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ON THE UNIFORM DISTRIBUTION PROPERTY OF CERTAIN LINEAR ALGEBRAIC GROUPS

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Let G be a connected semisimple linear algebraic group defined over an algebraic number field k. Denote by G_k and G_A the group of k-rational points of G and its adelization. In this paper, we prove, under suitable assumptions on G, a uniformity of distribution of G_k in G_A with respect to the Haar measure on G_A.

Introduction. Let G be a connected semisimple linear algebraic group defined over an algebraic number field k. We denote by G_k the group of k-rational points of G, and we write G_A for its adelization.

The purpose of this paper is to show, under suitable assumptions on G, that G_k is, in a sense, "uniformly distributed" in G_A with respect to a Haar measure on G_A.

For each place v of k, let G_{k_v} be the group of k_v-rational points of G where k_v is the v-completion of k. If v is a finite place, let O_v be the maximal compact subring of k_v. Then G_{O_v}, the group of O_v-rational points of G, is an open compact subgroup of G_{k_v}. We set

$$G_\infty = \prod_{v \in \mathscr{P}_\infty} G_{k_v},$$

$$G_{A_f} = \prod_{v \in \mathscr{P}_f}' G_{k_v} \text{ (restricted direct product).}$$

Here $\mathscr{P}_\infty$ (resp. $\mathscr{P}_f$) denotes the set of all infinite (resp. finite) places of k. Then we have

$$G_A = G_{A_f} G_\infty \text{ (direct product).}$$

Let $\mathscr{S}$ be a finite subset of $\mathscr{P}_f$. Furthermore, for each $\mathfrak{g} \in \mathscr{S}$, let $K_\mathfrak{g}$ be an open compact subgroup of $G_{O_\mathfrak{g}}$ and let $\{S_\mathfrak{g}(j)\}_{j=1}^\infty$ be a sequence of nonempty compact subsets of $G_{k_\mathfrak{g}}$ satisfying the following conditions:

$$K_\mathfrak{g} S_\mathfrak{g}(j) K_\mathfrak{g} = S_\mathfrak{g}(j) \qquad (j = 1, 2, \cdots).$$

We set

$$S(j) = \prod_{\mathfrak{g} \in \mathscr{S}} S_\mathfrak{g}(j) \times \prod_{\mathfrak{g} \in \mathscr{P}_f - \mathscr{S}} G_{o_\mathfrak{g}}.$$

Then $S(j)$ is a compact subset of G_{A_f}. For a relatively compact domain S_∞ in G_∞, let $N(S(j), S_\infty)$ be the number of points in the set $(S(j) \times S_\infty) \cap G_k$. It is easy to see that $N(S(j), S_\infty)$ is finite.

We say that a sequence $\{S(j)\}_{j=1}^\infty$ has the uniform distribution property with respect to a Haar measure dg on G_A if the following equality holds for any relatively compact domain S_∞ in G_∞:

$$\lim_{j\to\infty} N(S(j), S_\infty)/\int_{S(j)\times S_\infty} dg = 1/\int_{Gk\backslash G_A} dg .$$

Note that the above statement does not depend on the choice of a Haar measure on G_A.

Let dg_f be a Haar measure on G_{A_f}. Then our main result is stated as follows.

THEOREM 1. *Notation being as above, assume that G is anisotropic (namely that $G_k\backslash G_A$ is compact). Furthermore, we assume that G is absolutely almost simple*[1] *and simply connected*[2]. *Then the sequence $\{S(j)\}_{j=1}^\infty$ has the uniform distribution property with respect to a Haar measure dg, if the equality* (0.1) *is satisfied*:

$$\lim_{j\to\infty} \int_{S(j)} dg_f = +\infty . \tag{0.1}$$

REMARK 1. The additional assumption that G is absolutely almost simple can be replaced by the following weaker assumption (A).

(A) For $\mathfrak{g} \in \mathscr{S}$, if $G_{k_\mathfrak{g}}$ is noncompact then G is $k_\mathfrak{g}$-almost simple (namely that G has no proper closed connected normal subgroups defined over $k_\mathfrak{g}$) and $G_k G_{k_\mathfrak{g}}$ is dense in G_A.

Note that G has the property (A) if G is absolutely almost simple and simply connected, by virtue of the strong approximation theorem (cf. [9], [11], and [12]).

REMARK 2. There are numbers of examples of G satisfying the assumptions in Theorem 1 (e.g., quaternion unitary groups constructed by G. Shimura in [14]).

Even if G is not anisotropic, it is probable that an analogue of Theorem 1 is available. At present we can prove only the following:

THEOREM 2. *Let G be* SL_2 *(regarded as a linear algebraic group defined over k). Then the sequence $\{S(j)\}_{j=1}^\infty$ has the uniform distribution property if* (0.1) *is satisfied.*

We present an implication of our results. Assume that G

1) This implies that G has no proper closed connected normal subgroups.

2) For the definition, see [8], p. 189.

satisfies the assumptions in Theorem 1. Assume $k=\boldsymbol{Q}$ and $\mathscr{S}=\{p\}$[3]. Let S_j be the set of elements g in $G_{\boldsymbol{Q}}$ such that the coordinates of $p^j g$ are integral and moreover satisfy some prescribed congruence conditions. Let U, V be relatively compact domains in $G_{\boldsymbol{R}}$, and let $v(U)$, $v(V)$ be their volumes measured by a Haar measure on $G_{\boldsymbol{R}}$. We denote by $N_U(j)$ (resp. $N_V(j)$) the number of elements in the set $U\cap S_j$ (resp. $V\cap S_j$). Then we have

$$\lim_{j\to\infty} N_U(j)/N_V(j) = v(U)/v(V)^{4)}\ .$$

A special case of Theorem 1 was first obtained by M. Kuga in [10] when G is the group of indefinite division quaternions with reduced norm 1[5]. Several ideas in [10] together with recent results of Howe and Moore [7] are basic in our present paper.

Note that H. Yoshida shows in [16] that Theorem 2 for $k=\boldsymbol{Q}$ leads to his distribution law for PSL$(2, \boldsymbol{Z}[1/p])$-elliptic conjugacy classes.

I would like to express here my deep gratitude to Professor T. Shintani for his suggestion of the problem considered here, his many mathematical and linguistical advices and his constant encouragement during the preparation of this paper.

NOTATION. For a complex number s, we denote by Im s (resp. Re s) the imaginary (resp. real) part of s. For an algebraic number field k, we denote by $\boldsymbol{A}$ and $\boldsymbol{I}$ the adele ring of k and the idele group of k, respectively. We denote by $|a|_A$ the module of an idele a, given by the equality $d(ax)=|a|_A dx$ where dx is a Haar measure on $\boldsymbol{A}$. For a locally compact topological space X, we denote by $C^0(X)$ the space of continuous functions on X and denote by $C_c^0(X)$ the space consisting of $f\in C^0(X)$ with compact support. For

3) We assume that $G_{\boldsymbol{Q}_p}$ is noncompact.

4) In fact, our result implies the following asymptotic formula for $N_U(j)$. Let K be a sufficiently small open compact subgroup of $\prod_{l<\infty} G_{\boldsymbol{Z}_l}$, and put $\Gamma=(K\times G_{\boldsymbol{R}})\cap G_{\boldsymbol{Q}}$. We may assume that $\Gamma S_j\Gamma=S_j (j=1,2,\cdots)$. We denote by $|\Gamma\backslash S_j|$ the number of left Γ-cosets contained in S_j. Then we have

$$N_U(j)\sim \mathrm{vol}\,(\Gamma\backslash G_{\boldsymbol{R}})^{-1}\cdot v(U)\cdot|\Gamma\backslash S_j|$$

as $j\to\infty$.

5) C. Pommerenke obtained in [13] the following similar results, while his method seems to be different from ours.

Let A be a positive definite symmetric integral matrix of size $m\geqq 5$. Set $X=\{x\in\boldsymbol{R}^m|{}^t xAx=1\}\subset\boldsymbol{R}^m$. For a positive integer n, put

$$S_n=\{\xi/\sqrt{n}|\xi\in\boldsymbol{Z}^m,\ {}^t\xi A\xi=n\}\subset X\,.$$

Let Λ be the set consisting of positive integers n such that $S_n\neq\phi$. Then the sequence $\{S_n\}_{n\in\Lambda}$ is uniformly distributed in X with respect to a suitable measure on X.

a continuous function f on a locally compact group G, and for a compact subgroup M of G, we say that f is *right M-finite* if the set $\{R_m f; m \in M\}$ spans a finite dimensional subspace in $C^0(G)$, where we set $R_m f(g) = f(gm)$. For a bounded linear operator T on a Hilbert space H, we denote by $\|T\|$ the operator norm of T, given by

$$\|T\| = \sup_{v \in H, v \neq 0} \|Tv\|/\|v\| .$$

For a finite dimensional vector space V over $\boldsymbol{C}$, we denote by $\mathrm{End}_{\boldsymbol{C}}(V)$ the $\boldsymbol{C}$-algebra of $\boldsymbol{C}$-endomorphisms on V. If T is a $\boldsymbol{C}$-endomorphism on a $\boldsymbol{C}$-vector space with an inner product, we denote by T^* the adjoint of T with respect to the inner product. If τ is an unitary representation of a compact group M on a finite dimensional vector space V over $\boldsymbol{C}$, we set $\dim \tau = \dim_{\boldsymbol{C}} V$.

