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HEIGHT ESTIMATES FOR CAPILLARY SURFACES

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In this paper estimates are obtained for any scalar function $u(x)$ that staisfies the equation

$$\text{(1a)} \qquad \operatorname{div}(Tu) = \kappa u \quad \textbf{in} \quad \Omega$$

and the boundary condition

$$\text{(1b)} \qquad Tu \cdot \nu = \cos\gamma \quad \textbf{on} \quad \Sigma = \partial\Omega .$$

Here κ is a positive constant, Ω is an open domain in n-dimensional Euclidean space, ν is the exterior unit normal on Σ, and Tu is the vector operator

$$Tu = \frac{1}{\sqrt{1 + |\nabla u|^2}} \nabla u .$$

For $n = 2$, $u(x)$ can be interpreted physically as the height of a capillary surface above the undisturbed fluid level when a vertical cylindrical tube with section Ω is dipped into a large reservoir. The "capillarity constant" κ and the "contact angle" γ are determined physically; $\kappa = (\rho - \rho_0)\sigma^{-1}g$, where ρ is the density of the fluid, ρ_0 is the density of the gas, g is the acceleration due to gravity, and σ is the surface tension. If the tube is homogeneous γ is constant [4].

The operator $Nu = \operatorname{div}(Tu)$ is n times the mean curvature of the surface $x_{n+1} = u(x)$. Geometrically stated, a capillary surface has mean curvature proportional to its height above a horizontal reference plane and it meets a vertical cylinder in a prescribed angle.

We shall distinguish three types of domains: "interior", "exterior", and "general exterior" corresponding to Ω bounded, the complement of Ω bounded, and Σ unbounded, respectively. Existence, uniqueness, and regularity of solutions to problem (1) have been established under fairly general conditions on Ω and γ. However, for a general exterior domain uniqueness has not been established and for an exterior domain uniqueness has been established only under the condition $u = o(1)$ as $|x| \to \infty$ [see 6, 7, 8, 9].

For simplicity we shall only consider solutions to problem (1) in domains with piecewise smooth boundaries, with boundary condition (1b) holding on the smooth part Σ^* of Σ. In § 2, we extend the existence theory for smooth bounded domains to the case of piecewise smooth bounded or unbounded domains.

We now state our main results. In what follows $\kappa = 1$, and γ is constant, $0 \leqq \gamma \leqq \pi/2$.

The simplest examples of interior, exterior, and general exterior domains are $B_R(0) = \{x: |x| < R\}$, $B_R^c(0) = \{x: |x| > R\}$ and $H = \{x: x_1 > 0\}$. The solutions to problem (1) in these domains are $v(r, R, \gamma)$, $w(r, R, \gamma)$, and $z(x_1, \gamma)$, with $r = |x|$. An explicit formula is known for the "one-dimensional" solution $z(x_1, \gamma)$. We have

$$(2) \qquad z(x_1, \gamma) \sim C_1(\gamma)e^{-x_1} \quad \text{as} \quad x_1 \longrightarrow \infty$$

where $C_1(\gamma)$ is an explicitly known constant.

Chapter II is devoted to the study of v and w. General estimates are given for v and w that improve upon those given by Finn [5]. For small R, v is shown to be close to a spherical cap. Laplace's asymptotic formula for the center height is proved correct. For large R, near the boundary v and w are shown to be close to a one-dimensional solution, i.e.,

$$\lim_{R\to\infty} v(R - l, R, \gamma) = \lim_{R\to\infty} w(R + l, R, \gamma) = z(l, \gamma) .$$

Away from the boundary, estimates are also given, in particular

$$(3) \qquad v(0, R, \gamma) \sim C(\gamma)\frac{R^{(n-1)/2}}{\exp R} \quad \text{as} \quad R \longrightarrow \infty$$

$$(4) \qquad w(r, R, \gamma) \sim C(R, \gamma)\frac{\exp(-r)}{r^{(n-1)/2}} \quad \text{as} \quad r \longrightarrow \infty$$

where $C(\gamma)$ is an explicitly given constant and $C(R, \gamma)$ is determined asymptotically as $R \to \infty$. Monotonicity and continuity properties of $C(R, \gamma)$ are given. Estimates on the derivatives of v and w are also given.

In Chapter III, solutions to problem (1) are estimated in terms of v, w, and z. This is done by use of an appropriate comparison principle due to Concus and Finn [3].

General estimates are given that apply to any solution of equation (1a): $|u(x)| \leqq v(0, d, 0)$, d is the distance from x to Σ; $|u(x)| \leqq w(r, R, 0)$ if $B_R^c(0) \subset \Omega$; and $|u(x)| \leqq z(x_1, 0)$ if $H \subset \Omega$. Combining these general estimates with results (2)-(3)-(4), we see that solutions to equation (1a) decay exponentially away from the boundary of the domain. The derivatives of a solution decay at the same rate at which the solution decays.

The third general estimate improves on an estimate given by Gerhardt [8]. The first general estimate implies that for an exterior domain, problem (1) has a unique solution.

We now consider estimating solutions to problem (1). To improve upon the above general estimates or to study solutions in general exterior domains one needs to know when $B_R(y) \subset \Omega$, $r = |x - y|$, implies that $u(x) \leqq v(r, R, \gamma)$. An example shows that this is not always true. We prove that if $\Omega \cap B_{\bar{R}}(y)$ is convex, where $B_{\bar{R}}(y)$ is the maximal domain of existence of v, then the assertion is true. Alternatively, if Σ is smooth and bounded and $R \leqq \mathscr{R}_1$, where $\mathscr{R}_1$ is the "interior rolling number" of Ω, then the assertion is also true. Analogously, if $B_R^c(y) \subset \Omega$, $r = |x - y|$, if Σ is smooth bounded and strictly convex, and if $R \geqq \mathscr{R}_2$, where $\mathscr{R}_2$ is the "exterior rolling number", then $u(x) \leqq w(r, R, \gamma)$. Again, an example shows that this inequality is not always true when $B_R^c(y) \subset \Omega$.

Estimates for solutions to problem (1) in certain general exterior domains are given. We show that $z(x_1, \gamma)$ is the unique solution to problem (1) in H. Consider the two-dimensional domain $\mathscr{K}_\alpha = \{(x_1, x_2): x_1 > |x| \cos \alpha\}$, with $0 < \alpha < \pi$. For $0 < \gamma < \pi/2$ there exists a unique positive solution to problem (1) in $\mathscr{K}_\alpha$. Furthermore, for $\alpha < \pi/2$: on Σ^*, $u(x) \geqq z(0, \gamma)$ and $\lim_{|x| \to \infty} u(x) = z(0, \gamma)$. For $\alpha > \pi/2$: on Σ^*, $u(x) \leqq z(0, \gamma)$ and $\lim_{|x| \to \infty} u(x) = z(0, \gamma)$. For $\alpha + \gamma \geqq \pi/2$, $\alpha < \pi/2$:

$$\liminf_{x \to 0} u(x) \geqq \sqrt{2}\,[1 - (1 - k^2)^{1/2}]^{1/2}$$

and

$$\limsup_{x \to 0} u(x) \leqq 2\sqrt{2}\,[1 - (1 - k^2)^{1/2}]^{1/2}$$

where $k = \cos \gamma / \sin \alpha$. These inequalities show there is a "rise" at the corner. Some higher dimensional generalizations are given.

Chapter I contains preliminaries that are needed in the other chapters.

Chapter I
Preliminaries

1. The comparison principle. Our basic tool will be the comparison principle (CP):

THEOREM 1. *Let $\Sigma = \Sigma^0 + \Sigma^\alpha + \Sigma^\beta$ be a decomposition of Σ, such that Σ^β is C^1 and Σ^0 can be covered from within Ω by a sequence of smooth surfaces $\{\Lambda\}$, each of which meets Σ in a set of zero $(n - 1)$ dimensional measure, and such that $\Lambda \to \Sigma^0$ and the area of Λ tends to zero. Let $u, v \in C^2(\Omega)$ and suppose*

(i) *$Nu \geqq \kappa u$ and $Nv \leqq \kappa v$ on Ω.*

(ii) *for any approach to Σ^α from within Ω* $\limsup\, [u - v] \leqq 0$.

(iii) *on Σ^β, $(Tu - Tv) \cdot \nu \leqq 0$ almost everywhere as a limit from points of Ω.*

(iv) *if Ω is unbounded, $u - v \leqq o(1)$ as $|x| \to \infty$.*

Then $u \leqq v$ *on* Ω; *if equality holds at any point then* $u \equiv v$ *on* Ω.

Proof (see [3]). We have added condition (iv), but precisely the same proof holds.

Note. In condition (iii), ν has been extended continuously into a neighbordood of Σ^{β}.

The following corollary was proved in [3] in a different way.

COROLLARY. *If* $u(x)$ *satisfies* (1a) *in* Ω *and* $B_R(y) \subset \Omega$ *then* $|u(x)| < 2/\kappa R + R$ *in* $B_R(y)$.

Proof. Let $v(x) = n/\kappa R + R - \sqrt{R^2 - |x-y|^2}$. Then $Nv = n/R = \kappa \min v(x) \leqq \kappa v$ and $\lim Tv \cdot \nu = 1$. Thus $\limsup (Tu - Tv) \cdot \nu \leqq 0$. By CP $u(x) < v(x) < \sup v(x) = n/\kappa R + R$ on $B_R(y)$. Replace v by $-v$ to get the other inequality.

When Ω is contained in a symmetric domain we have the following comparison principle (CPS):

THEOREM 2. *Let* Ω *be bounded and* Σ *piecewise smooth. Suppose* $u(x)$ *is a solution to problem* (1) *with boundary data* $\gamma(\sigma)$, $0 \leqq \gamma(\sigma) \leqq \gamma_0$, $0 \leqq \gamma_0 \leqq \pi/2$, *on* Σ^*.

(i) *If* $\Omega \subset B_R(0)$ *then* $v < u$ *or* $v \equiv u$ *on* Ω.

(ii) *If* $\Omega \subset B_R^c(0)$ *then* $w < u$ *on* Ω.

(iii) *If* $\Omega \subset H$ *then* $z(x_1) < u(x)$ *on* Ω.

Here $v = v(r, R, \gamma_0)$, $w = w(r, R, \gamma_0)$, *and* $z(x_1) = z(x_1, \gamma_0)$.

