or (iv). On the other hand: (RNP) implies a certain completeness condition, since, in proving (RNP) we have to prove the existence of a μ-integrable function, being the Radon-Nikodym-derivative of a certain vector measure, w.r.t. a scalar measure. We first state some lemmas. Some of them have independent interest.

LEMMA 1. *Let $\sum$ be a separable σ-algebra. Suppose $\sum = \sigma(A)$ (the σ-algebra generated by A) where A is an algebra. Then there is a countable $B \subset A$ such that $\sum = \sigma(B)$.*

LEMMA 2. *Let X be a sequentially complete l.c.s., and $(x_i, \sum_i)_{i \in I}$ a uniformly bounded martingale. Put $\sum = \sigma(\bigcup_i \sum_i)$. Let $(\sum_{i_n})_{n=1}^{\infty}$ be a sequence such that $\sum = \sigma(\bigcup_{n=1}^{\infty} \sum_{i_n})$. Let $F: \sum \to X$ be the limit measure of $(x_{i_n}, \sum_{i_n})_{n=1}^{\infty}$. Then F is also the limitmeasure of $(x_i, \sum_i)_{i \in I}$.*

The proofs of Lemma 1 and 2 are easily made. From them we have:

LEMMA 3. *Let X be a sequentially complete l.c.s., and $(x_i, \sum_i)_{i \in I}$ a uniformly bounded martingale. Suppose $\sum = \sigma(\bigcup_i \sum_i)$ separable. Then the limit measure of $(x_i, \sum_i)$ exists on $\sum$.*

Let $(\Omega, \sum, \mu)$ be a separable positive finite measure space. Let F be a vectormeasure on $\sum$ into X, such that $A_\Omega(F)$ is bounded. Put:

$$x_\pi = \sum_{A \in \pi} \frac{F(A)}{\mu(A)} \chi_A$$

where π runs through Π (the set of all finite partitions of Ω into elements of $\sum$, directed in the usual way). Since $(\Omega, \sum, \mu)$ is countably generated, we have: $\sum$ is the σ-algebra generated by an increasing sequence of finite partitions π_n of Ω.

LEMMA 4. *$(x_\pi)_{\pi \in \Pi}$ is L_X^1-Cauchy iff every sequence $(x_{\pi_n})_{n=1}^{\infty}$ is L_X^1-Cauchy, with $(\pi_n)_{n=1}^{\infty}$ increasing such that $\sum = \sigma(\bigcup_n \pi_n)$. In this case we have that for any two such sequences $(\pi_n)_{n=1}^{\infty}$, $(\pi'_n)_{n=1}^{\infty}$:*

$$L_X^1 - \lim_{n\to\infty} (x_{\pi_n} - x_{\pi'_n}) = 0 .$$

In case only one such sequence $(x_{\pi_n})_{n=1}^{\infty}$ is L_X^1-convergent, then they all are convergent (to the same limit). This limit is also $L_X^1 - \lim_{\pi \in \Pi} x_\pi$.

Proof. Denote $\sum_n = \sigma(\pi_n)$: the σ-algebra generated by $\pi_n \cdot (x_\pi)_{\pi \in \Pi}$ is L_X^1-Cauchy. Hence for every continuous seminorm q on L_X^1, there is a $\pi_0 \in \Pi$, such that for every $\pi \geqq \pi_0$:

$$(1) \qquad q(x_\pi - x_{\pi_0}) \leqq \frac{1}{4} .$$

Let $\pi_0 = \{A_1, \cdots, A_n\}$. By a well-known theorem ([3], p. 76), we can construct

$$\left\{A'_1, \cdots, A'_n, \Omega \backslash \bigcup_{i=1}^{n} A'_i\right\}$$

in $\bigcup_{n=1}^{\infty} \pi_n$, such that $\mu(A_i \Delta A'_i) < 1/24.n.M_p$, for every $i = 1, \cdots, n$, where M_p is a p-bound of $(x_{\pi_n})_{n=1}^{\infty}$ $\left(\text{and where } q(f) = \int_\Omega p(f) d\mu\right)$. Making the usual arrangements:

$$A''_1 = A'_1,\ A''_i = A'_i \backslash \bigcup_{j=1}^{i-1} A'_j \quad (n \geqq i > 1)$$

$$A''_{n+1} = \Omega \backslash \bigcup_{i=1}^{n} A''_i$$

we get $\pi''_0 = \{A''_1, \cdots, A''_n, A''_{n+1}\}$.