1. We keep notation in the introduction without further comment. From now on, we always assume that G is a connected semisimple linear algebraic group defined over an algebraic number field k. We set

$$K = \prod_{\mathfrak{p} \in \mathscr{P}_f - \mathscr{S}} G_{\mathfrak{O}_\mathfrak{p}} \times \prod_{\mathfrak{p} \in \mathscr{S}} K_\mathfrak{p} .$$

Then K is an open compact subgroup of G_{A_f}. We normalize the Haar measure dg_f on G_{A_f} so that

$$\int_K dg_f = 1 .$$

Choose a Haar measure dg_∞ on G_∞ and fix the Haar measure dg on G_A by setting

$$dg = dg_f dg_\infty (g = g_f g_\infty,\ g_f \in G_{A_f},\ g_\infty \in G_\infty) .$$

Then dg induces an invariant measure $d\dot{g}$ on $G_k \backslash G_A$ in a natural manner.

Let $L^2(G_k \backslash G_A / K)$ be the Hilbert space of right K-invariant square integrable functions on $G_k \backslash G_A$. Note that constant functions are in $L^2(G_k \backslash G_A / K)$, since the volume of the quotient space $G_k \backslash G_A$ is finite (cf. [2], 5.6).

Let ξ_j be the characteristic function of $S(j)$. Then ξ_j is K-biinvariant, continuous, and compactly supported on G_{A_f}. For each $f \in L^2(G_k \backslash G_A / K)$, set

$$f * \xi_j(g) = \int_{G_{A_f}} f(g h_f^{-1}) \xi_j(h_f) dh_f . \tag{1.1}$$

Then the mapping $f \mapsto f * \xi_j$ gives rise to a bounded linear operator on $L^2(G_k\backslash G_A/K)$. We set

$$\deg \xi_j = \int_{G_{A_f}} \xi_j(g_f)dg_f = \int_{S(j)} dg_f . \tag{1.2}$$

Then $\deg \xi_j$ is equal to the number of left K-cosets contained in $S(j)$. We denote by $\|\ \|$ and $(,)$ the norm and the inner product in $L^2(G_k\backslash G_A/K)$, respectively. Set

$$v = \int_{G_k\backslash G_A} d\dot{g} .$$

Then the following proposition plays a basic role in the present paper.

PROPOSITION 1. *The sequence $\{S(j)\}_{j=1}^{\infty}$ has the uniform distribution property with respect to a Haar measure on G_A, if the following equality holds for any $f \in L^2(G_k\backslash G_A/K)$;*

$$\lim_{j\to\infty} \| f*\xi_j/\deg \xi_j - (f, 1)/v \| = 0 . \tag{1.3}$$

To prove the proposition, we need the next lemma.

LEMMA 1. *Under the assumption of Proposition 1, we have, for any $\varphi \in C_c^0(G_k\backslash G_A/K)$ and $g \in G_A$,*

$$\lim_{j\to\infty} \varphi*\xi_j(g)/\deg \xi_j = (\varphi, 1)/v . \tag{1.4}$$

Proof. Assume that the lemma is false. Then there exists $g_0 \in G_A$ such that the equality (1.4) does not hold for g_0. We have

$$\limsup_{j\to\infty} |\varphi*\xi_j(g_0)/\deg \xi_j - (\varphi, 1)/v| = \eta > 0 .$$

We can choose a subsequence $\{\xi_{j_k}\}_{k=1}^{\infty}$ of $\{\xi_j\}_{j=1}^{\infty}$ such that

$$\lim_{k\to\infty} |\varphi*\xi_{j_k}(g_0)/\deg \xi_{j_k} - (\varphi, 1)/v| = \eta .$$

Let $S(j) = \sum_{l=1}^{N_j} K\sigma_l^{(j)} (\sigma_l^{(j)} \in G_{A_f} (1 \leqq l \leqq N_j))$ be a decomposition of $S(j)$ into a disjoint union of left K-cosets. (The number N_j equals $\deg \xi_j$.) It is easily verified that

$$\varphi*\xi_j(g) = \sum_{l=1}^{N_j} \varphi(g\sigma_l^{(j)-1}) .$$

Hence we have, for $g \in G_A$,

$$\begin{aligned} &|\varphi*\xi_{j_k}(g)/\deg \xi_{j_k} - (\varphi, 1)/v| \\ &\qquad \geqq |\varphi*\xi_{j_k}(g_0)/\deg \xi_{j_k} - (\varphi, 1)/v| \end{aligned}$$

$$- \frac{1}{N_{j_k}} |\varphi * \xi_{j_k}(g) - \varphi * \xi_{j_k}(g_0)|$$
$$\geqq |\varphi * \xi_{j_k}(g_0)/\deg \xi_{j_k} - (\varphi, 1)/v|$$
$$- \frac{1}{N_{j_k}} \sum_{l=1}^{N_{j_k}} |\varphi(g\sigma_l^{(j)-1}) - \varphi(g_0\sigma_l^{(j)-1})| .$$

Since the function φ is continuous and compactly supported on $G_k \backslash G_A$, there exists an open neighborhood U of 1 in G_A such that $g_1^{-1}g_2 \in U$ always implies

$$|\varphi(g_1) - \varphi(g_2)| < \eta/2 .$$

Suppose that $g \in g_0(G_\infty \cap U)$. Then we have $(g_0\sigma_l^{(j_k)-1})^{-1} \times (g\sigma_l^{(j_k)-1}) = \sigma_l^{(j_k)}g_0^{-1}g\sigma_l^{(j_k)-1} = g_0^{-1}g \in U$. (Note that $g_0^{-1}g \in G_\infty$ commutes with every element in G_{A_f}.) Thus we have, for $g \in g_0(G_\infty \cap U)$,

$$\frac{1}{N_{j_k}} \sum_{l=1}^{N_{j_k}} |\varphi(g\sigma_l^{(j_k)-1}) - \varphi(g_0\sigma_l^{(j_k)-1})| < \eta/2 .$$

Hence, for any $g \in g_0(G_\infty \cap U)$, the following inequality holds:

$$\liminf_{k\to\infty} |\varphi * \xi_{j_k}(g)/\deg \xi_{j_k} - (\varphi, 1)/v| \geqq \eta - \eta/2 = \eta/2 . \tag{1.5}$$

In fact, this inequality (1.5) holds for any $g \in G_k g_0(G_\infty \cap U)K$ since $\varphi * \xi_{j_k}$ is left G_k-invariant and right K-invariant. Since $(G_\infty \cap U)K$ is an open set in G_A, we have, by virtue of Fatou's lemma,

$$\liminf_{k\to\infty} ||\varphi * \xi_{j_k}/\deg \xi_{j_k} - (\varphi, 1)/v||^2$$
$$\geqq \int_{G_k\backslash G_k g_0(G_\infty \cap U)K} \liminf_{k\to\infty} |\varphi * \xi_{j_k}(g)/\deg \xi_{j_k} - (\varphi, 1)/v|^2 \, d\dot{g}$$
$$\geqq (\eta/2)^2 \int_{G_k\backslash G_k g_0(G_\infty \cap U)K} d\dot{g} > 0 .$$

Contradiction! The lemma has been established.

Proof of Proposition 1. Let us consider any two relatively compact domains S'_∞, S''_∞ in G_∞ satisfying $\overline{S'_\infty} \subset S''_\infty$ (we denote by $\overline{S'_\infty}$ the closure of S'_∞ in G_∞). We choose a real-valued continuous function ψ_∞ on G_∞ satisfying the following conditions (1.6) and (1.7):

$$0 \leqq \psi_\infty(g_\infty) \leqq 1 \quad \text{for any } g_\infty \in G_\infty , \tag{1.6}$$

$$\psi_\infty(g_\infty) = \begin{cases} 1 & \text{if } g_\infty \in S'_\infty \\ 0 & \text{if } g_\infty \notin S''_\infty . \end{cases} \tag{1.7}$$

We set $\psi(g) = \psi_f(g_f)\psi_\infty(g_\infty)$ for $g = g_f g_\infty (g_f \in G_{A_f}$ and $g_\infty \in G_\infty)$, where

ψ_f denotes the characteristic function of K. Then $\psi(g)$ is continuous and compactly supported on G_A. It is easy to see that the series

$$\varphi(g) = \sum_{\gamma \in G_k} \psi(\gamma g)$$

converges absolutely and uniformly on any compact subset of G_A, and that $\varphi(g) \in C_c^0(G_k \backslash G_A / K)$. Applying Lemma 1 to φ, we obtain

$$\lim_{j \to \infty} \varphi * \xi_j(1) / \deg \xi_j = (\varphi, 1)/v \, . \tag{1.8}$$

We have

$$(\varphi, 1) = \int_{G_k \backslash G_A} \varphi(g) d\dot{g} = \int_{G_A} \psi(g) dg$$
$$= \int_{G_{A_f}} \psi_f(g_f) dg_f \cdot \int_{G_\infty} \psi_\infty(g_\infty) dg_\infty \, .$$

In view of the conditions (1.6) and (1.7) imposed on ψ_∞, we have

$$\mu(S'_\infty) \leqq (\varphi, 1) \leqq \mu(S''_\infty) \, , \tag{1.9}$$

where we set $\mu(S_\infty^{(i)}) = \int_{S_\infty^{(i)}} dg_\infty (i = 1, 2)$.