Proof. The proofs for parts (i) and (ii) are in [5]. We present the proof for (iii) which is very similar. We introduce the angle $\psi_1(x_1)$ between the curve and the positive x_1 direction

$$\sin \psi_1(x_1) = \frac{z_{x_1}}{\sqrt{1 + z_{x_1}^2}} .$$

It is known that $\sin \psi_1(x_1)$ is negative and increasing in x_1 (see § 5). On Σ^*, $Tu \cdot \nu = \cos \gamma \geqq \cos \gamma_0 = -\sin \psi_1(0) \geqq -\sin \psi_1(x_1) \geqq Tz \cdot \nu$. The last inequality is true because $|Tz| = -\sin \psi_1(x_1)$. Apply CP to obtain $z(x_1) < u(x)$ on Ω.

We shall need the following consequence of the boundary point form of Hopf's maximum principle [11].

LEMMA 1. *Let* $u \in C^1(\bar{B}_R) \cap C^2(B_R)$ *and* $v \in C^2(\bar{B}_R)$. *Suppose* u *and* *v satisfy condition* (i) *of Theorem* 1, $u < v$ *on* B_R *and* $u(x_0) = v(x_0)$ *at* $x_0 \in \partial B_R$. *Then* $(\partial u/\partial \nu)(x_0) \neq (\partial v/\partial \nu)(x_0)$.

Proof. Let $w = u - v$, then w satisfies

$$Lw = \sum_{i,j} a_{ij} w_{x_i x_j} + \sum_k b_k w_{x_k} \geqq \kappa w \tag{5}$$

where

$$a_{ij} = a_{ij}(\nabla u)\,, \qquad Nu = \sum a_{ij}(\nabla u) u_{x_i x_j}$$

$$b_k = \sum_{i,j} v_{x_i x_j} \int_0^1 \frac{\partial a_{ij}}{\partial p_k} [\nabla v + t(\nabla u - \nabla v)] dt\,.$$

Under the hypotheses L is uniformly elliptic with bounded coefficients on B_R. Since $w < 0$ on B_R and $w(x_0) = 0$, Hopf's theorem gives $(\partial w/\partial \nu)(x_0) \neq 0$.

REMARK. The smoothness properties of u and v can be interchanged in Lemma 1.

2. Existence theorems. Gerhardt [9] proves the following result:

THEOREM 3. *If* Ω *is bounded,* $\Sigma \in C^{2,\lambda}$, $0 < \gamma(\sigma) < \pi$, $\gamma(\sigma) \in C^{1,\lambda}$, *then there exists a unique solution* $u(x)$ *to problem* (1) *with* $u(x) \in C^{2,\alpha}(\bar{\Omega})$.

Simon and Spruck [16] prove a similar theorem under the condition $\Sigma \in C^4$. They give a local estimate of the gradient up to the boundary. With this estimate we prove:

THEOREM 4. *Let* Ω *be a domain with a piecewise smooth boundary* Σ. *Let* $\Sigma = \Sigma^0 + \Sigma^*$, *where* Σ^* *is open in* Σ *and* $\Sigma^* \in C^4$. *Suppose that on* Σ^*, $\gamma(\sigma) \in C^{1,\alpha}$ *and* $0 < \gamma(\sigma) < \pi$. *Then there exists a solution* $u(x)$ *to* (1), *with boundary condition* (1b) *holding on* Σ^*.

Proof.

Step 1. We construct approximating domains Ω_i with C^4 boundaries Σ_i. For Ω bounded we require:

(a) if $x \in \Sigma$ and $\operatorname{dist}(x, \Sigma^0) > \varepsilon_i$ then $x \in \Sigma_i$

(b) $\operatorname{dist}(\Sigma, \Sigma_i) < \varepsilon_i$

with $\lim_{i\to\infty} \varepsilon_i = 0$. For Ω unbounded we require $\Omega_{R_i} = \Omega \cap B_{R_i}(0)$ to be piecewise C^4, with $\lim_{i\to\infty} R_i = \infty$, and that conditions (a) and (b) hold with Σ replaced by $\partial\Omega_{R_i}$.

Extend $\gamma(\sigma)$ to all of Σ_i so that $0 < \gamma(\sigma) < \pi$ and $\gamma(\sigma) \in C^{1,\alpha}$. Let u_i be the solution to problem (1) with this data, $u_i \in C^2(\bar{\Omega})$.

Step 2. We now obtain a local Hölder estimate of the gradient of u_i up to Σ^*, independent of i.

Take $x_0 \in \Sigma^*$, choose $\delta > 0$ so that $\bar{B}_{4\delta}(x_0) \cap \bar{\Omega} \subset \Omega \cup \Sigma^*$, choose N so that $i > N$ implies $\bar{B}_{4\delta} \cap \Sigma^* \subset \Sigma_i$.

We estimate $|u_i(x)| \leqq M$ on $\bar{B}_{3\delta}(x_0) \cap \bar{\Omega}$. By the corollary to Theorem 1 we can choose $M = n(\kappa R)^{-1} + R$ if each point of $\bar{B}_{3\delta} \cap \bar{\Omega}$ is contained in a closed ball of radius R lying in $\bar{B}_{4\delta} \cap \bar{\Omega}$. One can choose $R = \min[\delta, (\bar{k})^{-1}]$ if $\bar{k} > 0$, $R = \delta$ otherwise, where $\bar{k} = \max k(\sigma)$ over $\bar{B}_{4\delta} \cap \Sigma^*$, $k(\sigma) =$ the maximum principal curvature of Σ at σ with respect to the interior normal.

By Theorem 3 of [16]: $|\nabla u_i|_{0,\bar{\Omega} \cap B_{2\delta}(x_0)} < \Gamma_1$.

By the argument on pp. 467–8 of [13]: $|u_i|_{1,\alpha,\bar{\Omega} \cap B_{\delta}(x_0)} < \Gamma_2$. Here Γ_1 and Γ_2 are independent of i.

Step 3. We now obtain interior estimates on the derivatives of u_i independent of i.

Take $x_0 \in \Omega$, choose $\delta > 0$ so that $\bar{B}_{4\delta}(x_0) \subset \Omega$, choose N so that $i > N$ implies $B_{4\delta} \subset \Omega_i$.

We estimate $|u_i| \leqq n(4\kappa\delta)^{-1} + 4\delta$ on $\bar{B}_{4\delta}$.

The gradient can be estimated on $B_{3\delta}$ since

$$|\nabla u_i(y)| \leqq C_1 \exp\{C_2 \sup[u_i - u_i(y)]/\delta\}$$

where C_1 and C_2 depend on n and $\delta \sup|u_i|$ (see [10]). This gives $|\nabla u_i|_{0,B_{3\delta}(x_0)} < \Gamma_3$.

The interior Hölder estimate for divergence structure equations [11, p. 265] yields $|\nabla u_i|_{\beta,B_{2\delta}(x_0)} < \Gamma_4$.

Finally, we apply the interior Schauder estimate, since we can treat the equation as a linear, uniformly elliptic equation with C^β coefficients. This gives $|u_i|_{2,\beta,B_\delta(x_0)} < \Gamma_5$. Here Γ_3, Γ_4, and Γ_5 are independent of i.

Step 4. Because of the boundary and interior estimates we can choose a subsequence of $\{u_i\}$ that converges in C^2 on every compact set $k \subset \Omega$, and in C^1 on every compact set $k \subset \Omega \cup \Sigma^*$. The limit function $u(x)$ will belong to $C^2(\Omega) \cap C^1(\Omega \cup \Sigma^*)$ and will be a solution to problem (1).

REMARK. If Ω is bounded the solution to problem (1) is unique. This is an immediate consequence of CP.

3. Two normalizations. From now on we shall take κ to be one. This is no loss in generality because of the following lemma:

LEMMA 2. *Let $u(x)$ satisfy $Nu = u$ in Ω and $Tu \cdot \nu = \cos\gamma$ on Σ. Define $v(x) = (1/\sqrt{\kappa})u(\sqrt{\kappa}x)$. Let Ω', Σ', and ν' be the images of Ω, Σ, and ν under the transformation $x \to (1/\sqrt{\kappa})x$. Then $Nv = \kappa v$ in Ω' and $Tv \cdot \nu' = \cos\gamma$ on Σ'.*

Proof. From the expression

$$Nu = \frac{\Delta u(1 + |\nabla u|^2) - \Sigma u_{x_i} u_{x_j} u_{x_i x_j}}{(1 + |\nabla u|^2)^{3/2}}$$

we see that $Nv(x) = \sqrt{\kappa}\, Nu(\sqrt{\kappa}x) = \sqrt{\kappa}\, u(\sqrt{\kappa}x) = \kappa v(x)$. Also, $Tv(x/\sqrt{\kappa}) = Tu(x)$ in Ω and $\nu'(x/\sqrt{\kappa}) = \nu(x)$ on Σ. Thus, $Tv \cdot \nu' = \cos\gamma$ on Σ'.

If γ is constant we shall assume $0 \leqq \gamma \leqq \pi/2$. This is no loss in generality because if $u(x)$ satisfies $Nu = u$ in Ω and $Tu \cdot \nu = \cos\gamma$ on Σ, then $v(x) = -u(x)$ satisfies $Nv = v$ in Ω and $Tv \cdot \nu = \cos(\pi - \gamma)$ on Σ.

4. Solutions to the linearized equation. We examine the solutions to the linearized equation $\Delta u = u$ that depend only on $r = |x|$; u must satisfy

$$u_{rr} + \frac{n-1}{r} u_r = u \,. \tag{6}$$

The general solution is $u(r) = AI(r; n) + BK(r; n)$; A and B are arbitrary constants, $I(r; n) = r^{-m} I_m(r)$, $K(r; n) = r^{-m} K_m(r)$, $m = (n-2)/2$, I_m is a modified Bessel function of the first kind, and K_m is a MacDonald's function.

To be explicit (see [14]):

$$I_m(r) = \sum_{k=0}^{\infty} \frac{(r/2)^{m+2k}}{\Gamma(k+1)\Gamma(k+m+1)} \tag{7}$$

$$K_m(r) = \int_0^{\infty} e^{-r\cosh u} \cosh(mu) du \,. \tag{8}$$

The following recurrence relations hold:

$$(r^{-m} I_m)_r = r^{-m} I_{m+1} \tag{9}$$

$$(r^{-m} K_m)_r = -r^{-m} K_{m+1} \,. \tag{10}$$

From equations (7) through (10) we conclude:

$$I_{rr} > 0 \quad \text{and} \quad I > 0 \quad \text{for} \quad r \geqq 0\,; \quad I_r > 0 \quad \text{for} \quad r > 0\,; \tag{11}$$
$$I_r(0) = 0 \qquad \text{and} \qquad I(0) = [2^{(n-2)/2}\Gamma(n/2)]^{-1}$$

$$K_{rr} > 0\,, \qquad K_r < 0\,, \qquad \text{and} \qquad K > 0 \qquad \text{for} \quad r > 0\,; \tag{12}$$
$$\lim_{r\to\infty} K(r) = \lim_{r\to\infty} K_r(r) = 0\,.$$

We shall employ I and K to estimate $v(r, R, \gamma)$ and $w(r, R, \gamma)$. In fact, AI and AK, for any positive constant A, satisfy a super solution condition:

LEMMA 3. *For any constant* A, $A > 0$, $N(AI) \leqq AI$, *and* $N(AK) \leqq AK$.