Let π' be any refinement of π'_0; $\pi' \in \Pi$

$$\pi' = \{B_{1,1}, \cdots, B_{1,p_1}; \cdots; B_{n,1}, \cdots, B_{n,p_n}; B_{n+1,1}, \cdots, B_{n+1,p_{n+1}}\} .$$

Choose $\pi'' = \pi' \vee \pi_0$ in Π. Then we consider three parts in π'':

(I) Those sets $B_{i,j}$ of π' which can also be taken in π'': i.e.: which are already part of one A_k. This part cancels in $x_{\pi'} - x_{\pi''}$.

(II) Those sets $B_{i,j}$ of π' which are in more than one A_k. As sets in π'' we have of course to choose $B_{i,j} \cap A_k (k = 1, \cdots, n)$.

(III) For those $B_{n+1,j}$, which are in more than one A_k, we take also $B_{n+1,j} \cap A_k (k = 1, \cdots, n)$ in π''.

We have:

$$\begin{aligned}
&q(x_{\pi'} - x_{\pi''}) \\
&\quad = q\left(\sum_{(\mathrm{II})} \frac{F(B_{i,j})}{\mu(B_{i,j})} \chi_{B_{i,j}} - \sum_{(\mathrm{II})} \sum_{k=1}^{n} \frac{F(B_{i,j} \cap A_k)}{\mu(B_{i,j} \cap A_k)} \chi_{B_{i,j} \cap A_k}\right) \\
&\qquad + q\left(\sum_{(\mathrm{III})} (\text{the same})\right) \\
&\quad \leqq \sum_{(\mathrm{II})} q\left(\sum_{k=1}^{n} \left(\frac{F(B_{i,j})}{\mu(B_{i,j})} - \frac{F(B_{i,j} \cap A_k)}{\mu(B_{i,j} \cap A_k)}\right) \chi_{B_{i,j} \cap A_k}\right) \\
&\qquad + \sum_{(\mathrm{III})} (\text{the same}) \\
&\quad \leqq \sum_{(\mathrm{II})} p\left(\frac{F(B_{i,j})}{\mu(B_{i,j})} - \frac{F(B_{i,j} \cap A_i)}{\mu(B_{i,j} \cap A_i)}\right) \mu(B_{i,j} \cap A_i) \\
&\qquad + \sum_{(\mathrm{II})} \sum_{k \neq i} p\left(\frac{F(B_{i,j})}{\mu(B_{i,j})} - \frac{F(B_{i,j} \cap A_k)}{\mu(B_{i,j} \cap A_k)}\right) \mu(B_{i,j} \cap A_k)
\end{aligned}$$

$$+\sum_{(\mathrm{III})}\sum_{k=1}^{n} p\left(\frac{F(B_{n+1,j})}{\mu(B_{n+1,j})}-\frac{F(B_{n+1,j})\cap A_k)}{\mu(B_{n+1,j}\cap A_k)}\right)\mu(B_{n+1,j}\cap A_k)$$
$$=: (1)+(2)+(3).$$

We remark that, when E, G are arbitrary in $\sum$, $\mu(E)>0$, $\mu(G)>o$, we have:

$$\frac{F(E)}{\mu(E)}=\frac{F(G)}{\mu(G)}+\frac{\mu(G)F(E)-F(G)\mu(E)}{\mu(G)\mu(E)}$$
$$=\frac{F(G)}{\mu(G)}+\frac{F(E\backslash G)}{\mu(E)}-\frac{F(G\backslash E)}{\mu(E)}-\frac{F(G)\mu(E\backslash G)}{\mu(E)\mu(G)}+\frac{F(G)\mu(G\backslash E)}{\mu(E)\mu(G)}\,.$$

Now, here, we put $E=B_{i,j}$, $G=B_{i,j}\cap A_i$. We can suppose $\mu(B_{i,j})>0$, $\mu(B_{i,j}\cap A_i)>0$, since we consider only partitions, $\mu-$a.e.. Hence:

$$p\left(\frac{F(B_{i,j})}{\mu(B_{i,j})}-\frac{F(B_{i,j}\cap A_i)}{\mu(B_{i,j}\cap A_i)}\right)$$
$$\leqq\frac{|F|_p(B_{i,j}\Delta(B_{i,j}\cap A_i))}{\mu(B_{i,j})}+p\left(\frac{F(B_{i,j}\cap A_i)}{\mu(B_{i,j}\cap A_i)}\right)\cdot\frac{\mu(B_{i,j}\Delta(B_{i,j}\cap A_i))}{\mu(B_{i,j})}\,,$$

where $|F|_p$ denotes the p-variation on F.
So

$$(1)\leqq\sum_{(\mathrm{II})}[M_p\mu(B_{i,j}\Delta(B_{i,j}\cap A_i))+M_p\mu(B_{i,j}\Delta(B_{i,j}\cap A_i))]$$
$$\leqq\sum_{i=1}^{n}2M_p\mu(A_i''\Delta(A_i''\cap A_i))$$
$$<\frac{1}{12}\,.$$