Next we have

$$\varphi * \xi_j(1) = \int_{G_{A_f}} \varphi(g_f^{-1}) \xi_j(g_f) dg_f = \int_{G_{A_f}} \sum_{\gamma \in G_k} \psi(\gamma g_f^{-1}) \xi_j(g_f) dg_f$$
$$= \sum_{\gamma \in G_k} \psi_\infty(\gamma_\infty) \int_{G_{A_f}} \psi_f(\gamma_f g_f^{-1}) \xi_j(g_f) dg_f \, ,$$

where we write $\gamma = \gamma_f \gamma_\infty (\gamma_f \in G_{A_f}$ and $\gamma_\infty \in G_\infty)$. Since $\psi_f(\gamma_f g_f^{-1})\xi_j(g_f)$ is, as a function of g_f, the characteristic function of $K\gamma_f \cap S(j)$, we have

$$\int_{G_{A_f}} \psi_f(\gamma_f g_f^{-1}) \xi_j(g_f) dg_f = \begin{cases} 1 \text{ if } \gamma_f \in S(j) \\ 0 \text{ otherwise.} \end{cases}$$

Hence

$$\varphi * \xi_j(1) = \sum_{\gamma \in G_k \cap (S(j) \times G_\infty)} \psi_\infty(\gamma_\infty) \, .$$

Then (1.6) and (1.7) imply that

$$\sum_{\gamma \in G_k \cap (S(j) \times S'_\infty)} 1 \leqq \varphi * \xi_j(1) \leqq \sum_{\gamma \in G_k \cap (S(j) \times S''_\infty)} 1 \, .$$

Thus,

$$N(S(j), S'_\infty) \leqq \varphi * \xi_j(1) \leqq N(S(j), S''_\infty) \, . \tag{1.10}$$

Combining two inequalities (1.9) and (1.10), we get

$$N(S(j),\ S'_\infty)/\mu(S''_\infty) \deg \xi_j \leqq \varphi*\xi_j(1)/(\varphi,\ 1) \deg \xi_j \\ \leqq N(S(j),\ S''_\infty)/\mu(S'_\infty) \deg \xi_j\ .$$

It follows from (1.8) that

$$\limsup_{j\to\infty} N(S(j),\ S'_\infty)/\mu(S''_\infty) \deg \xi_j \leqq 1/v \\ \leqq \liminf_{j\to\infty} N(S(j),\ S''_\infty)/\mu(S'_\infty) \deg \xi_j\ .$$

It is easy to see

$$\int_{S(j)\times S^{(i)}_\infty} dg = \mu(S^{(i)}_\infty)\cdot \deg \xi_j \quad (i = 1,\ 2)\ .$$

Hence we get the following two inequalities;

$$\limsup_{j\to\infty} N(S(j),\ S'_\infty)\Big/\int_{S(j)\times S'_\infty} dg \leqq 1/v\cdot \mu(S''_\infty)/\mu(S'_\infty)\ , \tag{1.11}$$

$$\liminf_{j\to\infty} N(S(j),\ S''_\infty)\Big/\int_{S(j)\times S''_\infty} dg \geqq 1/v\cdot \mu(S'_\infty)/\mu(S''_\infty)\ . \tag{1.12}$$

For a given relatively compact domain S_∞ in G_∞, apply (1.11), setting $S'_\infty = S_\infty$. For any $\varepsilon > 0$, there exists a relatively compact domain S''_∞ such that $1 \leqq \mu(S''_\infty)/\mu(S_\infty) \leqq 1 + \varepsilon$ and that $S''_\infty \supset \overline{S_\infty}$. Hence we have

$$\limsup_{j\to\infty} N(S(j),\ S_\infty)\Big/\int_{S(j)\times S_\infty} dg \leqq 1/v\ . \tag{1.13}$$

Similar arguments for the inequality (1.12) (in this case, we set $S''_\infty = S_\infty$) lead to

$$\liminf_{j\to\infty} N(S(j),\ S_\infty)\Big/\int_{S(j)\times S_\infty} dg \geqq 1/v\ . \tag{1.14}$$

Thus we have

$$\lim_{j\to\infty} N(S(j),\ S_\infty)\Big/\int_{S(j)\times S_\infty} dg = 1/v\ .$$

This implies that the sequence $\{S(j)\}_{j=1}^\infty$ has the uniform distribution property with respect to a Haar measure on G_A, and hence the proposition has been proved.

2. In this section, we assume that G is anisotropic, simply connected, and satisfies the condition (A).

We shall prove the following:

PROPOSITION 2. *If* (0.1) *is satisfied, the equality*

(2.1) $$\lim_{j\to\infty} \| f*\xi_j/\deg \xi_j - (f, 1)/v \| = 0$$

holds for any $f \in L^2(G_k\backslash G_A/K)$.

By virtue of Proposition 1, Theorem 1 is an immediate consequence of this result.

In this section we keep the normalization of Haar measures on G_{A_f}, G_∞, and G_A given in §1. We set $K_\mathfrak{g} = G_{O_\mathfrak{g}}$ for $\mathfrak{g} \in \mathscr{P}_f - \mathscr{S}$ and normalize the Haar measure $dg_\mathfrak{g}$ on $G_{k_\mathfrak{g}}$ for $\mathfrak{g} \in \mathscr{P}_f$ so that $\int_{K_\mathfrak{g}} dg_\mathfrak{g} = 1$. The product measure $\prod_{\mathfrak{g}\in\mathscr{P}_f} dg_\mathfrak{g}$ is equal to the previously normalized Haar measure dg_f on G_{A_f}.

Let $L^2(G_k\backslash G_A)$ be the Hilbert space of square integrable functions on $G_k\backslash G_A$. Then $L^2(G_k\backslash G_A/K)$ is the closed subspace of right K-invariant functions in $L^2(G_k\backslash G_A)$.

The next lemma is easily verified.

LEMMA 2. *Let* $\{T_j\}_{j=1}^{\infty}$ *be a sequence of bounded linear operators on a Hilbert space* H *such that*

$$\sup_{j\geq 1} \| T_j \| < \infty .$$

Let $\{F_n\}_{n=1}^{\infty}$ *be a countable orthonormal basis of* H. *If we have*

$$\lim_{j\to\infty} \| T_j F_n \| = 0$$

for all n, *then we have, for any* $F \in H$,

$$\lim_{j\to\infty} \| T_j F \| = 0 .$$

For each $f \in L^2(G_k\backslash G_A/K)$, set

$$T_j f = f*\xi_j/\deg \xi_j - (f, 1)/v .$$

Then the mapping $f \mapsto T_j f$ gives rise to a bounded linear operator on $L^2(G_k\backslash G_A/K)$. For any $x \in G_A$ and $f \in L^2(G_k\backslash G_A)$, we set $R_x f(g) = f(gx)$. Then R_x is a norm preserving linear operator on $L^2(G_k\backslash G_A)$. Since we can write, for any $f \in L^2(G_k\backslash G_A/K)$,

$$f*\xi_j = \sum_{l=1}^{\deg \xi_j} R_{\sigma_l^{(j)-1}} f,$$

we have

$$\| f*\xi_j \| \leq \sum_{l=1}^{\deg \xi_j} \| R_{\sigma_l^{(j)-1}} f \| = \deg \xi_j \, \| f \| .$$

Hence

$$\|T_j f\| \leqq \|f\| + \frac{|(f, 1)|}{v} \|1\| \leqq 2\|f\| .$$

This implies that

$$\sup_{j \geqq 1} \|T_j\| \leqq 2 .$$

We shall pick up a well-chosen orthonormal basis of $L^2(G_k \backslash G_A/K)$ and prove that the equality (2.1) holds for each member of this basis. Then Lemma 2 implies that the equality (2.1) will hold for any $f \in L^2(G_k \backslash G_A/K)$ and hence Proposition 2 will be proved.

Let Π be the right regular representation of G_A on $L^2(G_k \backslash G_A)$. Since $G_k \backslash G_A$ is compact, the unitary representation Π decomposes into a direct sum of at most countable irreducible unitary representations of G_A with finite multiplicities (cf. [4], Chap. I, § 2.3, Theorem). We write

$$L^2(G_k \backslash G_A) = \sum_{n=0}^{\infty} H^{(n)} , \quad \Pi = \sum_{n=0}^{\infty} \pi^{(n)}$$

where $H^{(n)}$ is a closed G_A-invariant subspace of $L^2(G_k \backslash G_A)$, and $\pi^{(n)}$ is the restriction of π to $H^{(n)}$ ($\pi^{(n)}$ is irreducible). We may assume that $H^{(0)} = \boldsymbol{C} \cdot 1$ and that $\pi^{(0)}$ is trivial. Furthermore each representation $\pi^{(n)}$ is factorizable. That is to say, there exists an irreducible unitary representation $\pi_v^{(n)}$ of G_{k_v} for every place v of k satisfying the following conditions (2.2) and (2.3).

(2.2) Except for a finite number of v, the representation space $H_v^{(n)}$ of $\pi_v^{(n)}$ contains a G_{O_v}-invariant unit vector f_0^v which is unique up to a scalar multiple.

(2.3) The restricted tensor product $\bigotimes_v \pi_v^{(n)}$ with respect to the family $\{f_0^v\}$ is unitarily equivalent to $\pi^{(n)}$ (cf. [4], Chap. III, § 6.2).