Proof. For a function $u = u(r)$

$$Nu = \frac{1}{r^{n-1}}\left(\frac{r^{n-1}u_r}{\sqrt{1+u_r^2}}\right)_r\,. \tag{13}$$

We use the properties listed under (11) and (12):

$$\begin{aligned} N(AI) &= \frac{AI_{rr}}{[1+(AI_r)^2]^{3/2}} + \frac{(n-1)}{r}\,\frac{AI_r}{[1+(AI_r)^2]^{1/2}} \\ &\leqq AI_{rr} + \frac{(n-1)}{r}\,AI_r = AI \\ N(AK) &= \frac{AK_{rr} + \frac{(n-1)}{r}(AK_r)[1+(AK_r)^2]}{[1+(AK_r)^2]^{3/2}} \\ &= \frac{AK + \frac{(n-1)}{r}(AK_r)^3}{[1+(AK_r)^2]^{3/2}} < AK\,. \end{aligned}$$

The asymptotic behavior of I_m and K_m has been studied (see [14]). As a consequence of these estimates we obtain

$$I(r) = \frac{1}{\sqrt{2\pi}}\,\frac{e^r}{r^{(n-1)/2}}\,[1 + O(1/r)] \qquad \text{as} \quad r \longrightarrow \infty \tag{14}$$

$$K(r) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-r}}{r^{(n-1)/2}}\,[1 + O(1/r)] \qquad \text{as} \quad r \longrightarrow \infty \tag{15}$$

$$I_r(r) = \frac{1}{\sqrt{2\pi}}\,\frac{e^r}{r^{(n-1)/2}}\,[1 + O(1/r)] \qquad \text{as} \quad r \longrightarrow \infty \tag{16}$$

$$K_r(r) = -\sqrt{\frac{\pi}{2}}\,\frac{e^{-r}}{r^{(n-1)/2}}\,[1 + O(1/r)] \qquad \text{as} \quad r \longrightarrow \infty\,. \tag{17}$$

We now give explicit estimates on I_r/I and K_r/K. For any solution u of equation (6), $v = u_r/u$ satisfies a nonlinear first order differential equation

$$v_r = 1 - \frac{(n-1)}{r} v - v^2 . \tag{18}$$

Let

$$\mathscr{P} = \left\{(r, v): r > 0 \text{ and } 1 - \frac{(n-1)}{r} v - v^2 > 0\right\}$$
$$= \{(r, v): r > 0 \text{ and } v_1(r) < v < v_2(r)\}$$

where

$$v_1(r) = -\frac{(n-1)}{2r} - \left[1 + \left(\frac{n-1}{2r}\right)^2\right]^{1/2}$$

and

$$v_2(r) = -\frac{(n-1)}{2r} + \left[1 + \left(\frac{n-1}{2r}\right)^2\right]^{1/2} .$$

With this notation, we have:

LEMMA 4. *For* $r > 0$:
(i) I_r/I *and* K_r/K *are strictly increasing in* r.
(ii) $0 < I_r/I < v_2(r)$.
(iii) $v_1(r) < K_r/K < -1$.

Proof. Part (i) is equivalent to saying that $(r, I_r/I)$ and $(r, K_r/K)$ are contained in $\mathscr{P}$, thus we need only prove parts (ii) and (iii).

We note that $v_1(r)$ and $v_2(r)$ are increasing functions of r.

Let $v = I_r/I$, then $v(0) = 0$ and $v(r) > 0$ for $r > 0$. Suppose $v(r_1) \geqq v_2(r_1)$, then $v(r) \geqq v_2(r_1) > 0$ for $r < r_1$, contradicting that $v(0) = 0$. This proves part (ii).

Let $w = K_r/K$, then $w < 0$. Since $K_r < 0$, we have $K_{rr} = K - ((n-1)/r)K_r > K$ and $K_r K_{rr} < K_r K$. Integrating from r to ∞ gives $K_r^2 > K^2$ and hence $K_r/K < -1$.

Suppose $w(r_1) \leqq v_1(r_1)$, then for $r > r_1$, $w(r) < v_1(r_1) = -a$, with $a > 1$. Integrating from r_1 to r gives $K(r) < K(r_1) \exp a(r_1 - r)$ for for $r > r_1$. This contradicts equation (15) as $r \to \infty$. The proof of part (iii) is complete.

REMARK. The upper bound on I_r/I and the lower bound on K_r/K are asymptotically exact to order $1/r$ since

$$\frac{I_r}{I} = 1 - \frac{(n-1)}{2r} + O\left(\frac{1}{r^2}\right) \quad \text{as} \quad r \longrightarrow \infty$$

and

$$\frac{K_r}{K} = -1 - \frac{(n-1)}{2r} + O\left(\frac{1}{r^2}\right) \qquad \text{as} \quad r \longrightarrow \infty \,.$$

5. The one-dimensional solution. There is only one explicitly known solution to problem (1). We denote by $z(x;\gamma)$ the unique solution to

$$\left(\frac{z_x}{\sqrt{1+z_x^2}}\right)_x = z \qquad \text{for} \quad 0 < x < \infty \tag{19a}$$

$$z_x(0+) = -\cot\gamma \tag{19b}$$

$$\lim_{x\to\infty} z(x) = 0 \,. \tag{19c}$$

Here, z can be interpreted physically as the height of a capillary surface on one side of an infinite vertical plate.

An explicit solution with x as a function of z is given by

$$\begin{aligned} x &= -2\sqrt{1-\frac{z^2}{4}} - \ln\frac{z}{2} + \ln\left(1+\sqrt{1-\frac{z^2}{4}}\right) + D \\ D &= \sqrt{2(1+\sin\gamma)} + \ln\sqrt{\frac{1-\sin\gamma}{2}} - \ln\left(1+\sqrt{\frac{1+\sin\gamma}{2}}\right). \end{aligned} \tag{20}$$

THEOREM 5. *For* $0 \leqq \gamma < \pi/2$:

(i) *There is a unique solution* z *to problem* (19) *given by* (20), $z(x) \in C^2(0,\infty) \cap C[0,\infty)$, *with* $z > 0$ *and* $z_x < 0$.

(ii) $z = C_1(\gamma)e^{-x}[1 + O(e^{-2x})]$ *as* $x \to \infty$ *with* $C_1(\gamma) = 4\exp(D-2)$.

(iii) $\sin\psi_1(x) = -C_1(\gamma)e^{-x}[1 + O(e^{-2x})]$ *as* $x \to \infty$ *and* $\cos\psi_1(x) = 1 + O(e^{-2x})$ *as* $x \to \infty$.

(iv) $z < z_0 e^{-x}$ *for* $x > 0$, $z_0 = z(0,\gamma) = \sqrt{2(1-\sin\gamma)}$.

(v) $z(x) \in C^2[0,\infty)$ *if* $0 < \gamma < \pi/2$.

Here $\sin\psi_1(x) = z_x/\sqrt{1+z_x^2}$.

Proof.

Part (i). From equation (20) x is defined for $0 < z < z_0$ and

$$x_z = \frac{z^2/2 - 1}{z\sqrt{1 - z^2/4}} \,. \tag{21}$$

Thus $x_z < 0$ for $0 < z < z_0$. Inverting, z is defined for $0 < x < \infty$ and $z_x < 0$. From equation (21):

$$\sin\psi_1 = -z\sqrt{1 - z^2/4} \tag{22}$$

and

(23) $$\cos\psi_1 = 1 - z^2/2 .$$

We can rewrite equation (19a) as

$$-(\cos\psi_1)_z = z .$$

Thus, by equation (23) we see that z satisfies equation (19a). By equation (22) we see that z satisfies equation (19b). That z satisfies condition (19c) comes from equation (20).

Part (ii). We rewrite equation (20) as

(24) $$\frac{e^x z}{C_1(\gamma)} = \frac{1}{2}(\sqrt{1 - z^2/4} + 1)\exp 2(1 - \sqrt{1 - z^2/4}) .$$

By Part (i) this quantity is $1 + o(1)$, thus $z = O(e^{-x})$; putting this back into equation (24) gives

$$\frac{e^x z}{C_1(\gamma)} = 1 + O(e^{-2x}) .$$

Part (iii). These estimates follow from Part (ii) and equations (22) and (23).

Part (iv). Let $v(x) = z_0 e^{-x}$, then

$$Nv = \frac{z_0 e^{-x}}{(1 + z_0^2 e^{-2x})^{3/2}} < z_0 e^{-x} = v$$

$v(0) = z(0)$, and $z - v = o(1)$ as $x \to \infty$. By CP $z < v$ for $x > 0$.

Part (v). For $0 < \gamma < \pi/2$, equation (21) shows that $z_x \in C[0, \infty)$ and by equation (19a) $z_{xx} = z(1 + z_x^2)^{3/2} \in C[0, \infty)$.

REMARK. If $\gamma = \pi/2$, $z(x, \pi/2) \equiv 0$ is the unique solution to problem (21). This is an immediate consequence of CP.

Chapter II
Solutions in Symmetric Domains

6. General estimates—interior case. We obtain estimates for the function $v(r; R, \gamma)$ that satisfies

(25a) $$\frac{1}{r^{n-1}}\left(\frac{r^{n-1}v_r}{\sqrt{1 + v_r^2}}\right)_r = v \qquad \text{for} \quad 0 < r < R$$

(25b) $$v_r(0+) = 0$$

(25c) $$v_r(R-) = \cot\gamma .$$

Johnson and Perko [12] prove by the method of successive approximations:

THEOREM 6. *Problem* (25) *has a unique solution* $v(r, R, \gamma)$ *which is continuous in* (r, R, γ) *for* $0 \leqq r \leqq R$, $R > 0$, *and* $0 \leqq \gamma \leqq \pi/2$. *If* $0 \leqq \gamma < \pi/2$ *then* $v > 0$, $v_r > 0$, *and* $v_{rr} > 0$ *for* $0 < r < R$. *If* $\gamma = \pi/2$ *then* $v \equiv 0$.