Now:

$$(2)\leqq 2M_p\sum_{(\mathrm{II})}\sum_{k\neq i}\mu(B_{i,j}\cap A_k)$$
$$\leqq 2M_p\sum_{i=1}^{n}\sum_{k\neq i}\mu(A_i''\cap A_k)$$
$$\leqq 2M_p\cdot n\cdot\frac{1}{24nM_p}\left(\text{since }\bigcup_{k\neq i}A_i''\cap A_k\subset A_i''\backslash A_i\right)$$
$$=\frac{1}{12}\,.$$

$$(3)\leqq 2M_p\mu(A_{n+1}'')$$
$$=2M_p\mu\left(\Omega\backslash\bigcup_{i=1}^{n}A_i'\right)$$
$$\leqq 2M_p\cdot n\left(\frac{1}{24.n.M_p}\right)$$
$$=\frac{1}{12}\,.$$

Thus $q(x_\pi - x_{\pi''}) < 1/4$.
We have also by (1): $q(x_{\pi''} - x_{\pi_0}) < 1/4$.
Now $\pi_0'' \subset \bigcup_n \sum_n$. Hence there exists a $n_0 \in N$ such that $\pi_{n_0} \geqq \pi_0''$.
So $q(x_{\pi_{n_0}} - x_{\pi_0}) < 1/2$.
When $\pi_n \geqq \pi_{n_0}$, we have also $\pi_n \geqq \pi_0''$. Hence also $q(x_{\pi_n} - x_{\pi_0}) < 1/2$.
Hence $q(x_{\pi_n} - x_{\pi_{n_0}}) < 1$, $\forall n \geqq n_0$. So $(x_{\pi_n})_{n=1}^\infty$ is L_X^1-Cauchy.

$\Leftarrow$ Let $\sum = \sigma(\bigcup_{n=1}^\infty \pi_n)$ where (π_n) is an increasing sequence of finite partitions of Ω. Supposing $(x_\pi)_{\pi \in \Pi}$ not L_X^1-Cauchy, we have: there is a continuous seminorm q on $L_X^1(\mu)$ such that for every $\pi \in \Pi$, $\exists \pi', \pi'' \in \Pi$, $\pi', \pi'' \geqq \pi$, with $q(x_{\pi'} - x_{\pi''}) > 2$. Let π''' be π' or π'' according to $q(x_\pi - x_{\pi'''}) > 1$.

We start the induction with $\pi = \pi_1$; we call π''' now: π_1'. Then for $\pi = \pi_1' \vee \pi_2$; we call π''' now: π_2', and so on. Hence we have $(x_{\pi_k''})_{k=1}^\infty$ with $\pi_{2n}'' = \pi_n'$

$$\pi_{2n-1}'' = \pi_n \vee \pi_{n-1}'$$

for every $n = 1, 2, 3, \cdots$; It is trivial that $(x_{\pi_k''})_{k=1}^\infty$ is not L_X^1-Cauchy, although $\sigma(\bigcup_{k=1}^\infty \pi_k'') = \sum$, since $\pi_{2n}'' = \pi_n' \geqq \pi_n$ for every n in N.

So, the two assertions are equivalent. In this case, since $(x_\pi)_{\pi \in \Pi}$ is L_X^1.Cauchy, we have, for every continuous seminorm p on X, $\exists \pi_0 \in \Pi$ such that for any $\pi \geqq \pi_0$:

$$q(x_\pi - x_{\pi_0}) = \int_\Omega p(x_\pi - x_{\pi_0}) < \frac{1}{4} . \tag{1}$$

Let $(\pi_n)_{n=1}^\infty$ and $(\pi_n')_{n=1}^\infty$ be two increasing sequences, consisting of finite partitions of Ω into elements of $\sum$, such that $\sum = \sigma(\bigcup_n \pi_n) = \sigma(\bigcup_n \pi_n')$. From the first part of the proof of this lemma, and (1), we deduce: There is a π_{n_0} such that

$$\text{for every} \quad n \geqq n_0\colon q(x_{\pi_n} - x_{\pi_0}) < \frac{1}{2} \tag{2}$$

and a π_{n_1}' such that for every $n \geqq n_1$: $q(x_{\pi_n'} - x_{\pi_0}) < 1/2$.