Then we have the decomposition;

$$L^2(G_k \backslash G_A/K) = \sum_{n=0}^{\infty} (L^2(G_k \backslash G_A/K) \cap H^{(n)}) .$$

Now we shall choose an orthonormal basis of $L^2(G_k \backslash G_A/K) \cap H^{(n)}$ for each n. Then the union of bases for all n forms an orthonormal basis of $L^2(G_k \backslash G_A/K)$. When $n = 0$, we have

$$L^2(G_k \backslash G_A/K) \cap H^{(0)} = H^{(0)} ,$$

and $\{1/\sqrt{v}\}$ can be taken as its orthonormal basis. It is obvious that

$$T_j(1/\sqrt{v}) = 1/\sqrt{v} - (1/\sqrt{v}, 1)/v = 0 .$$

From now on, we fix an index $n \geqq 1$, and, for simplicity, we drop the index n. Hence we write H, π, π_v, and H_v for $H^{(n)}, \pi^{(n)}, \pi_v^{(n)}$, and $H_v^{(n)}$ respectively. Note that, for any $F \in H \cap L^2(G_k \backslash G_A/K)$, we have $(F, 1) = 0$ and $T_j F = F * \xi_j / \deg \xi_j$. Let us take an isometric linear mapping T from the restricted tensor product $\bigotimes_v H_v$ to H, intertwining $\bigotimes_v \pi_v$ and π. For $\mathfrak{g} \in \mathscr{P}_f$, let $V_\mathfrak{g}$ be the space of $K_\mathfrak{g}$-invariant vectors in $H_\mathfrak{g}$. Then $V_\mathfrak{g}$ is finite dimensional (cf. [1], Theorem 1). In view of (2.2), there exists a finite subset $\mathscr{S}'$ of $\mathscr{P}_f$ containing $\mathscr{S}$ such that $V_\mathfrak{g}$ is one dimensional if $\mathfrak{g} \in \mathscr{P}_f - \mathscr{S}'$. Then we can take as a countable orthonormal basis of $H \cap L^2(G_k \backslash G_A/K)$ a set of elements of the form $T(\otimes \varphi^v)$ where $\varphi^v \in H_v$ for every place v and satisfies the following condition (C):

$$(\mathrm{C}) \quad \begin{cases} (\mathrm{i}) & \|\varphi^v\|_v = 1 \text{ for any place } v, \text{ where } \|\ \|_v \text{ denotes the norm of } H_v. \\ (\mathrm{ii}) & \varphi^\mathfrak{g} \in V_\mathfrak{g} \text{ for } \mathfrak{g} \in \mathscr{S}'. \\ (\mathrm{iii}) & \varphi^\mathfrak{g} = f_0^\mathfrak{g} \text{ for } \mathfrak{g} \in \mathscr{P}_f - \mathscr{S}'. \end{cases}$$

Hence, to show Proposition 2, it is enough to establish the following:

PROPOSITION 3. *Notation being as above, for any element $F \in H$ of the form $T(\bigotimes_v \varphi_F^v)$ where φ_F^v satisfies the condition* (C), *we have*

$$\lim_{j \to \infty} \| F * \xi_j / \deg \xi_j \| = 0 . \tag{2.4}$$

Proof. We set $S_\mathfrak{g}(j) = K_\mathfrak{g}$ for all j if $\mathfrak{g} \in \mathscr{P}_f - \mathscr{S}$. Let $\xi_j^\mathfrak{g}$ be the characteristic function of $S_\mathfrak{g}(j)$ for $\mathfrak{g} \in \mathscr{P}_f$. Then $\xi_j^\mathfrak{g}$ is $K_\mathfrak{g}$-biinvariant, continuous, and compactly supported on $G_{k_\mathfrak{g}}$. For $g_f = (\cdots, g_\mathfrak{g}, \cdots) \in G_{A_f}$, we have

$$\xi_j(g_f) = \prod_{\mathfrak{g} \in \mathscr{P}_f} \xi_j^\mathfrak{g}(g_\mathfrak{g}) .$$

For $\mathfrak{g} \in \mathscr{S}'$, choose an orthonormal basis $\{f_l^\mathfrak{g} \mid 1 \leqq l \leqq \dim V_\mathfrak{g}\}$ of $V_\mathfrak{g}$. We may assume that $f_1^\mathfrak{g} = \varphi_F^\mathfrak{g}$. Then, for $\mathfrak{g} \in \mathscr{S}'$, the integral

$$f_l^\mathfrak{g} * \xi_j^\mathfrak{g} = \int_{G_{k_\mathfrak{g}}} \xi_j^\mathfrak{g}(g_\mathfrak{g}^{-1}) \pi_\mathfrak{g}(g_\mathfrak{g}) f_l^\mathfrak{g} dg_\mathfrak{g}$$

belongs to $H_\mathfrak{g}$ and is invariant under the action of $K_\mathfrak{g}$ through $\pi_\mathfrak{g}$. Thus it belongs to $V_\mathfrak{g}$ and hence can be written as a $\boldsymbol{C}$-linear combination of $f_m^\mathfrak{g} (1 \leqq m \leqq \dim V_\mathfrak{g})$. That is to say, we have

$$f_l^\mathfrak{g} * \xi_j^\mathfrak{g} = \sum_{m=1}^{\dim V_\mathfrak{g}} f_m^\mathfrak{g} \cdot \lambda_{m,l}^\mathfrak{g}(\xi_j)$$

where

$$\lambda_{m,l}^{\mathfrak{g}}(\xi_j) = \int_{G_{k_\mathfrak{g}}} \xi_j^{\mathfrak{g}}(g_\mathfrak{g}^{-1})(\pi_\mathfrak{g}(g_\mathfrak{g})f_l^\mathfrak{g}, f_m^\mathfrak{g})_\mathfrak{g} dg_\mathfrak{g}$$
$$= \int_{S_\mathfrak{g}(j)} (\pi_\mathfrak{g}(g_\mathfrak{g}^{-1})f_l^\mathfrak{g}, f_m^\mathfrak{g})_\mathfrak{g} dg_\mathfrak{g} .$$

(Here $(,)_\mathfrak{g}$ denotes the inner product of $H_\mathfrak{g}$.) Now we have, for any $x \in G_A$,

$$F*\xi_j(x) = \int_{G_{A_f}} F(xg_f)\xi_j(g_f^{-1})dg_f$$
$$= \int_{G_{A_f}} (\pi(g_f)T(\otimes \varphi_F^v))(x)\xi_j(g_f^{-1})dg_f$$
$$= T\Big(\bigotimes_{v\in\mathscr{P}_\infty} \varphi_F^v \otimes \bigotimes_{\mathfrak{g}\in\mathscr{P}_f} \int_{G_{k_\mathfrak{g}}} \xi_j^\mathfrak{g}(g_\mathfrak{g}^{-1})\pi_\mathfrak{g}(g_\mathfrak{g})\varphi_F^\mathfrak{g} dg_\mathfrak{g}\Big)(x)$$
$$= T\big(\bigotimes_{v\in\mathscr{P}_\infty} \varphi_F^v \otimes \bigotimes_{\mathfrak{g}\in\mathscr{P}_f-\mathscr{S}'} f_0^\mathfrak{g} \otimes \bigotimes_{\mathfrak{g}\in\mathscr{S}'} f_1^\mathfrak{g}*\xi_j^\mathfrak{g}\big)(x) .$$

Since T is norm-preserving, we get

$$||F*\xi_j|| = \prod_{v\in\mathscr{P}_\infty} ||\varphi_F^v||_v \times \prod_{\mathfrak{g}\in\mathscr{P}_f-\mathscr{S}'} ||f_0^\mathfrak{g}||_\mathfrak{g} \times \prod_{\mathfrak{g}\in\mathscr{S}'} ||f_1^\mathfrak{g}*\xi_j^\mathfrak{g}||_\mathfrak{g}$$
$$= \prod_{\mathfrak{g}\in\mathscr{S}'} \Big|\Big|\sum_{m=1}^{\dim V_\mathfrak{g}} f_m^\mathfrak{g}\cdot\lambda_{m,1}^\mathfrak{g}(\xi_j)\Big|\Big|_\mathfrak{g}$$
$$\leqq \prod_{\mathfrak{g}\in\mathscr{S}'} \Big(\sum_{m=1}^{\dim V_\mathfrak{g}} |\lambda_{m,1}^\mathfrak{g}(\xi_j)|\Big) .$$

On the other hand, if we put

$$\deg \xi_j^\mathfrak{g} = \int_{G_{k_\mathfrak{g}}} \xi_j^\mathfrak{g}(g_\mathfrak{g})dg_\mathfrak{g} = \int_{S_\mathfrak{g}(j)} dg_\mathfrak{g}$$

for $\mathfrak{g} \in \mathscr{P}_f$, it is easy to see (recall the definition (1.2))

$$\deg \xi_j = \prod_{\mathfrak{g}\in\mathscr{S}'} \deg \xi_j^\mathfrak{g}$$

(in fact it equals $\prod_{\mathfrak{g}\in\mathscr{S}} \deg \xi_j^\mathfrak{g}$). Thus we have

$$\text{(2.5)} \qquad ||F*\xi_j/\deg \xi_j|| \leqq \prod_{\mathfrak{g}\in\mathscr{S}'} \Big(\sum_{m=1}^{\dim V_\mathfrak{g}} |\lambda_{m,1}^\mathfrak{g}(\xi_j)|/\deg \xi_j^\mathfrak{g}\Big) .$$

Since $S_\mathfrak{g}(j) = K_\mathfrak{g}$ for $\mathfrak{g} \in \mathscr{S}' - \mathscr{S}$, we get finally

$$\text{(2.6)} \quad ||F*\xi_j/\deg \xi_j|| \leqq C \prod_{\mathfrak{g}\in\mathscr{S}} \int_{S_\mathfrak{g}(j)} \sum_{m=1}^{\dim V_\mathfrak{g}} |(\pi_\mathfrak{g}(g_\mathfrak{g}^{-1})f_1^\mathfrak{g}, f_m^\mathfrak{g})_\mathfrak{g}|\, dg_\mathfrak{g} \Big/ \int_{S_\mathfrak{g}(j)} dg_\mathfrak{g}$$

where $C = \prod_{\mathfrak{g}\in\mathscr{S}'-\mathscr{S}} \dim V_\mathfrak{g}$.