Note. Johnson and Perko give the proof for $n = 2$, but the same proof holds for $n > 2$.

Introducing $\sin \psi(r) = v_r/\sqrt{1 + v_r^2}$ we have

$$\frac{1}{r^{n-1}} (r^{n-1} \sin \psi)_r = v \tag{26a}$$

$$\sin \psi(0+) = 0 \tag{26b}$$

$$\sin \psi(R-) = \cos \gamma . \tag{26c}$$

Equation (26a) can be rewritten as

$$(\sin \psi)_r + \frac{(n-1)}{r} \sin \psi = v . \tag{27}$$

Partially inverting equation (27):

$$-(\cos \psi)_v + \frac{(n-1)}{r} \sin \psi = v . \tag{28}$$

We shall estimate $\sin \psi/r$ and then use equations (28) to estimate v.

LEMMA 5.

$$\int_{r_1}^{r_2} r^{n-1} v(r) dr = r_2^{n-1} \sin \psi(r_2) - r_1^{n-1} \sin \psi(r_1) .$$

Proof. Multiply equation (26a) by r^{n-1} and integrate.

LEMMA 6.

(i) $\dfrac{\sin \psi}{r} < \dfrac{v}{n}$ *for* $0 < r \leqq R$

(ii) $\left(\dfrac{\sin \psi}{r}\right)_r > 0$ *for* $0 < r \leqq R$.

Proof.

Part (i). From Lemma 5 and Theorem 6:

$$r^{n-1} \sin \psi = \int_0^r s^{n-1} v(s) ds < \frac{r^n}{n} v(r) .$$

Part (ii).

$$\left(\frac{\sin \psi}{r}\right)_r = \frac{r(\sin \psi)_r - \sin \psi}{r^2} = \frac{-n \sin \psi + rv}{r^2} > 0 .$$

The last equality comes from equation (27); the inequality comes from Part (i).

LEMMA 7.

$$\frac{v(0)}{n} < \frac{\sin \psi}{r} < \min \left\{\frac{v}{n}, \frac{\cos \gamma}{R}\right\}$$

for $0 < r < R$.

Proof. From Lemma 6 (ii)

$$\lim_{s \to 0+} \frac{\sin \psi(s)}{s} < \frac{\sin \psi(r)}{r} < \lim_{s \to R-} \frac{\sin \psi(s)}{s} .$$

We note that

$$\lim_{s \to 0+} \frac{\sin \psi(s)}{s} = \lim_{s \to 0+} [\sin \psi(s)]_s .$$

Thus, using equation (27):

$$n \cdot \lim_{s \to 0+} \frac{\sin \psi(s)}{s} = \lim_{s \to +0} v(s) = v(0) .$$

By condition (26b)

$$\lim_{s \to R-} \frac{\sin \psi(s)}{s} = \frac{\cos \gamma}{R} .$$

We have shown

$$\frac{v(0)}{n} < \frac{\sin \psi(r)}{r} < \frac{\cos \gamma}{R} .$$

Combining this inequality with Part (i) of Lemma 6 gives the stated inequality.

THEOREM 7. *For* $0 \leqq \gamma < \pi/2$:

(i) $\max \left\{\sqrt{2(1 - \sin \gamma)}, \frac{n \cos \gamma}{R}\right\} < v(R, R, \gamma) .$

(ii) $v(R, R, \gamma) < (n-1)\frac{\cos\gamma}{R} + \sqrt{2(1-\sin\gamma) + \frac{\cos^2\gamma}{R^2}}$

(iii) $2(1-\sin\gamma) + \frac{2(n-1)}{n}[v(R) - v(0)]v(0) < v^2(R) - v^2(0) < 2n(1-\sin\gamma)$.

Proof. Parts (i) and (iii) are due to Finn [5]. Integrate equation (28) from $v(0)$ to $v(R)$:

$$\frac{v^2(R) - v^2(0)}{2} = 1 - \sin\gamma + (n-1)\int_{v(0)}^{v(R)} \frac{\sin\psi}{r}\, dv \;. \tag{29}$$

Employ the estimates of Lemma 7:

$$\int_{v(0)}^{v(R)} \frac{\sin\psi}{r}\, dv > \frac{v(0)}{n}[v(R) - v(0)] \tag{30}$$

$$\begin{aligned}\int_{v(0)}^{v(R)} \frac{\sin\psi}{r}\, dv &< \int_{v(0)}^{n\cos\gamma/R} \frac{v}{n}\, dv + \int_{n\cos\gamma/R}^{v(R)} \frac{\cos\gamma}{R}\, dv \\ &= -\frac{n}{2}\left(\frac{\cos\gamma}{R}\right)^2 - \frac{v^2(0)}{2n} + v(R)\frac{\cos\gamma}{R} \;.\end{aligned} \tag{31}$$

Part (i). Equation (29) gives $v(R) > \sqrt{2(1-\sin\gamma)}$; Part (i) of Lemma 6 with $r = R$ gives $v(R) > n\cos\gamma/R$.

Part (ii). Combining equation (29) and inequality (31) gives

$$\begin{aligned}\frac{v^2(R)}{2} &< 1 - \sin\gamma + \frac{1}{2n}v^2(0) - \frac{n(n-1)}{2}\frac{\cos^2\gamma}{R^2} \\ &\quad + v(R)(n-1)\frac{\cos\gamma}{R} \;.\end{aligned}$$

Solving this quadratic inequality:

$$v(R) < (n-1)\frac{\cos\gamma}{R} + \left[2(1-\sin\gamma) + (1-n)\frac{\cos^2\gamma}{R^2} + \frac{v^2(0)}{n}\right]^{1/2} \;.$$

Estimating $v(0) < n(\cos\gamma/R)$ gives the stated inequality.

Part (iii). Combining equation (29) and inequality (30) gives the left-hand side of inequality (iii). The right-hand side is obtained by employing $\sin\psi/r < v/n$; this estimate yields

$$\int_{v(0)}^{v(R)} \frac{\sin\psi}{r}\, dv < \frac{v^2(R) - v^2(0)}{2n} \;.$$

Combining this with equation (29) gives the result.

Inequalities (i) and (ii) are sharp for small and large R, more precisely:

COROLLARY.

$$v(R, R, \gamma) = z(0, \gamma) + O(1/R) \qquad as \quad R \longrightarrow \infty$$

$$v(R, R, \gamma) = n\frac{\cos\gamma}{R} + O(R) \qquad as \quad R \longrightarrow 0 \,.$$

7. **The narrow tube.** A study is made of $v(r, R, \gamma)$ for small R. We show that the solution is close to a spherical cap.

First, we prove a technical lemma:

LEMMA 8. *Suppose* $Nf(r) \geqq Ng(r)$ *for* $0 \leqq r \leqq R$, *then* $f(r_2) - f(r_1) \geqq g(r_2) - g(r_1)$ *for* $0 \leqq r_1 \leqq r_2 \leqq R$.

Proof.

$$\frac{1}{r^{n-1}}\left(\frac{r^{n-1}f_r}{\sqrt{1+f_r^2}}\right)_r \geqq \frac{1}{r^{n-1}}\left(\frac{r^{n-1}g_r}{\sqrt{1+g_r^2}}\right)_r .$$

Multiply by r^{n-1} and integrating from 0 to r gives

$$\frac{r^{n-1}f_r}{\sqrt{1+f_r^2}} \geqq \frac{r^{n-1}g_r}{\sqrt{1+g_r^2}} \,.$$

Simplifying: $f_r \geqq g_r$. The conclusion follows upon integrating from r_1 to r_2.

We take $0 \leqq \gamma < \pi/2$ in what follows. Introduce

$$S_1(r) = R\tan\gamma - \sqrt{(R\sec\gamma)^2 - r^2} + n\,\frac{\cos\gamma}{R}$$

and

$$S_2(r) = S_1(r) + R\sec\gamma(1 - \sin\gamma) \,.$$

LEMMA 9. $S_1(r) < v(r) < S_2(r)$ *for* $0 \leqq r < R$.

Proof. To apply CP, we check that

$$NS_1 = n\,\frac{\cos\gamma}{R} \geqq S_1(r)$$

$$NS_2 = n\,\frac{\cos\gamma}{R} \leqq S_2(r)$$

for $0 \leqq r < R$, and $TS_1 \cdot \nu = TS_2 \cdot \nu = \cos\gamma$ for $r = R$.

Two special cases of Lemma 9 are

$$n\frac{\cos\gamma}{R} \leqq v(R, R, \gamma) \leqq R\sec\gamma(1-\sin\gamma) + n\frac{\cos\gamma}{R} \tag{32}$$

$$n\frac{\cos\gamma}{R} > v(0, R, \gamma) > n\frac{\cos\gamma}{R} - R\sec\gamma(1-\sin\gamma)\ . \tag{33}$$

Introduce $R_1 = R\sec\gamma[1-(R^2/n)\sec^2\gamma(1-\sin\gamma)]^{-1}$ and $R_2 = R\sec\gamma[1+(R^2/n)\sec^2\gamma(1-\sin\gamma)]^{-1}$.

THEOREM 8. $(0 \leqq \gamma < \pi/2)$ $v(r, R, \gamma) > S_3(r) = H_2(R, \gamma) + R_1 - \sqrt{R_1^2 - r^2}$ *and* $v(r, R, \gamma) = S_3(r) + O(R^3)$ *as* $R \to 0$, *where*

$$H_2(R, \gamma) = n\frac{\cos\gamma}{R} - R_2 + \frac{n}{R^n}\int_0^R r^{n-1}\sqrt{R_2^2 - r^2}dr$$

for $\gamma \neq 0$, *and*

$$H_2(R, 0) = \frac{n}{R}\left(1+\frac{R^2}{n}\right) - R\left(1+\frac{R^2}{n}\right)^n - R_2 + \frac{n}{R_2^n}\int_0^{R_2} r^{n-1}\sqrt{R_2^2 - r^2}dr\ .$$

Proof.