Choose $m = \max(n_0, n_1)$. So, there is a m in N such that for every $n \geqq m$: $q(x_{\pi_n} - x_{\pi_n'}) < 1$, for every p. Hence:

$$L_X^1 - \lim_{n \to \infty} (x_{\pi_n} - x_{\pi_n'}) = 0 .$$

Now suppose that there is at least one sequence $(x_{\pi_n})_{n=1}^\infty$ with $\sigma(\bigcup_n \pi_n) = \sum$, such that there is a x in $L_X^1(\mu)$ for which $L_X^1 - \lim_n x_{\pi_n} = x$. Let $(x_{\pi_n'})_{n=1}^\infty$ be another sequence with $\sum = \sigma(\bigcup_n \pi_n')$. It is immediate that $F(A) = \int_A x d\mu$, for every A in

$\bigcup_n \pi_n$. Hence $F(A) = \lim_n \int_A x_{\pi_n} d\mu$, for every A in $\bigcup_n \pi_n$. Since $A_\Omega(F)$ is bounded we have that $F(A) = \lim_n \int_A x_{\pi_n} d\mu$, for every A in $\sum$. Thus $F(A) = \int_A x d\mu$, for every A in $\sum$. So: $L_X^1 - \lim_n x_{\pi'_n} = x$, and $L_X^1 - \lim_{\pi \in \Pi} x_\pi = x$.

THEOREM 1. *Let X be a sequentially complete l.c.s.. The following assertions are equivalent:*

(1) *X has* (RNP).

(2a) *Every uniformly bounded martingale $(x_n, \sum_n)_{n=1}^\infty$ with $\sum = \sigma(\bigcup_n \sum_n)$ separable, is L_X^1-convergent.*

(2b) *Every uniformly bounded and finitely generated martingale $(x_n, \sum_n)_{n=1}^\infty$ is L_X^1-convergent.*

(2c) *Every uniformly bounded martingale $(x_i, \sum_i)_{i\in I}$, with $\sum = \sigma(\bigcup_i \sum_i)$ separable, is L_X^1-convergent.*

(2d) *Every uniformly bounded and finitely generated martingale $(x_i, \sum_i)_{i\in I}$ with $\sum = \sigma(\bigcup_i \sum_i)$ separable, is L_X^1-convergent.*

Proof. This proof is now done in the same way as in Banach spaces; We use now Lemmas 3 and 4.

REMARKS. (1) When the property "separable" is deleted in the definition of (RNP) we can prove in Theorem 1 only (1) $\Leftrightarrow$ (2c) $\Leftrightarrow$ (2d) (without the assumption $\sum$ separable). This we can do if X is supposed to be quasi-complete (to be sure of the existence of the limitmeasure). However Theorem 1 is much more useful as will be seen later on.

(2) When the property "$A_\Omega(F)$ bounded" in the definition (RNP) is changed into "F bounded variation and μ-continuous", we can prove Theorem 1 in the same way, but now using L_X^1-bounded and uniformly integrable martingales instead of uniformly bounded martingales: However Theorem 1 is more interesting in connection with σ-dentability. (See Theorem 2.)

We are now going to characterize σ-dentability in terms of martingale-Cauchy-properties.

THEOREM 2. *Let X be a sequentially complete l.c.s.. The following assertions are equivalent:*

(3) *X is σ-dentable.*

(4a) *Every uniformly bounded and finitely generated martingale $(x_n, \sum_n)_{n=1}^\infty$ is L_X^1-Cauchy.*

(4b) *Every uniformly bounded martingale $(x_n, \sum_n)_{n=1}^\infty$ is L_X^1-Cauchy.*

REMARKS. (1) As will follow from the proof of this theorem, we may also use in (4a) and (4b) martingales on a separable measure space only. We may even restrict the martingales to be defined on $([0, 1], B[0, 1], \lambda)(B[0, 1]$ = the Borelsets in $[0, 1]$ and λ denoting Lebesgue measure).

(2) In (4a) and (4b) we may also use martingales with an arbitrary index-set I. This is trivial, since we are looking at Cauchy-properties.

Proof of Theorem 2.

(4) $\Rightarrow$ (3). This a adaptation of the proof of Huff [7] to our case: Now supposing X not being σ-dentable, we can construct a seminorm-independent uniformly bounded and finitely generated martingale, which is not L^1_X-Cauchy.