Now we shall show that $\pi_\mathfrak{g}$ is not one dimensional if $\mathfrak{g} \in \mathscr{S}$ and $G_{k_\mathfrak{g}}$ is noncompact. More generally, we shall prove the following:

LEMMA 3. *Let H be a connected, simply connected semisimple* *linear algebraic group defined over an algebraic number field k, and let v be a place of k. Assume that H is k_v-almost simple and that $H_k H_{k_v}$ is dense in H_A. Let ρ be an irreducible unitary representation of H_A realized on a closed subspace $\mathscr{H}$ of $L^2(H_k\backslash H_A)$ by right translation. Furthermore assume that ρ decomposes into a restricted tensor product $\otimes \rho_w$ of irreducible unitary representations ρ_w of H_{k_w}. If ρ is nontrivial, then ρ_v is not one dimensional.*

Proof. Since $H_k H_{k_v}$ is dense in H_A, H_{k_v} is not compact, and hence $\mathrm{rank}_{k_v} H \geqq 1$ (cf. [9], p. 187). It is known that, if X is a semisimple, simply connected, almost simple linear algebraic group defined over a local field K with positive K-rank, then X_K coincides with its own commutator (cf. [3], 6–4 and 6–15; see also [6], Appendix II, Theorem). Hence H_{k_v} has no nontrivial unitary characters. Assume that ρ_v is trivial. Then every element in $\mathscr{H}$ is right H_{k_v}-invariant as a function on H_A. There exists $\varphi \in C_c^0(H_A)$ and $f \in \mathscr{H}$ such that

$$F = \int_{H_A} \varphi(h)\rho(h) f dh \neq 0 \text{ (as an element of } \mathscr{H}) .$$

It is easy to see that F is, as a function on H_A, continuous. Since F is right H_{k_v}-invariant and left H_k-invariant, F is constant on $H_k H_{k_v}$ which is dense in H_A. Hence F is a nonzero constant function on H_A. Since ρ is irreducible, we have $\mathscr{H} = \boldsymbol{C}\cdot 1$ and ρ is trivial. Contradiction! The lemma is proved.

Applying Lemma 3 for $(H, \rho, v) = (G, \pi, \mathfrak{g})$ ($\mathfrak{g} \in \mathscr{S}$ and $G_{k_\mathfrak{g}}$ is noncompact), we see that $\pi_\mathfrak{g}$ is not one dimensional.

On the other hand, Howe and Moore proved the following:

LEMMA 4 (cf. [6] Theorem 5.2).[6] *Let π be an irreducible unitary representation of k-almost simple, simply connected linear algebraic group G defined over a local field k on a Hilbert space H. For any $x, y \in H$, set $\rho_{x,y}(g) = (\pi(g)x, y)$. Then $\rho_{x,y}$ vanishes at infinity if π is not one dimensional.*

(Here we say that a continuous function f on a locally compact group G *vanishes at infinity*, if, for any $\varepsilon > 0$, there exists a compact subset C of G such that $\sup_{g \in G-C} |f(g)| < \varepsilon$. In case that G is

[6] In case $G=\mathrm{SL}(2)$ over a $\mathfrak{g}$-adic field $k_\mathfrak{g}$, this result easily follows from the existence of the Kirillov model of ρ (that is to say, ρ is realized on a closed subspace $\mathscr{H}$ of $L^2(k_\mathfrak{g}^\times)$, and, for any $f \in \mathscr{H}$ and for any $a \in k_\mathfrak{g}^\times$, we have

$$\rho\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} f(x) = f(a^2 x) \quad (x \in k_\mathfrak{g}^\times)$$

(cf. [4], Appendix to Chapter II, n°2 and n°5)).

compact, every continuous function is said to vanish at infinity.)

Thus we conclude that the function on $G_{k_{\mathfrak{g}}}$ given by $g_{\mathfrak{g}} \mapsto |(\pi_{\mathfrak{g}}(g_{\mathfrak{g}}^{-1})f_1^{\mathfrak{g}}, f_m^{\mathfrak{g}})_{\mathfrak{g}}|$ $(1 \leqq m \leqq \dim V_{\mathfrak{g}})$ vanishes at infinity. In view of (2.6), the proof of Proposition 3 has now been reduced to the following lemma.

LEMMA 5. *Let $G_l (1 \leqq l \leqq m)$ be a locally compact group with a left invariant measure dg_l. Let $f_l (1 \leqq l \leqq m)$ be a continuous function on G_l vanishing at infinity. Let $\{S_l(j)\}_{j=1}^{\infty}$ be a sequence of open compact subsets of G_l such that*

$$\inf_{j \geqq 1} \int_{S_l(j)} dg_l = \eta_l > 0 \quad (1 \leqq l \leqq m) .$$

If

$$\lim_{j \to \infty} \prod_{l=1}^{m} \int_{S_l(j)} dg_l = +\infty , \tag{2.7}$$

then the following equality holds:

$$\lim_{j \to \infty} \prod_{l=1}^{m} \left(\int_{S_l(j)} f_l(g_l) dg_l \Big/ \int_{S_l(j)} dg_l \right) = 0 . \tag{2.8}$$

Proof. Note that (2.7) implies that, for at least one $l \in M = \{1, 2, \cdots, m\}$, G_l is noncompact. Observe that

$$\left| \int_{S_l(j)} f_l(g_l) dg_l \Big/ \int_{S_l(j)} dg_l \right|$$

is bounded if G_l is compact. Hence we may assume that G_l is noncompact for every $l \in M$. Set $N_l = \sup_{g \in G_l} |f_l(g)|$. Then $N_l < \infty (1 \leqq l \leqq m)$. For any $\varepsilon > 0$, there is a compact subset C_l of G_l such that $|f_l(g)| < \varepsilon$ for every $g \in G_l - C_l$. By a simple calculation, we have

$$\begin{aligned} &\prod_{l \in M} \int_{S_l(j)} f_l(g_l) dg_l \\ &\quad = \sum_{A \subset M} \prod_{l \in A} \int_{S_l(j) \cap C_l} f_l(g_l) dg_l \\ &\quad \cdot \prod_{l \in M - A} \int_{S_l(j) \cap (G_l - C_l)} f_l(g_l) dg_l \end{aligned}$$

where A ranges over the collection of all subsets of M. To simplify the notation, we set

$$J_l = \int_{C_l} dg_l \text{ and } K_l(j) = \int_{S_l(j)} dg_l .$$

Since

$$\left|\int_{S_l(j)\cap C_l} f_l(g_l)dg_l\right| \leqq \mathrm{Min}\,(N_lJ_l,\ N_lK_l(j)) ,$$

and

$$\left|\int_{S_l(j)\cap(G_l-C_l)} f_l(g_l)dg_l\right| \leqq \varepsilon K_l(j) ,$$

we have

$$\begin{aligned}\Big|\prod_{l\in M}\Big\{\int_{S_l(j)} f_l(g_l)dg_l/K_l(j)\Big\}\Big| \\ &\leqq \prod_{l\in M} N_lJ_l/\prod_{l\in M} S_l(j) \\ &+ \sum_{\Lambda\subsetneqq M}\prod_{l\in\Lambda} N_lK_l(j) \prod_{l\in M-\Lambda}\varepsilon\cdot K_l(j)\cdot \prod_{l\in M} K_l(j)^{-1} \\ &= \prod_{l\in M} N_lJ_l/\prod_{l\in M} S_l(j) + \sum_{\Lambda\subsetneqq M}\varepsilon^{|M-\Lambda|}\cdot\prod_{l\in\Lambda} N_l .\end{aligned}$$

Here $|M-\Lambda|$ denotes the number of elements in the set $M-\Lambda$. Hence, if $\varepsilon < 1$, we have

$$\begin{aligned}\limsup_{j\to\infty}\Big|\prod_{l\in M}\Big\{\int_{S_l(j)} f_l(g_l)dg_l/K_l(j)\Big\}\Big| \\ &\leqq \varepsilon\sum_{\Lambda\subsetneqq M}\prod_{l\in\Lambda} N_l .\end{aligned}$$

Since we can choose arbitrary small $\varepsilon > 0$, we obtain the equality (2.8).

Thus Theorem 1 has been established.

3. In this section, we always assume $G = \mathrm{SL}_2$ (regarded as a linear algebraic group defined over an algebraic number field k). We shall prove Proposition 2 under our assumptions. Note that Proposition 2 for $G = \mathrm{SL}_2$ implies Theorem 2, by virtue of Proposition 1. We set

$$U = \left\{\begin{pmatrix}1 & u\\ & 1\end{pmatrix}\right\},\quad H = \left\{\begin{pmatrix}t & \\ & t^{-1}\end{pmatrix}\Big|\, t\neq 0\right\},\quad \text{and } P = UH .$$

These groups can be naturally regarded as k-subgroups of G. For any $F\in C_c^0(U_AH_k\backslash G_A)$, set

$$\theta_F(g) = \sum_{\gamma\in P_k\backslash G_k} F(\gamma g) . \tag{3.1}$$

The series (3.1) converges absolutely and uniformly on any compact subset of G_A. The function θ_F is, as a function on $G_k\backslash G_A$, continuous and compactly supported, and hence square integrable on $G_k\backslash G_A$ (cf. [5], § 6). Let Θ be the closure of the subspace of $L^2(G_k\backslash G_A)$ spanned by all elements of the form θ_F with $F\in C_c^0(U_AH_k\backslash G_A)$. Let $\mathscr{H}$ be the closed subspace of $L^2(G_k\backslash G_A)$ consisting of all elements f such that the integral $\int_{U_k\backslash U_A} f(ug)du$ vanishes for almost all $g \in G_A$. Then Θ and $\mathscr{H}$ are both right G_A-invariant. It is known that $L^2(G_k\backslash G_A)$ is the direct orthogonal sum of Θ and $\mathscr{H}$ (cf. [5], § 7). It follows that $L^2(G_k\backslash G_A/K)$ is the direct orthogonal sum of $\tilde{\Theta} = \Theta \cap L^2(G_k\backslash G_A/K)$ and $\tilde{\mathscr{H}} = \mathscr{H} \cap L^2(G_k\backslash G_A/K)$; $L^2(G_k\backslash G_A/K) = \tilde{\Theta} \oplus \tilde{\mathscr{H}}$ (direct orthogonal sum). Hence, for any $f \in L^2(G_k\backslash G_A/K)$, we can write $f = \varphi + \psi$ where $\varphi \in \tilde{\Theta}$ and $\psi \in \tilde{\mathscr{H}}$. As is well-known, $\tilde{\Theta}$ contains constant functions. Hence $\tilde{\mathscr{H}}$ is orthogonal to $\boldsymbol{C}\cdot 1$. Thus we have

$$\|f*\xi_j/\deg \xi_j - (f, 1)/v\| \leqq \|\varphi*\xi_j/\deg \xi_j - (\varphi, 1)/v\| + \|\psi*\xi_j/\deg \xi_j\| .$$

Hence the proof of Proposition 2 in our case has now been reduced to the verification of the following two propositions.