Case $\gamma \neq 0$. Let $f_1(r) = R_1 - \sqrt{R_1^2 - r^2}$ and $f_2(r) = R_2 - \sqrt{R_2^2 - r^2}$. Clearly, $R_1 > R$ and $R_2 > R$ for $R \leqq R_0$, with R_0 sufficiently small. By estimates (32) and (33)

$$Nf_1(r) = \frac{n}{R_1} < v(0) \leqq v(r) = Nv$$

$$Nf_2(r) = \frac{n}{R_2} \geqq v(R) \geqq v(r) = Nv\ .$$

Hence, by Lemma 8

$$R_1 - \sqrt{R_1^2 - r^2} < v(r) - v(0) < R_2 - \sqrt{R_2^2 - r^2}\ . \tag{34}$$

Let

$$H_1(R, \gamma) = n\frac{\cos\gamma}{R} - R_1 + \frac{n}{R^n}\int_0^R r^{n-1}\sqrt{R_1^2 - r^2}dr\ .$$

Multiplying estimate (34) by r^{n-1} and integrating from 0 to R gives

$$H_2 < v(0) < H_1\ . \tag{35}$$

Let

$$S_3(r) = H_2 + f_1(r)$$

and

$$S_4(r) = H_1 + f_2(r) .$$

By inequalities (34) and (35): $S_3(r) < v(r) < S_4(r)$. Now

$$S_4(r) - S_3(r) = R_2 - R_1 + \frac{n}{R^n}\int_0^R r^{n-1}(\sqrt{R_1^2 - r^2} - \sqrt{R_2^2 - r^2})dr + (\sqrt{R_1^2 - r^2} - \sqrt{R_2^2 - r^2}) .$$

Clearly $R_2 - R_1 = O(R^3)$ as $R \to 0$, and

$$\sqrt{R_1^2 - r^2} - \sqrt{R_2^2 - r^2} = \frac{(R_1 - R_2)(R_1 + R_2)}{\sqrt{R_1^2 - r^2} + \sqrt{R_2^2 - r^2}} = O(R^3)$$

as $R \to 0$. Hence $S_4(r) - S_3(r) = O(R^3)$ as $R \to 0$.

Case $\gamma = 0$. We must modify the above proof. We still have $v(0) < H_1(R, 0)$ and $R_1 - \sqrt{R_1^2 - r^2} < v(r) - v(0)$; however, $R_2 < R$ and $v(r) - v(0) < R_2 - \sqrt{R_2^2 - r^2}$ only for $0 \leqq r < R_2$. We obtain

$$v(0) > \frac{n}{R^2}\sin\psi(R_2) - R_2 + \frac{n}{R_2^n}\int_0^{R_2} r^{n-1}\sqrt{R_2^2 - r^2}dr . \tag{36}$$

Here

$$R_1 = R\left(1 - \frac{R^2}{n}\right)^{-1} \quad \text{and} \quad R_2 = R\left(1 + \frac{R^2}{n}\right)^{-1} .$$

We estimate $\sin\psi(R_2)$; Lemma 5 gives

$$\sin\psi(R_2) = \left(\frac{R}{R_2}\right)^{n-1} - \frac{1}{R_2^{n-1}}\int_{R_2}^R r^{n-1}v(r)dr .$$

Thus

$$\sin\psi(R_2) > \left(\frac{R}{R_2}\right)^{n-1} - \left(\frac{n}{R} + R\right)\frac{R^n - R_2^n}{nR_2^{n-1}} = 1 - \frac{R^2}{n}\left(1 + \frac{R^2}{n}\right)^{n-1} .$$

Putting this into estimate (36) gives $v(0) > H_2(R, 0)$. Thus $v(r) > S_3(r)$. Now

$$H_2(R, 0) - H_1(R, 0) = R_1 - R_2 + R\left[1 - \left(1 + \frac{R^2}{n}\right)^n\right] + n\int_0^{R_2} r^{n-1}\left(\frac{\sqrt{R_2^2 - r^2}}{R_2^n} - \frac{\sqrt{R_1^2 - r^2}}{R^n}\right)dr$$

$$+ \frac{n}{R^n} \int_{R_2}^{R} r^{n-1} \sqrt{R_1^2 - r^2} dr$$
$$= O(R^3) \quad \text{as} \quad R \longrightarrow 0 .$$

We estimate $v(R, R, \gamma)$ by Part (iii) of Theorem 7:

$$(37) \qquad v(R) - v(0) < \frac{2n}{v(R) + v(0)} < \frac{2n}{\frac{n}{R} + \frac{n}{R} - R} = R\left(1 - \frac{R^2}{2n}\right)^{-1}$$

$$(38) \qquad v(R) - v(0) > \left[\frac{v(R) + v(0)}{2} - \frac{(n-1)}{n} v(0)\right]^{-1}$$
$$> R\left[1 + R^2\left(1 - \frac{1}{n}\right)\right]^{-1} .$$

We have employed estimates (32) and (33) for $v(R)$ and $v(0)$. Estimates (37) and (38) show that $v(R) = v(0) + R + O(R^3)$ as $R \rightarrow 0$.

Since $Nv = v > NS_3$ for $0 \leqq r < R$, Lemma 8 gives $v(R) - v(r) > S_3(R) - S_3(r)$. Thus $v(r) < v(R) - S_3(R) + S_3(r)$. We note that $S_3(R) - S_3(0) = R_1 - \sqrt{R_1^2 - R^2} = R + O(R^3)$ as $R \rightarrow 0$. Therefore

$$\begin{aligned} v(r) &< [v(0) + R] - [S_3(0) + R] + S_3(r) + O(R^3) \\ &= S_3(r) + [v(0) - S_3(0)] + O(R^3) \\ &< S_3(r) + O(R^3) \quad \text{as} \quad R \longrightarrow 0 . \end{aligned}$$

We have used that $v(0) - S_3(0) = v(0) - H_2(R, 0) < H_1(R, 0) - H_2(R, 0) = O(R^3)$. Since we started with $S_3(r) < v(r)$ the proof is complete.

COROLLARY. *For $n = 2$, $H_2(R, \gamma)$ can be calculated explicitly, yielding*

$$(39) \qquad v(0, R, \gamma) = 2\frac{\cos\gamma}{R} - \frac{R}{\cos\gamma}\left(1 - \frac{2}{3}\frac{1 - \sin^3\gamma}{\cos^2\gamma}\right) + O(R^3)$$

as $R \rightarrow 0$.

REMARK. The formula (39) is known as Laplace's formula. Ferguson has given a formula differing from formula (39) [1]. We have settled the question.

8. The wide tube.

8A. *Estimate near the boundary.* We show that near the boundary the solution v approaches the one-dimensional solution as $R \rightarrow \infty$.

LEMMA 10. *For $0 \leqq \gamma < \pi/2$ and $0 \leqq l \leqq R$:*

(40) $$v^2(R-l) < v^2(R) + 2\sin\gamma - 2\cos\psi(R-l)$$

(41) $$\sin\psi(R-l) < -\sin\psi_1(l) + \frac{l}{R}(n-1)\cos\gamma\ .$$

Proof. Integrating equation (28) from $v(R-l)$ to $v(R)$ and estimating $\sin\psi > 0$ gives inequality (40).

By CPS, $z(x_1, \gamma) < v(|x-y|, R, \gamma)$ in $B_R(y)$ if $B_R(y) \subset H$. Choose $y = (R, 0, \cdots, 0)$ and $x = (x_1, 0, \cdots, 0)$. Thus

(42) $$z(x_1, \gamma) < v(R - x_1, R, \gamma) \qquad \text{for} \quad 0 < x_1 \leqq R\ .$$

Let $r = R - x_1$. Then $[\sin\psi_1(x_1)]_{x_1} = -[\sin\psi(x_1)]_r$. Therefore $(\sin\psi)_r + (n-1)(\sin\psi/r) > -[\sin\psi_1(x_1)]_r$. Integrating from $R-l$ to R and estimating $\sin\psi/r < \cos\gamma/R$ gives inequality (41).

THEOREM 9. *For* $0 \leqq \gamma < \pi/2$ *and* $l \geqq 0$:
(i) $v(R-l, R, \gamma)$ *is strictly decreasing in* R.
(ii) $\lim_{R\to\infty} v(R-l, R, \gamma) = z(l, \gamma)$.

Proof.

Part (i). By CPS, $v(|x-y_2|, R_2, \gamma) < v(|x-y_1|, R_1, \gamma)$ in $B_{R_1}(y_1)$ if $B_{R_1}(y_1) \subset B_{R_2}(y_2)$. Choosing $y_i = (R_i, 0, \cdots, 0)$ $i = 1$ or 2, and $x = (l, 0, \cdots, 0)$ gives $v(R_2 - l, R_2, \gamma) < v(R_1 - l, R_1, \gamma)$ for $R_1 < R_2$ and $0 < l \leqq R_1$.

For $l = 0$ we have $v(R_2, R_2, \gamma) \leqq v(R_1, R_1, \gamma)$. If $\gamma \neq 0$ Lemma 1 rules out equality.

A special argument is needed for the case $\gamma = 0$ and $l = 0$: Let $u_1(r) = v(r, R_1, 0)$ and $u_2(r) = v(r + R_2 - R_1, R_2, 0)$. Suppose $v(R_2, R_2, 0) = v(R_1, R_1, 0)$, then $u_1(R_1) = u_2(R_1)$. Also $u_1(r) > u_2(r)$ for $0 \leqq r < R_1$. At R_1

$$(\sin\varphi_1)_r + \frac{(n-1)}{R_1} = (\sin\varphi_2)_r + \frac{(n-1)}{R_2}$$

since $Nu_1 = Nu_2$ at R_1; here φ_i ($i = 1$ or 2) is the angle between the curve given by $u_i(r)$ and the positive r direction. Thus, $(\sin\varphi_1)_r < (\sin\varphi_2)_r$ at R_1. This inequality must hold for $R_1 - \varepsilon \leqq r \leqq R_1$, for some $\varepsilon > 0$. Integrating from r to R_1 gives $\sin\varphi_1(r) > \sin\varphi_2(r)$; this implies that $u_1(r) < u_2(r)$ for $R_1 - \varepsilon \leqq r \leqq R_1$ after another integration. This is a contradiction.

Part (ii). For R such that

(43) $$R > \frac{(n+1)l\cos\gamma}{1 + \sin\psi_1(l)}$$

the right-hand member of inequality (41) is <1. Combining inequalities (40) and (41) and noting by equation (23) that $2[\sin\gamma - \cos\psi_1(l)] = z^2(l) - z^2(0)$ gives

$$
(44) \qquad \begin{aligned} v^2(R-l) &< v^2(R) - z^2(0) + z^2(l) + 2\eta(l, R) \\ \eta(l, R) &= \cos\psi_1(l) - [\cos^2\psi_1(l) + 2\varepsilon\sin\psi_1(l) - \varepsilon^2]^{1/2} \end{aligned}
$$

where $\varepsilon = (n-1)(l/R)\cos\gamma$. Clearly, $\lim_{R\to\infty}\eta(l, R) = 0$. By the corollary to Theorem 7 $\lim_{R\to\infty} v^2(R) = z^2(0)$. Thus $\lim_{R\to\infty} v(R-l, R, \gamma) \leqq z(l, \gamma)$. By inequality (42) $v(R-l, R, \gamma) \geqq z(l, \gamma)$. Hence Part (ii) is proved.