(3) $\Rightarrow$ (4a). An application of Rieffel's theorem to our case shows that $(x_\pi)_{\pi \in \Pi}$ is L^1_X-Cauchy, with

$$x_\pi = \sum_{A \in \pi} \frac{\lim_n \int_A x_n d\mu}{\mu(A)} \chi_A$$

where $(x_n, \sum_n)_{n=1}^\infty$ is the given uniformly bounded and finitely generated martingale, and where $\Pi = \{\pi \| \pi$ is a finite partition of Ω into elements of $\Sigma\}$.

Then Lemma 4 gives the result.

The proof of (4a) $\Leftrightarrow$ (4b) is easily made.

COROLLARY. *Let X be a quasi-complete* (BM)*-space. Then all the assertions in Theorem* 1 *are equivalent with all the assertions in Theorem* 2 (*and equivalent with dentability*).

Proof. This is easily seen by the result of Saab [10].

We also see that in a quasi-complete (BM)-space, we get an equivalent formulation of (RNP), by deleting the word "separable" in our definition.

The proof of the following lemma is immediate:

LEMMA 5. *Let $(x_n)_{n=1}^\infty$ be a sequence of step-functions which is $L^1_X(\mu)$-Cauchy. Then there is a martingale $(y_n, \sum_n)_{n=1}^\infty$, such that*

$$L^1_X(\mu) - \lim_{n\to\infty} (y_n - x_n) = 0 .$$

From this lemma and Theorems 1 and 2 we have now:

THEOREM 3. *σ-dentability is equivalent with* (RNP)*(in sequentially complete l.c.s.) iff every uniformly bounded L^1_X-Cauchy sequence of (step-) functions in $L^1_X(\Omega, \sum, \mu)$ is L^1_X-convergent.* (*$(\Omega, \sum, \mu)$: any separable positive finite measure space.*)

Hence the Radon-Nikodym-property's equivalence with σ-dentability depends critically on the sequential completeness of $L^1_X(\mu)$.

For the remainder of this article, we intend to prove that there is a sequentially complete l.c.s. X for which L^1_X is not sequentially complete: We shall even show that there is a Schur space X for which $L^1_{X,\sigma(X,X')}$ is not sequentially complete. This is done by proving that these X are σ-dentable and have not (RNP). We first recall the definition of a weak-Radon-Nikodym-Banach space.

DEFINITION. Let X be a Banach space. X is said to have the weak-Radon-Nikodym property (WRNP), w.r.t. the measure space $(\Omega, \sum, \mu)$, if for every X-valued measure F on $\sum$, which is μ-continuous and of finite variation, there is a Pettis-integrable function $f: \Omega \to X$ such that

$$F(A) = P - \int_A f d\mu$$

for every A in $\sum$. (Here $P - \int_A f d\mu$ denotes the Pettis-integral of f over A.)

The following lemma is immediately seen:

LEMMA 6. *Let the Banach space X be weakly sequentially complete. If X, $\sigma(X, X')$ has* (RNP) *then X has* (WRNP) *w.r.t. separable measure spaces.*

We denote by JH the space constructed by Hagler [6].

LEMMA 7 ([1], [2], [6]). *JH' is a Schur space without* (RNP). *L^1 is a weakly sequentially complete Banach space without* (RNP). *Every Schur space is trivially weakly sequentially complete.*

In Theorems 4 and 5, X denotes a weakly sequentially complete Banach space without (RNP).

THEOREM 4. *There is a closed separable subspace Y of X such that Y, $\sigma(Y, Y')$ is σ-dentable and has not* (RNP).

Proof. Since X does not have (RNP), there exists a closed separable subspace Y of X without (RNP), hence without (RNP)w.r.t. ([0, 1], B[0, 1], λ). (Here B[0, 1] denotes the class of the Borel subset of [0, 1] and λ denotes Lebesgue measure on [0, 1]). Since Y is separable, Y has not (WRNP)w.r.t. ([0, 1], B[0, 1], λ). By Lemma 6: $Y, \sigma(Y, Y')$ has not (RNP)w.r.t. ([0, 1], B[0, 1], λ). Furthermore $Y, \sigma(Y, Y')$ is sequentially complete, and by [5] (Cor. 3 of Theorem 1) is σ-dentable.

From Theorems 1, 2 and 4, we have now:

THEOREM 5. *There is a sequentially complete l.c.s. X such that L^1_λ is not sequentially complete.*

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