PROPOSITION 4. *If* (0.1) *is satisfied, the equality*

$$\lim_{j\to\infty} \|\varphi*\xi_j/\deg \xi_j - (\varphi, 1)/v\| = 0 \tag{3.2}$$

holds for any $\varphi \in \tilde{\Theta}$.

PROPOSITION 5. *If* (0.1) *is satisfied, the equality*

$$\lim_{j\to\infty} \|\psi*\xi_j/\deg \xi_j\| = 0 \tag{3.3}$$

holds for any $\psi \in \tilde{\mathscr{H}}$.

It is known that the right regular representation on $\mathscr{H}$ decomposes into a direct orthogonal sum of at most countable irreducible and factorizable unitary representations with finite multiplicities (cf. [5], § 2 and [4], Chap. III, § 3-3, Theorem). Then, to prove Proposition 5, we just repeat the argument of the proof of Proposition 2 in § 2, replacing $L^2(G_k\backslash G_A/K)$ with $\tilde{\mathscr{H}}$.

In order to show Proposition 4, we need several results about the spectral decomposition of Θ.

Let $M = \prod_v M_v$ be the maximal compact subgroup of G_A, where we set

$$M_v = \begin{cases} G_{\mathcal{O}_v} & \text{if } v \in \mathscr{P}_f \\ SO(2) & \text{if } v \in \mathscr{P}_\infty \text{ and } k_v \cong \boldsymbol{R} \\ SU(2) & \text{if } v \in \mathscr{P}_\infty \text{ and } k_v \cong \boldsymbol{C} . \end{cases}$$

Then we have $G_A = U_A H_A M$. We fix, once and for all, the Iwasawa decomposition of $g \in G_A$ given by

$$g = \underline{u}(g)\underline{h}(g)\underline{m}(g) ,$$

where $\underline{u}(g) \in U_A$, $\underline{h}(g) \in H_A$, and $\underline{m}(g) \in M$. We normalize the Haar measure dg on G_A by putting, for any $f \in C_c^0(G_A)$,

$$\int_{G_A} f(g)dg = \int_M dm \int_{H_A} |\beta(h)|_A^{-1} dh \int_{U_A} f(uhm)du . \tag{3.4}$$

Here we set

$$\beta(h) = t^2$$

for

$$h = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in H_A ,$$

and we denote by du, dh, and dm Haar measures on U_A, H_A, and M respectively, which are normalized by the following conditions;

$$\int_{U_k \backslash U_A} du = 1, \int_{H_k \backslash H_A, |\beta(h)| \leq 1} |\beta(h)|_A^s dh = 1/s \quad (\mathrm{Re}\, s > 0) ,$$

and

$$\int_M dm = 1 .$$

From now on, we normalize the Haar measure $dg_\mathfrak{g}$ on $G_{k_\mathfrak{g}} (\mathfrak{g} \in \mathscr{P}_f)$ so that $\int_{M_\mathfrak{g}} dg_\mathfrak{g} = 1$. Let dg_f be a Haar measure on G_{A_f} given by

$$dg_f = \prod_{\mathfrak{g} \in \mathscr{P}_f} dg_\mathfrak{g} \quad (g_f = \prod_{\mathfrak{g} \in \mathscr{P}_f} g_\mathfrak{g} \in G_{A_f}) . \tag{3.5}$$

We normalize the Haar measure dg_∞ on G_∞ so that $dg = dg_\infty dg_f$ $(g = g_\infty g_f)$, where dg and dg_f are given by (3.4) and (3.5), respectively.

Let I_1 be the subgroup of I consisting of ideles with module 1. For a positive real number λ, we denote by $\xi(\lambda)$ the idele such that $\xi(\lambda)_\mathfrak{g} = 1$ for every $\mathfrak{g} \in \mathscr{P}_f$ and $\xi(\lambda)_v = \lambda$ for every $v \in \mathscr{P}_\infty$. Let N be the image of $\{\xi(\lambda); \lambda > 0\}$ by the natural projection from I to $k^\times \backslash I$. Then we have

$$k^\times \backslash I = (k^\times \backslash I_1) \times N \quad \text{(direct product)} .$$

Let X_1 be the set of all unitary characters on $H_k \backslash H_A$ which are trivial on the image of N by the natural isomorphism from $k^\times \backslash I$ to $H_k \backslash H_A$. Then X_1 can be identified with the dual of $k^\times \backslash I_1$. Since $k^\times \backslash I_1$ is compact, X_1 is discrete (cf. [15], Chap. VII, § 4).

We fix a complete system $M^{\wedge}$ of representatives of equivalence classes of finite dimensional irreducible unitary representations of M. Let H_τ be the representation space of $\tau \in M^{\wedge}$. For $\chi \in X_1$, let $H_\tau(\chi)$ be the subspace of H_τ consisting of all vectors $v \in H_\tau$ which satisfy the following equality:

$$v \cdot \tau(uh) = v \cdot \chi^{-1}(h) \quad (\forall uh \in P_A \cap M) .$$

We denote by $X_1(\tau)$ the set of all elements $\chi \in X_1$ such that $H_\tau(\chi) \neq 0$. It is easy to see that $X_1(\tau)$ is a finite set.

For $F \in C_c^0(U_A H_k \backslash G_A)$, $\tau \in M^{\wedge}$, $\chi \in X_1(\tau)$, and for $s \in \boldsymbol{C}$, we set

$$F^{\wedge}(s, \chi, \tau) = \int_M \int_{H_k/H_A} F(hm)\tau(m^{-1})\chi(h) \, |\beta(h)|_A^{-s} dh dm . \tag{3.6}$$

The integral (3.6) converges for any $s \in \boldsymbol{C}$. As a function of s, $F^{\wedge}(s, \chi, \tau)$ is a holomorphic function in $\boldsymbol{C}$ with values in $\mathrm{End}_{\boldsymbol{C}}(H_\tau)$. Set

$$\theta_F^{\wedge}(s, \chi, \tau) = \int_M \int_{H_k \backslash H_A} \int_{U_k \backslash U_A} \theta_F(uhm)\tau(m^{-1})\chi(h) \, |\beta(h)|_A^{s-1} du dh dm , \tag{3.7}$$

where θ_F is given by (3.1). The integral (3.7) converges absolutely and uniformly on any compact subset of the domain $\{s \in \boldsymbol{C} \mid \mathrm{Re}\, s > 1\}$. As a function of s, $\theta_F^{\wedge}(s, \chi, \tau)$ is continued to a meromorphic function in $\boldsymbol{C}$ with values in $\mathrm{End}_{\boldsymbol{C}}(H_\tau)$ (cf. [5], § 6). It is known that

$$\theta_F^{\wedge}(s, \chi, \tau) = F^{\wedge}(1-s, \chi, \tau) + F^{\wedge}(s, \chi^{-1}, \tau)\Phi(s; \chi, \tau) , \tag{3.8}$$

where $\Phi(s; \chi, \tau)$ is a meromorphic function of s in $\boldsymbol{C}$ with values in $\mathrm{End}_{\boldsymbol{C}}(H_\tau)$ (cf. [5], § 6). Furthermore suppose that, as a function on G_A, $F(g)$ depends smoothly with respect to the archimedean components of g. Then the norm of θ_F in $L^2(G_k \backslash G_A)$ is given by the following formula (cf. [5], § 7, (7.8)):

$$\|\theta_F\|^2 = \frac{1}{2\pi\sqrt{-1}} \sum_{\tau \in M^{\wedge}} \sum_{\chi \in X_1(\tau)} \int_J \|\theta_F^{\wedge}(s, \chi, \tau)\|_\tau^2 ds + \frac{1}{v} |(\theta_F, 1)|^2 . \tag{3.9}$$

Here $\|T\|_\tau^2$ denotes $\dim \tau \cdot \mathrm{Tr}(TT^*)$ for $T \in \mathrm{End}_{\boldsymbol{C}}(H_\tau)$, and we set

$$J = \left\{ s \in \boldsymbol{C} \,\middle|\, \mathrm{Re}\, s = \frac{1}{2},\ \mathrm{Im}\, s < 0 \right\} . \tag{3.10}$$

The following lemma is easily proved.

LEMMA 6. *Let $\{T_j\}_{j=1}^{\infty}$ be a sequence of bounded linear operators on a Hilbert space H such that $\sup_{j \geq 1} \|T_j\| < \infty$, and let H' be a dense subspace of H. Assume that, for any $v \in H'$,* (3.11) $$\lim_{j\to\infty} \| T_j v \| = 0 .$$

Then the equality (3.11) *holds for any* $v \in H$.

Now we are ready to prove Proposition 4. Let T_j be a linear operator on $\tilde{\Theta}$ given by

$$T_j\varphi = \varphi * \xi_j / \deg \xi_j - (\varphi, 1)/v \quad (\varphi \in \tilde{\Theta}) .$$

We have already seen that $\| T_j \| \leqq 2 (j = 1, 2, \cdots)$. We set $M_\infty = \prod_{v \in \mathscr{P}_\infty} M_v$. Then M_∞ is a maximal compact subgroup of G_∞. Let $\mathscr{D}$ be the space consisting of all continuous functions on $U_A H_k \backslash G_A / K$ satisfying the following conditions (3.12) and (3.13).