8B. *Estimate of $v(r, R, \gamma)$ from above.*

THEOREM 10. *For $0\leqq\gamma<\pi/2$: $v(r, R, \gamma)<H(R, \gamma)I(r)[1+O(1/R^{1/2})]$ for $0 \leqq r \leqq R - (1/4)\ln R$ as $R \to \infty$, where*

$$
H(R, \gamma) = \sqrt{2\pi}C_1(\gamma)\frac{R^{(n-1)/2}}{\exp R}.
$$

Proof. By CP and Lemma 3, $v(r, R) < (I(r)/I(R-l))v(R-l, R)$ for $0 \leqq r < R - l$. By inequality (44)

$$
v(R-l) < z(l)\{1 + [2z(l)]^{-2}[v^2(R) - z^2(0) + 2\eta(l, R)]\}^{1/2}.
$$

Choose $l = (1/4)\ln R$, for R sufficiently large condition (43) shows that η is defined. By the estimates of Chapter I we find

$$
I(R-l) = \frac{1}{\sqrt{2\pi}}\frac{\exp(R-l)}{R^{(n-1)/2}}\left[1 + O\left(\frac{\ln R}{R}\right)\right]
$$

and

$$
z(l) = C_1(\gamma)\exp(-l)\left[1 + O\left(\frac{1}{R^{1/2}}\right)\right].
$$

To estimate η, we note that

$$
0 < \eta(l, R) < \varepsilon[-2\sin\psi_1(l) + \varepsilon][\cos\psi_1(l)]^{-1}.
$$

Thus

$$
\eta(l, R) = O\left(\frac{\ln R}{R^{5/4}}\right).
$$

Also $v^2(R) - z^2(0) = O(1/R)$ by the corollary to Theorem 7. Combining estimates and noting that $z(l) = O(1/R^{1/4})$ gives the result.

8C. *Estimate of $v(r, R, \gamma)$ from below.*

THEOREM 11. *For $0\leqq\gamma<\pi/2$: $v(r,R,\gamma)>H(R,\gamma)I(r)[1+O(\ln R/R)]$*

for $0 \leqq r \leqq R - (5/2) \ln R$ *as* $R \to \infty$.

Proof. By Theorem 9, $v(R - l, R, \gamma) > z(l, \gamma)$ for $0 \leqq l \leqq R$. We construct a subsolution $u(r)$ for $0 \leqq r \leqq R - l$ with $u(R - l) = z(l, \gamma)$, where l will be chosen later. Let

$$u(r) = h\left(1 - \frac{\varepsilon}{r^2 + a}\right) I(r) = hB(r)I(r)$$

where h, ε, and a are positive constants with $\varepsilon < a$.

$$u_r = h\left[B(r)I_r + \frac{2\varepsilon r}{(r^2 + a)^2} I\right].$$

We note: $u > 0$ and $u_r \geqq 0$. The subsolution condition $Nu \geqq u$ becomes

$$u_{rr} + \frac{(n-1)}{r} u_r + \frac{(n-1)}{r} u_r^3 \geqq u(1 + u_r^2)^{3/2}.$$

We require the stronger inequalities

$$u_{rr} + \frac{(n-1)}{r} u_r \geqq u(1 + 2u_r^2) \tag{45a}$$

$$\max u_r < 1 . \tag{45b}$$

These are stronger because $(1 + x)^{3/2} \leqq 1 + 2x$ for $0 \leqq x \leqq 1$. Condition (45a) reduces to

$$u_r^2 < \frac{\varepsilon A B^{-1}}{(r^2 + a)^2} \tag{46}$$

$$A(r) = 2r \frac{I_r}{I} + \frac{a - 3r^2}{r^2 + a} + (n - 1) .$$

By Lemma 4 we can choose b so that

$$b \frac{I_r(b)}{I(b)} \geqq \frac{5 - n}{2} .$$

Let $a = 3b^2$. With this choice for a, $A(r) > 1$ for all $r \geqq 0$, as can be seen by checking the cases $r > b$ and $r < b$. Choose $h = z(l)[I(R - l)B(R - l)]^{-1}$. Let

$$c = \max_{r>0} \frac{2r}{(r^2 + a)^2} .$$

By Lemma 4 we can estimate $I_r < I$, thus

$$\begin{aligned}\max_{0\leqq r\leqq R-l} u_r &< h[B(R-l)+\varepsilon c]I(R-l)\\ &= z(l)[1+\varepsilon cB^{-1}(R-l)]\\ &< z(l)\left[1+\varepsilon c\left(1-\frac{\varepsilon}{a}\right)^{-1}\right] \qquad (47)\\ \min_{0\leqq r\leqq R-l}\frac{\varepsilon AB^{-1}}{(r^2+a)^2} &> \frac{\varepsilon B^{-1}(R)}{(R^2+a)^2}\,.\end{aligned}$$

We can satisfy condition (46) by choosing l so that

$$2e^{-2l}\left[1+\varepsilon c\left(1-\frac{\varepsilon}{a}\right)^{-1}\right]^2 < \frac{\varepsilon B^{-1}(R)}{(R^2+a)^2}\,. \tag{48}$$

We have estimated $z(l) < \sqrt{2}\,e^{-l}$. Finally, choose $\varepsilon = \ln R/R$ and $l = (5/2)\ln R$, then $l > (1/2)\ln[2R(R^2+a)^2/\ln R] + o(1)$ as $R\to\infty$. Thus condition (48) is satisfied for R sufficiently large. With estimate (47) we check that condition (45b) holds for R sufficiently large.

By CP, $u(r) < v(r)$ for $0\leqq r\leqq R-l$. Thus

$$\begin{aligned}v(r) &> hB(r)I(r)\\ &> \frac{z(l)}{I(R-l)}B(0)B^{-1}(R-l)I(r)\,.\end{aligned}$$

By the estimates in Chapter I we find

$$z(l) = C_1(\gamma)e^{-l}\left[1+O\left(\frac{1}{R^{5/2}}\right)\right]$$

$$I(R-l) = \frac{1}{\sqrt{2\pi}}\frac{\exp(R-l)}{R^{(n-1)/2}}\left[1+O\left(\frac{\ln R}{R}\right)\right].$$

Combining estimates and noting that $B^{-1}(R-l) = 1+O(\ln R/R)$ gives the result.

Corollary 1. *$v(r, R, \gamma) = H(R,\gamma)I(r)[1+O(1/R^{1/2})]$ for $0\leqq r\leqq R-(5/2)\ln R$ as $R\to\infty$.*

Proof. Combine Theorems 10 and 11.

We can estimate the first derivative as follows:

Corollary 2. *For $0\leqq r\leqq R-(5/2)\ln R$*

$$\sin\psi(r) = H(R,\gamma)I_r(r)\left[1+O\left(\frac{1}{R^{1/2}}\right)\right]$$

for $R\to\infty$.

Proof. By Lemma 5,

$$\sin \psi(r) = \frac{1}{r^{n-1}} \int_0^r s^{n-1} v(s) ds .$$

Estimating $v(r)$ with Theorem 10 gives

$$\sin \psi(r) < H(R, \gamma) I_r(r) \left[1 + O\left(\frac{1}{R^{1/2}} \right) \right]$$

for $0 \leqq r \leqq R - (1/4) \ln R$. Estimating $v(r)$ with Theorem 11 gives

$$\sin \psi(r) > H(R, \gamma) I_r(r) \left[1 + O\left(\frac{\ln R}{R} \right) \right]$$

for $0 \leqq r \leqq R - (5/2) \ln R$. Combining estimates gives the result.

REMARK. One expects the error in Corollaries 1 and 2 to be $O(\ln R/R)$. Perko [15] obtains this error estimate for $\gamma > 0$ by a completely different method. The case $\gamma = 0$ is important in view of the general estimates of Chapter III.

9. The exterior problem. We obtain estimates for the function $w(r, R, \gamma)$ which satisfies

(49a) $$\frac{1}{r^{n-1}} \left(\frac{r^{n-1} w_r}{\sqrt{1 + w_r^2}} \right)_r = w \qquad \text{for} \quad r > R$$

(49b) $$w_r(R+) = -\cot \gamma$$

(49c) $$\lim_{r \to \infty} w(r) = 0 .$$

Johnson and Perko [12] prove by a "shooting argument":

THEOREM 12. *Problem* (49) *has a unique solution* $w(r, R, \gamma)$ *which is continuous in* (r, γ) *for* $r \geqq R$ *and* $0 < \gamma \leqq \pi/2$; $w > 0$, $w_r < 0$ *if* $0 \leqq \gamma < \pi/2$; $w \equiv 0$ *if* $\gamma = \pi/2$; $w(r, R, 0) = \lim_{\gamma \to 0+} w(r, R, \gamma)$; *and* $\lim_{r \to \infty} w_r = 0$.

REMARK. The note after Theorem 6 applies here as well.

9A. *Continuity with respect to* (r, R, γ). We fill in a gap in the continuous dependence properties.

LEMMA 11. *For* $0 \leqq \gamma < \pi/2$, $R_1 < R_2$ *and* $r > R_2$

(50) $$w(r + R_1 - R_2, R_1, \gamma) < w(r, R_2, \gamma) < \frac{R_2}{R_1} w\left(\frac{R_1}{R_2} r, R_1, \gamma \right) .$$

Proof. Let $v_i = w(r_i, R_i, \gamma)$ and $r_i = |x - y_i|$ for $i = 1$ or 2. If

$B^c_{R_2}(y_2) \subset B^c_{R_1}(y_1)$ then the proof of Theorem 2 shows that $Tv_1 \cdot \nu \leqq \cos\gamma = Tv_2 \cdot \nu$ on $\partial B^c_{R_2}$. Since $v_1 = o(1)$ and $v_2 = o(1)$ as $|x| \to \infty$, CP gives $v_1 < v_2$ on $B^c_{R_1}$. Choose $y_2 = 0$, $y_1 = (R_2 - R_1, 0, \cdots, 0)$, and $x = (x_1, 0, \cdots, 0)$; thus $w(x_1 + R_1 - R_2, R_1, \gamma) < w(x_1, R_2, \gamma)$ for $x_1 > R_2$.

Let

$$u = \frac{R_2}{R_1} w\left(\frac{R_1}{R_2} r, R_1, \gamma\right).$$

Then by Lemma 2

$$Nu = \left(\frac{R_1}{R_2}\right)^2 u < u$$

in $B^c_{R_2}(0)$ and $Tu \cdot \nu = \cos\gamma$ on $\partial B^c_{R_2}$. Thus by CP, $u > w$ in $B^c_{R_2}(0)$.

LEMMA 12. *For* $0 \leqq \gamma \leqq \pi/2$:

$$0 < \varepsilon < \sqrt{2}, \qquad \sigma(\varepsilon) = \frac{\varepsilon^3}{3}.$$

If $|r_1 - r_2| < \sigma(\varepsilon)$ *and* $r_1, r_2 \geqq R$ *then* $|w(r_2, R, \gamma) - w(r_1, R, \gamma)| < \varepsilon$.

Proof. See [5].

THEOREM 13. *For* $0 \leqq \gamma \leqq \pi/2$: $w(r, R, \gamma)$ *is continuous in* (r, R) *independent of* γ.

Proof. We show continuity at (r_1, R_1).

For $R > R_1$: By inequality (50)

$$w(r, R_1, \gamma) < w(r, R, \gamma) < \frac{R}{R_1} w\left(\frac{R_1}{R} r, R_1, \gamma\right).$$

Suppose

$$\left.\begin{aligned} |r - r_1| &< \sigma(\varepsilon) \\ \left|\frac{R_1}{R} r - r_1\right| &< \sigma(\varepsilon) \end{aligned}\right\}. \tag{51}$$

By Lemma 12

$$w(r_1, R_1, \gamma) - \varepsilon < w(r, R, \gamma) < \frac{R}{R_1}[w(r_1, R_1, \gamma) + \varepsilon]. \tag{52}$$

For $R < R_1$: By inequality (50)

$$\frac{R}{R_1} w\left(\frac{R_1}{R} r, R_1, \gamma\right) < w(r, R, \gamma) < w(r, R_1, \gamma).$$

Supposing condition (51):

$$\frac{R}{R_1}[w(r_1, R_1, \gamma) - \varepsilon] < w(r, R, \gamma) < w(r_1, R_1, \gamma) + \varepsilon \ . \tag{53}$$

Inequalities (52) and (53) show continuity at (r_1, R_1) independent of γ.

COROLLARY. *$w(r, R, \gamma)$ is continuous in (r, R, γ) for $r \geqq R$, $R > 0$, and $0 \leqq \gamma \leqq \pi/2$.*

Proof. We show continuity at (r_1, R_1, γ_1).

Note:

$$\begin{aligned} |w(r, R, \gamma) - w(r_1, R_1, \gamma_1)| &\leqq |w(r, R, \gamma) - w(r_1, R_1, \gamma)| \\ &\quad + |w(r_1, R_1, \gamma) - w(r_1, R_1, \gamma_1)| \\ &= I + II \ . \end{aligned}$$

For $|\gamma - \gamma_1| < \delta_1$: $II < \varepsilon$ by Theorem 12. For $|r - r_1| < \delta_2$ and $|R - R_1| < \delta_3$: $I < \varepsilon$ by Theorem 13. Hence $I + II < 2\varepsilon$.

9B. *General estimates—exterior case.* Introducing $\sin\varphi(r) = w_r/\sqrt{1 + w_r^2}$ we have

$$\frac{1}{r^{n-1}}(r^{n-1}\sin\varphi)_r = w \qquad \text{for} \quad r > R \tag{54a}$$

$$\sin\varphi(R-) = -\cos\gamma \ . \tag{54b}$$

Equation (54a) can be rewritten as

$$(\sin\varphi)_r + \frac{(n-1)}{r}\sin\varphi = w \ . \tag{55}$$

Partially inverting equation (55):

$$-(\cos\varphi)_w + \frac{(n-1)}{r}\sin\varphi = w \ . \tag{56}$$

LEMMA 13. *For $0 \leqq \gamma < \pi/2$ and $r > R$, $(\sin\varphi)_r > 0$ and*

$$-\frac{\cos\gamma}{R} < \frac{\sin\varphi}{r} < 0 \ . \tag{57}$$

Proof. Since $\sin\varphi < 0$, $(\sin\varphi)_r = w - (n-1)/r\sin\varphi > 0$. Thus $-\cos\gamma < \sin\varphi(r) < 0$ for $r > R$ implying inequality (57).

THEOREM 14. *For $0 \leqq \gamma < \pi/2$:*

(i) $$w(R, R, \gamma) < \sqrt{2(1 - \sin\gamma)}\ .$$

(ii) $$w(R, R, \gamma) > -(n-1)\frac{\cos\gamma}{R} + \sqrt{2(1-\sin\gamma) + \left[\frac{(n-1)\cos\gamma}{R}\right]^2}\ .$$

Proof. Part (i) is due to Finn [5]. Integrate equation (56) from $w(r)$ to $w(R)$:

$$\frac{w^2(R) - w^2(r)}{2} = -\sin\gamma + \cos\varphi + (n-1)\int_{w(r)}^{w(R)} \frac{\sin\varphi}{r}\, dw\ . \tag{58}$$

Let $r \to \infty$:

$$w^2(R) = 2(1 - \sin\gamma) + 2(n-1)\int_0^{w(R)} \frac{\sin\varphi}{r}\, dw\ .$$

Estimating with inequality (57) gives

$$w^2(R) < 2(1 - \sin\gamma)$$

$$w^2(R) > 2(1 - \sin\gamma) - 2(n-1)\frac{\cos\gamma}{R} w(R)\ .$$

Solving this quadratic inequality gives part (ii).

COROLLARY. *$w(R, R, \gamma) = \sqrt{2(1 - \sin\gamma)} + O(1/R)$ as $R \to \infty$.*

9C. *Estimate near the boundary.*

LEMMA 14. *For $0 \leqq \gamma < \pi/2$:*

$$w(r, R, \gamma) < z(r - R, \gamma) \quad \textit{for} \quad r > R \tag{59}$$

$$\sin\varphi(R + l) < \sin\psi_1(l) + (n-1)\frac{l\cos\gamma}{R} \quad \textit{for} \quad l > 0\ . \tag{60}$$

Proof. Let $u(x) = z(x_1 - R, \gamma)$. By the proof of Theorem 2: $Tw \cdot \nu \leqq \cos\gamma = Tu \cdot \nu$ on $\{x : x_1 = R\}$. Since $w = o(1)$ and $u > 0$ we have $w - u \leqq o(1)$ as $|x| \to \infty$. By CP,

$$w(r, R, \gamma) < z(x_1 - R, \gamma) \leqq z(r - R, \gamma) \quad \text{for} \quad r > R\ .$$

Let $x_1 = r - R$, then (cf. Lemma 10)

$$(\sin\varphi)_r + \frac{(n-1)}{r}\sin\varphi < -[\sin\psi_1(x_1)]_r\ .$$

Integrating from R to $R + l$ and employing inequality (57) gives inequality (60).

THEOREM 15. *For* $0 \leqq \gamma < \pi/2$ *and* $l \geqq 0$:
(i) $w(R + l, R, \gamma)$ *is strictly increasing in* R.
(ii) $\lim_{R\to\infty} w(R + l, R, \gamma) = z(l, \gamma)$.

Proof.

Part (i). For $l > 0$ and $R_1 < R_2$, take $r = R_2 + l$ in Lemma 11: $w(R_1 + l, R_1, \gamma) < w(R_2 + l, R_2, \gamma)$. For $l = 0$ we have $w(R_1, R_1, \gamma) \leqq w(R_2, R_2, \gamma)$. If $\gamma \neq 0$ Lemma 1 rules out equality. The case $l = 0$ and $\gamma = 0$ is handled as in the proof of Theorem 9.

Part (ii). From equation (58): $w^2(r) > w^2(R) + 2[\sin\gamma - \cos\varphi(r)]$. Let $r = R + l$ and combine this with inequality (60) noting that $2[\sin\gamma - \cos\psi_1(l)] = z^2(l) - z^2(0)$:

$$w^2(R + l) > w^2(R) - z^2(0) + z^2(l) + 2\eta(l, R) \tag{61}$$

where η is the function introduced in § 8. Now, by the corollary to Theorem 14 $\lim_{R\to\infty} w^2(R) = z^2(0)$. As before $\lim_{R\to\infty} \eta(l, R) = 0$, thus $\lim_{R\to\infty} w(R + l, R, \gamma) \geqq z(l)$. By inequality (59): $w(R + l, R, \gamma) \leqq z(l, \gamma)$. Thus $\lim_{R\to\infty} w(R + l, R, \gamma) \leqq z(l, \gamma)$.

9D. *Behavior at infinity.* We study $w(r, R, \gamma)$ for r large.

THEOREM 16. *For* $0 \leqq \gamma < \pi/2$:
(i) $C(R, \gamma) = \lim_{r\to\infty} \dfrac{w(r, R, \gamma)}{K(r)}$ *exists and* $C(R, \gamma) > 0$.
(ii) $w(r, R, \gamma) = C(R, \gamma)K(r)\{1 + O[K^2(r)]\}$ *as* $r \to \infty$.
(iii) $C(R, \gamma) < \sqrt{2/\pi}C_1(\gamma)R^{(n-1)/2}e^R[1 + O(\ln R/R)]$
$C(R, \gamma) > \sqrt{2/\pi}C_1(\gamma)R^{(n-1)/2}e^R[1 + O(1/R^{1/2})]$ *as* $R \to \infty$.

Proof.

Part (i). Since $K(r)$ is a supersolution (Lemma 3) CP gives

$$w(r) < \frac{w(R_1)}{K(R_1)} K(r) \qquad \text{for} \quad r > R_1 .$$

Thus $w(r)/K(r)$ is monotone decreasing in r and positive. Hence $C(R, \gamma)$ exists and $C(R, \gamma) \geqq 0$.

We now construct a subsolution. Let $u(r) = AK(r)[1 + aK^2(r)]$ for $r \geqq R_1 \geqq R$. Let $A = w(R_1, R, \gamma)\{K(R_1)[1 + aK^2(R_1)]\}^{-1}$. We will determine $a(R, R_1) > 0$ so that u satisfies the subsolution condition $Nu \geqq u$ for $r \geqq R_1$. By CP we will then have $u < v$ for $r > R_1$.