(3.12) $F(g)$ is compactly supported modulo $U_A H_k$.

(3.13) As a function on G_A, $F(g)$ depends smoothly on G_∞ and $F(g)$ is right M_∞-finite.

Let Θ' be the linear space spanned by elements θ_F with $F \in \mathscr{D}$. Then Θ' is a dense subspace of $\tilde{\Theta}$.

Now we shall prove that the following equality holds for any $\theta_F \in \Theta'$:

(3.14) $$\lim_{j\to\infty} \| T_j \theta_F \| = 0 .$$

Then, in view of Lemma 6, Proposition 4 will be proved. To show the equality (3.14), we need the next lemma.

LEMMA 7. *For any* $\theta_F \in \Theta'$, *we have*

(3.15) $$\| T_j \theta_F \|^2 = \frac{1}{2\pi\sqrt{-1}} \sum_{\tau \in M^\wedge} \sum_{\chi \in X_1(\tau)} \int_J \| (\theta_F * \xi_j)^\wedge (s, \chi, \tau) / \deg \xi_j \|_\tau^2 ds .$$

Proof. We have

$$\begin{aligned} \| T_j \theta_F \|^2 &= \| \theta_F * \xi_j / \deg \xi_j - (\theta_F, 1)/v \|^2 \\ &= \| \theta_F * \xi_j / \deg \xi_j \|^2 - 2 \operatorname{Re} \{ \overline{(\theta_F, 1)} v^{-1} / \deg \xi_j \cdot (\theta_F * \xi_j, 1) \} \\ &\quad + \| (\theta_F, 1)/v \|^2 . \end{aligned}$$

We set $\xi_j^\vee(g) = \xi_j(g^{-1})$. Then it is easily verified that

$$(f_1 * \xi_j, f_2) = (f_1, f_2 * \xi_j^\vee) \quad (f_1, f_2 \in L^2(G_k \backslash G_A / K)) ,$$

and that $\deg \xi_j = \deg \xi_j^\vee$. Hence we have

(3.16) $$(\theta_F * \xi_j, 1) = (\theta_F, 1 * \xi_j^\vee) = \deg \xi_j^\vee (\theta_F, 1) = \deg \xi_j (\theta_F, 1) .$$

Thus

$$\|T_j\theta_F\|^2 = \|\theta_F*\xi_j/\deg\xi_j\|^2 - |(\theta_F, 1)|^2/v .$$

Observe that $\theta_F*\xi_j$ also belongs to Θ'. Applying the formula (3.9) to $\theta_F*\xi_j$, we have

$$\|T_j\theta_F\|^2 = \frac{1}{2\pi\sqrt{-1}}\sum_{\tau\in M^\wedge}\sum_{\chi\in X_1(\tau)}\int_J \|(\theta_F*\xi_j)^\wedge(s, \chi, \tau)/\deg\xi_j\|_\tau^2 ds$$
$$+ \frac{1}{v}|(\theta_F*\xi_j/\deg\xi_j, 1)|^2 - |(\theta_F, 1)|^2/v .$$

The equality (3.16) implies that the last two terms of the right side of the above equality cancel each other, and hence the lemma is proved.

Since θ_F is, as a function on G_∞, right M_∞-finite, and since $\theta_F*\xi_j$ is right K-invariant, there exists a finite subset L of $M^\wedge$ such that $\tau\in M^\wedge - L$ always implies $(\theta_F*\xi_j)(s, \chi, \tau) = 0 (j = 1, 2, \cdots)$ for any $s\in \boldsymbol{C}$ and for any $\chi\in X_1(\tau)$. Thus the right side of (3.15) is a finite sum. Hence, to verify the equality (3.14), we have only to show that the following equality holds for any $\tau\in M^\wedge$ and any $\chi\in X_1(\tau)$:

$$\text{(3.17)} \qquad \lim_{j\to\infty}\frac{1}{2\pi\sqrt{-1}}\int_J \|(\theta_F*\xi_j)^\wedge(s, \chi, \tau)/\deg\xi_j\|_\tau^2 ds = 0 .$$

Observe that

$$\frac{1}{2\pi\sqrt{-1}}\int_J \|(\theta_F*\xi_j)^\wedge(s, \chi, \tau)/\deg\xi_j\|_\tau^2 ds$$
$$\leqq \frac{1}{2\pi\sqrt{-1}}\int_J\frac{1}{|s|^2}ds \times \sup_{s\in J}\|s\cdot(\theta_F*\xi_j)^\wedge(s, \chi, \tau)/\deg\xi_j\|_\tau^2$$
$$= \frac{1}{2}\sup_{s\in J}\|s\cdot(\theta_F*\xi_j)^\wedge(s, \chi, \tau)/\deg\xi_j\|_\tau^2 .$$

Hence the proof of (3.17), and hence of Proposition 4 has now been reduced to the verification of the following equality for any $F\in\mathscr{D}$, $\tau\in M^\wedge$, and for any $\chi\in X_1(\tau)$ under the assumption (0.1):

$$\text{(3.18)} \qquad \lim_{j\to\infty}\{\sup_{s\in J}\|s\cdot(\theta_F*\xi_j)^\wedge(s, \chi, \tau)/\deg\xi_j\|_\tau\} = 0$$

(recall that J is given by (3.10)).

To establish the equality (3.18), we need the following lemma.

LEMMA 8. *For $F\in\mathscr{D}$, $\tau\in M^\wedge$, and $\chi\in X_1(\tau)$, there exists a positive constant C such that the following inequality holds for any $s\in J = \{s\in\boldsymbol{C}\mid \operatorname{Re}s = 1/2, \operatorname{Im}s < 0\}$:*

$$\|s\cdot(\theta_F*\xi_j)^\wedge(s,\chi,\tau)\|_\tau \leqq C\int_M\Big(\int_{G_{A_f}} |\beta(\underline{h}(mg_f^{-1}))|_A^{1/2}\xi_j(g_f)dg_f dm \quad (j=1,2,\cdots). \tag{3.19}$$

Proof. We set

$$F*\xi_j(g)=\int_{G_{A_f}} F(gh_f^{-1})\xi_j(h_f)dh_f\,. \tag{3.20}$$

Then it is easily verified that $F*\xi_j$ also belongs to $\mathscr{D}$, and that $\theta_F*\xi_j=\theta_{F*\xi_j}$. Applying (3.8) to $\theta_F*\xi_j=\theta_{F*\xi_j}$, we obtain

$$\begin{aligned}(\theta_F*\xi_j)^\wedge(s,\chi,\tau) &= (F*\xi_j)^\wedge(1-s,\chi,\tau)\\ &\quad+(F*\xi_j)^\wedge(s,\chi^{-1},\tau)\Phi(s;\chi,\tau)\,.\end{aligned} \tag{3.21}$$

In view of (3.6) and (3.20), we have

$$\begin{aligned}&(F*\xi_j)^\wedge(s,\chi,\tau)\\ &\quad=\int_M\int_{H_k\backslash H_A}\int_{G_{A_f}} F(hmg_f^{-1})\xi_j(g_f)\tau(m^{-1})\chi(h)\,|\beta(h)|_A^{-s}dg_f dh dm\,.\end{aligned}$$

Observing that

$$hmg_f^{-1}=h\underline{u}(mg_f^{-1})h^{-1}\cdot h\underline{h}(mg_f^{-1})\cdot\underline{m}(mg_f^{-1})\,,$$

we have

$$\begin{aligned}&(F*\xi_j)^\wedge(s,\chi,\tau)\\ &=\int_M\int_{H_k\backslash H_A}\int_{G_{A_f}} F(h\underline{h}(mg_f^{-1})\underline{m}(mg_f^{-1}))\xi_j(g_f)\tau(m^{-1})\chi(h)\,|\beta(h)|_A^{-s}dg_f dh dm\\ &=\int_M\int_{G_{A_f}}\Big(\int_{H_k\backslash H_A} F(h\cdot\underline{m}(mg_f^{-1}))\chi(h)\,|\beta(h)|_A^{-s}dh\Big)\chi^{-1}(\underline{h}(mg_f^{-1}))\\ &\qquad\cdot|\beta(\underline{h}(mg_f^{-1}))|_A^{s}\xi_j(g_f)\tau(m^{-1})dg_f dm\end{aligned}$$

(note that $h\underline{u}(mg_f^{-1})h^{-1}\in U_A$ and that F is left U_A-invariant). Set

$$F^\wedge(g,s,\chi)=\int_{H_k\backslash H_A} F(hg)\chi(h)\,|\beta(h)|_A^{-s}dh\,.$$

This integral converges absolutely for any $s\in\boldsymbol{C}$ and for any $g\in G_A$. Then,

$$\begin{aligned}(F*\xi_j)^\wedge(s,\chi,\tau) &=\int_M\int_{G_{A_f}} F^\wedge(\underline{m}(mg_f^{-1}),s,\chi)\chi^{-1}(\underline{h}(mg_f^{-1}))\\ &\qquad\cdot|\beta(\underline{h}(mg_f^{-1}))|_A^{s}\xi_j(g_f)\tau(m^{-1})dg_f dm\,.\end{aligned} \tag{3.22}$$

Observe that

$$\int_M \tau(m^{-1})F^\wedge(m,s,\chi)dm=F^\wedge(s,\chi,\tau)\,.$$

Then applying Peter-Weyl's theorem, we have

$$F^{\wedge}(m, s, \chi) = \sum_{\tau \in M^{\wedge}} \dim \tau \cdot \mathrm{Tr}\,[\tau(m) F^{\wedge}(s, \chi, \tau)] \tag{3.23}$$

for $m \in M$. Since F is, as a function on G_A, right M-finite, the right side of (3.23) is a finite sum. Moreover it is known that $F^{\wedge}(s, \chi, \tau)$ is, as a function of s, rapidly decreasing at infinity in any vertical strip (cf. [5], § 7). Hence, if $P(s)$ is a polynomial of s, we have

$$\sup_{\mathrm{Re}\, s=1/2, m \in M} |P(s) \cdot F^{\wedge}(m, s, \chi)| < \infty .$$

We set

$$C_1 = \sup_{s \in J\ m \in M} |s \cdot F^{\wedge}(m, 1-s, \chi)|$$

and

$$C_2 = \sup_{s \in J \cdot m \in M} |s \cdot F^{\wedge}(m, s, \chi^{-1})| .$$

In view of (3.22), we have, for $s \in J = \{s \in \boldsymbol{C} \mid \mathrm{Re}\, s = 1/2,\ \mathrm{Im}\, s < 0\}$,

$$\begin{aligned} &\|s \cdot (F * \xi_j)^{\wedge}(1-s, \chi, \tau)\|_\tau \\ &\leqq \int_M \int_{G_{A_f}} |s \cdot F^{\wedge}(\underline{m}(m g_f^{-1}), 1-s, \chi)| \\ &\quad \cdot |\beta(\underline{h}(m g_f^{-1}))|_A^{1/2} \xi_j(g_f)\, \|\tau(m^{-1})\|_\tau dg_f dm \\ &\leqq C_1 \cdot \dim \tau \int_M \int_{G_{A_f}} |\beta(\underline{h}(m g_f^{-1}))|_A^{1/2} \xi_j(g_f) dg_f dm . \end{aligned} \tag{3.24}$$

Similarly we obtain the following inequality for $s \in J$:

$$\begin{aligned} &\|s \cdot (F * \xi_j)^{\wedge}(s, \chi^{-1}, \tau)\|_\tau \\ &\leqq C_2 \dim \tau \int_M \int_{G_{A_f}} |\beta(\underline{h}(m g_f^{-1}))|_A^{1/2} \xi_j(g_f) dg_f dm . \end{aligned} \tag{3.25}$$

On the other hand, it is known that, for any $s \in J$,

$$\|\Phi(s, \chi, \tau)\|_\tau = C_3 . \tag{3.26}$$

Here C_3 is a positive constant which depends only on τ and χ (cf. [5], § 6, (6.16)). Combining (3.21), (3.24), (3.25) and (3.26), we obtain the inequality (3.19) if we set $C = (C_1 + C_2 C_3) \dim \tau$. Hence the lemma has been proved.

We set, for $s \in \boldsymbol{C}$,

$$\Omega(s, \xi_j) = \int_M \int_{G_{A_f}} |\beta(\underline{h}(m g_f^{-1}))|_A^s \xi_j(g_f) dg_f dm .$$

By virtue of Lemma 8, the proof of (3.18), and hence of Proposition 4, has now been reduced to the verification of the following proposition.

PROPOSITION 6. *If* (0.1) *is satisfied, then we have*

$$\lim_{j\to\infty} \Omega\left(\frac{1}{2}, \xi_j\right)\Big/\deg \xi_j = 0 .$$

Proof. To prove the proposition, we shall express $\Omega(s, \xi_j)$ as a product of some integrals of zonal spherical functions on $G_{k_\mathfrak{g}} = SL(2, k_\mathfrak{g})$ for $\mathfrak{g} \in \mathscr{S}$. For $\mathfrak{g} \in \mathscr{P}_f$, we fix the Iwasawa decomposition of $g_\mathfrak{g} \in G_{k_\mathfrak{g}}$ given by $g_\mathfrak{g} = \underline{u}(g_\mathfrak{g})\underline{h}(g_\mathfrak{g})\underline{m}(g_\mathfrak{g})$, where $\underline{u}(g_\mathfrak{g}) \in U_{k_\mathfrak{g}}$, $\underline{h}(g_\mathfrak{g}) \in H_{k_\mathfrak{g}}$, and $\underline{m}(g_\mathfrak{g}) \in M_\mathfrak{g}$. We set

$$\beta(h) = t^2 \text{ for } h = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \in H_{k_\mathfrak{g}} .$$

We denote by $|\ \ |_\mathfrak{g}$ the module of $k_\mathfrak{g}$. Namely, for a prime element κ of $k_\mathfrak{g}$, $n \in \boldsymbol{Z}$, and for any element ε in the unit group of $O_\mathfrak{g}$, we put

$$|\kappa^n \varepsilon|_\mathfrak{g} = q^{-n} .$$

Here q denotes the order of the residue field of $k_\mathfrak{g}$. We normalize the Haar measure $dm_\mathfrak{g}$ on $M_\mathfrak{g}$ so that

$$\int_{M_\mathfrak{g}} dm_\mathfrak{g} = 1 .$$

We set, for $g \in G_{k_\mathfrak{g}}$ and $s \in \boldsymbol{C}$,

$$\omega_\mathfrak{g}(g, s) = \int_{M_\mathfrak{g}} |\beta(\underline{h}(m_\mathfrak{g} g))|_\mathfrak{g}^s dm_\mathfrak{g} . \tag{3.27}$$

The integral (3.27) converges absolutely for any $s \in \boldsymbol{C}$, and for any $g \in G_{k_\mathfrak{g}}$. We call $\omega_\mathfrak{g}(g, s)$ the zonal spherical function on $G_{k_\mathfrak{g}}$. This function is, as a function of g, $M_\mathfrak{g}$-biinvairant on $G_{k_\mathfrak{g}}$.

For $m = \prod_v m_v \in M$ and $g_f = \prod_{\mathfrak{g} \in \mathscr{P}_f} g_\mathfrak{g} \in G_{A_f}$, it is easily verified that

$$|\beta(\underline{h}(m g_f^{-1}))|_A = \prod_{\mathfrak{g} \in \mathscr{P}_f} |\beta(\underline{h}(m_\mathfrak{g} g_\mathfrak{g}^{-1}))|_\mathfrak{g} .$$

It follows that

$$\Omega(s, \xi_j) = \prod_{\mathfrak{g} \in \mathscr{S}} \int_{M_\mathfrak{g}} \int_{S_\mathfrak{g}(j)} |\beta(\underline{h}(m_\mathfrak{g} g_\mathfrak{g}^{-1}))|_\mathfrak{g}^s dg_\mathfrak{g} dm_\mathfrak{g}$$
$$\times \prod_{\mathfrak{g} \in \mathscr{P}_f - \mathscr{S}} \int_{M_\mathfrak{g}} \int_{M_\mathfrak{g}} |\beta(\underline{h}(m_\mathfrak{g} g_\mathfrak{g}^{-1}))|_\mathfrak{g}^s dg_\mathfrak{g} dm_\mathfrak{g} .$$

Note that $g_\mathfrak{g} \in M_\mathfrak{g}$ implies $|\beta(\underline{h}(m_\mathfrak{g} g_\mathfrak{g}^{-1}))|_\mathfrak{g} = 1$ for any $m_\mathfrak{g} \in M_\mathfrak{g}$. Thus

$$\Omega(s, \xi_j) = \prod_{\mathfrak{g} \in \mathscr{P}} \int_{M_\mathfrak{g}} \int_{S_\mathfrak{g}(j)} |\beta(\underline{h}(m_\mathfrak{g} g_\mathfrak{g}^{-1}))|_\mathfrak{g}^s dg_\mathfrak{g} dm_\mathfrak{g} .$$

Changing the order of integrations, we obtain

$$\Omega(s, \xi_j) = \prod_{\mathfrak{g} \in \mathscr{P}} \int_{S_\mathfrak{g}(j)} \omega_\mathfrak{g}(g_\mathfrak{g}^{-1}, s) dg_\mathfrak{g} .$$

Applying Lemma 5, we observe that it is enough to establish the following.

LEMMA 9. *For every* $\mathfrak{g} \in \mathscr{P}_f$, *the function on* $G_{k_\mathfrak{g}}$ *given by* $g \mapsto \omega_\mathfrak{g}(g, 1/2)$ *vanishes at infinity.*

Proof. As is well-known, the zonal spherical function $\omega_\mathfrak{g}(g, 1/2)$ is a matrix coefficient of an irreducible unitary representation of $SL_2(k_\mathfrak{g})$ belonging to the principal series. Hence the lemma follows from the general result of Howe and Moore (stated in § 2 as Lemma 4). However, in the following, we give a direct proof of the lemma based on the precise knowledge on the behavior of $\omega_\mathfrak{g}(g, 1/2)$ on $G_{k_\mathfrak{g}}$.

By virtue of the explicit formula for the zonal spherical function on $G_{k_\mathfrak{g}}$ (cf. [4], Chap. II, § 3.10), we have

$$\omega_\mathfrak{g}\left(\begin{pmatrix} \kappa^n & \\ & \kappa^{-n} \end{pmatrix}, \frac{1}{2}\right) = q^{-n}(1 + q)^{-1}\{(2n + 1)q - (2n - 1)\}$$

for $n \geqq 0$. Hence

$$\lim_{n \to \infty} \omega_\mathfrak{g}\left(\begin{pmatrix} \kappa^n & \\ & \kappa^{-n} \end{pmatrix}, \frac{1}{2}\right) = 0 . \tag{3.28}$$

Then the lemma follows from (3.28) together with the Cartan decomposition of $G_{k_\mathfrak{g}}$:

$$G_{k_\mathfrak{g}} = \bigcup_{n=0}^{\infty} M_\mathfrak{g} \begin{pmatrix} \kappa^n & \\ & \kappa^{-n} \end{pmatrix} M_\mathfrak{g} \text{ (disjoint union)} .$$

Thus Theorem 2 has been completely proved.

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