We calculate $u_r = AK_r(1 + 3aK^2)$ and note that $u_r < 0$. We estimate u_r by noting

$$\frac{w(R_1)}{K(R_1)} \leqq \frac{w(R)}{K(R)} < \frac{\sqrt{2}}{K(R)} .$$

Thus

$$-u_r < \sqrt{2}\left[-\frac{K_r(R)}{K(R)}\right]\frac{1 + 3aK^2(R_1)}{1 + aK^2(R_1)}$$
$$< 3\sqrt{2}\left[\frac{(n-1)}{2R} + \sqrt{1 + \left(\frac{n-1}{2R}\right)^2}\right] = b(R) .$$

The last inequality comes from Lemma 4. The subsolution condition becomes upon dividing by u:

$$(62) \qquad 1 + 2a(K^2 + 3K_r^2)(1 + aK^2)^{-1} + \frac{(n-1)}{r}\frac{K_r}{K}\frac{(1 + 3aK^2)}{(1 + aK^2)}u_r^2 \geqq (1 + u_r^2)^{3/2} .$$

We will require the stronger inequality obtained by replacing $(1 + u_r^2)^{3/2}$ with $1 + c(R)u_r^2$, with

$$c(R) = \frac{[b^2(R) + 1]^{3/2} - 1}{b^2(R)} .$$

We require

$$(63) \qquad u_r^2 f < 2ag(1 + aK^2)^{-1}$$

where

$$f = c + \frac{(n-1)}{r}\left(-\frac{K_r}{K}\right)\left(\frac{1 + 3aK^2}{1 + aK^2}\right)$$
$$g = K^2 + 3K_r^2 .$$

Rearranging condition (63);

$$(64) \qquad 2a > A^2K_r^2 fg^{-1}(1 + 3aK^2)^2(1 + aK^2) .$$

As above,

$$A < \frac{\sqrt{2}}{K(R)}[1 + aK^2(R_1)]^{-1} .$$

Thus

$$A^2(1 + 3aK^2)(1 + aK^2) < \frac{6}{K^2(R)} .$$

Also

$$f < \bar{f} = c + \frac{n-1}{r}\left(-\frac{K_r}{K}\right)(3) .$$

Thus

$$\max_{r \geqq R} K_r^2 fg^{-1} \leqq \max_{r \geqq R} K_r^2 \bar{f} g^{-1} = d(R) .$$

Choose $a = 3d(R)/[K^2(R) - 9d(R)K^2(R_1)]$ and condition (64) will be satisfied. We note that $b(R)$, $c(R)$ and $d(R)$ are decreasing in R.

Take R_1 sufficiently large, then $w(r) > u(r)$ for $r > R_1$ and thus

$$C(R, \gamma) = \lim_{r\to\infty} \frac{w(r)}{K(r)} \geqq A > 0 .$$

Part (ii). From Part (i) $C(R, \gamma) < w(r)/K(r)$ or

$$C(R, \gamma)K(r) < w(r, R, \gamma) \qquad \text{for} \quad r > R . \tag{65}$$

Next, note

$$w(r) > \frac{w(R_1)}{K(R_1)} \frac{K(r)[1 + aK^2(r)]}{[1 + aK^2(R_1)]}$$

for $r \geqq R_1 \geqq R$. Divide by $K(r)$ and let $r \to \infty$:

$$C(R, \gamma) \geqq \frac{w(R_1, R, \gamma)}{K(R_1)[1 + aK^2(R_1)]} . \tag{66}$$

Replacing R_1 by r gives

$$w(r, R, \gamma) < C(R, \gamma)K(r)[1 + a(R, r)K^2(r)] . \tag{67}$$

Combining estimates (65) and (67) gives Part (ii).

Part (iii). We find an *upper bound* as follows:

$$C(R, \gamma) < \frac{w(R_1, R, \gamma)}{K(R_1)} < \frac{z(l, \gamma)}{K(R_1)}$$

where $l = R_1 - R$. The second inequality comes from Lemma 14. Choose $l = (1/2)\ln R$, then by the estimates of Chapter I:

$$z(l) = C_1(\gamma)e^{-l}\left[1 + O\left(\frac{1}{R}\right)\right]$$

$$K(R_1) = \sqrt{\pi/2}\,\frac{\exp -(R + l)}{R^{(n-1)/2}}\left[1 + O\left(\frac{\ln R}{R}\right)\right] .$$

Thus

$$\frac{z(l)}{K(R_1)} = \sqrt{2/\pi}\,C_1(\gamma)e^R R^{(n-1)/2}\left[1 + O\left(\frac{\ln R}{R}\right)\right] .$$

We find a *lower bound* by combining inequalities (61) and (66) with $R_1 = R + l$:

$$C(R, \gamma) > \frac{z(l)}{K(R_1)} \frac{\{1 + z^{-2}(l)[w^2(R) - z^2(0) + 2\eta(l, R)]\}^{1/2}}{[1 + a(R, R_1)K^2(R_1)]} .$$

Choose $l = (1/4)\ln R$, then

$$z(l) = C_1(\gamma)e^{-l}\left[1 + O\left(\frac{1}{R^{1/2}}\right)\right] = O\left(\frac{1}{R^{1/4}}\right)$$

$$K^2(R_1)K^{-2}(R) = O\left(\frac{1}{R^{1/2}}\right)$$

$$\eta(l, R) = O\left(\frac{\ln R}{R^{5/4}}\right).$$

By the corollary to Theorem 14: $w^2(R) - z^2(0) = O(1/R)$. Noting that $d(R)$ is decreasing in R we have

$$C(R, \gamma) > \sqrt{2/\pi}\, C_1(\gamma)e^R R^{(n-1)/2}\left[1 + O\left(\frac{1}{R^{1/2}}\right)\right].$$

We can determine the asymptotic behavior of the derivatives of w:

COROLLARY.
(i) $w_r = C(R, \gamma)K_r(r)\{1 + O[K^2(r)]\}$ *as* $r \to \infty$.
(ii) $w_{rr} = C(R, \gamma)K_{rr}(r)\{1 + O[K^2(r)]\}$ *as* $r \to \infty$.

Proof.

Part (i). Integrate the equation $(r^{n-1}\sin\varphi)_r = r^{n-1}w$ from r to ∞:

$$-\sin\varphi(r) = \frac{1}{r^{n-1}}\int_r^\infty s^{n-1}w(s)ds .$$

Estimate the integral with $w(s) \leqq w(r)(K(s)/K(r))$ and

$$w(s) \geqq \frac{w(r)K(s)[1 + aK^2(s)]}{K(r)[1 + aK^2(r)]} > \frac{w(r)K(s)}{K(r)[1 + aK^2(r)]} .$$

Note:

$$\int_r^\infty s^{n-1}K(s)ds = -r^{n-1}K_r(r) .$$

Hence

$$-\frac{K_r}{K(1 + aK^2)}\cdot w \leqq -\sin\varphi \leqq -\frac{K_r}{K}w(r) .$$

Thus $\sin\varphi = C(R, \gamma)K_r[1 + O(K^2)]$ as $r \to \infty$. This implies Part (i).

Part (ii):

$$\begin{aligned}\frac{w_{rr}}{(1+w_r^2)^{3/2}} &= (\sin\varphi)_r = w - (n-1)\frac{\sin\varphi}{r} \\ &= C(\gamma, R)\left[K - \frac{(n-1)}{r}K_r\right][1 + O(K^2)] \\ &= C(\gamma, R)K_{rr}[1 + O(K^2)] \,.\end{aligned}$$

We give some further properties of $C(R, \gamma)$:

THEOREM 17.
(i) *$C(R, \gamma)$ is continuous in (R, γ) for $R > 0$ and $0 \leqq \gamma \leqq \pi/2$.*
(ii) *$C(R, \gamma)$ is strictly increasing in R for $0 \leqq \gamma < \pi/2$; $C(R, \gamma)$ is strictly decreasing in γ.*

Proof.

Part (i). We show continuity at (R_0, γ_0). For $\gamma_0 < \pi/2$ we combine inequalities (65) and (66):

$$\begin{aligned}[1 + a(R)K^2(R_1)]^{-1}\frac{w(R_1, R, \gamma)}{w(R_1, R_0, \gamma_0)} &\leqq \frac{C(R, \gamma)}{C(R_0, \gamma_0)} \\ &\leqq \frac{w(R_1, R, \gamma)}{w(R_1, R_0, \gamma_0)}[1 + a(R_0)K^2(R_1)] \,.\end{aligned}$$

Require $|R - R_0| < R_0/2$, then choose R_1 so that $a(R_0)K^2(R_1) < \varepsilon$ and $a(R)K^2(R_1) < \varepsilon$. Here $a(R_0) = a(R_0, R_1)$ and $a(R) = a(R, R_1)$. By the corollary to Theorem 13 there are $\delta_1 > 0$ and $\delta_2 > 0$ so that

$$\left|\frac{w(R_1, R, \gamma)}{w(R_1, R_0, \gamma_0)} - 1\right| < \varepsilon$$

for $|R - R_0| < \delta_1$ and $|\gamma - \gamma_0| < \delta_2$. Thus

$$\frac{1-\varepsilon}{1+\varepsilon} \leqq \frac{C(R, \gamma)}{C(R_0, \gamma_0)} \leqq (1+\varepsilon)^2 \,.$$

For $\gamma_0 = \pi/2$ and $\gamma < \pi/2$:

$$0 < C(R, \gamma) < \frac{w(R, R, \gamma)}{K(R)} < \frac{\sqrt{2(1-\sin\gamma)}}{K(R)} \,.$$

Require $|R - R_0| < R_0/2$. Clearly there is a $\delta > 0$ such that $0 < C(R, \gamma) < \varepsilon$ for $|\gamma - \pi/2| < \delta$. This shows continuity since $C(R_0, \pi/2) = 0$.

Part (ii). Let $R_1 < R_2$. By Lemma 11 $w(r, R_1, \gamma) < w(r, R_2, \gamma)$ for $r \geqq R_2$. Dividing by $K(r)$ and taking $r \to \infty$ gives $C(R_1, \gamma) \leqq C(R_2, \gamma)$. Likewise if $\gamma_1 < \gamma_2$, CP gives $w(r, R, \gamma_2) < w(r, R, \gamma_1)$ for $r > R$. Dividing by $K(r)$ and taking $r \to \infty$ gives $C(R, \gamma_2) \leqq C(R, \gamma_1)$.

To complete the proof we need only show that if w_1 and w_2 are two solutions of equation (49a) with $w_1 \sim CK(r)$ as $r \to \infty$ and $w_2 \sim CK(r)$ as $r \to \infty$ then $w_1 \equiv w_2$. Let $u = w_1 - w_2$; without loss of generality we can assume $u \geqq 0$, since by CP if $u(R) \geqq 0$ then $u(r) \geqq 0$ for $r \geqq R$. Suppose $w_1 \not\equiv w_2$, then we must have $u(r) > 0$ for $r > R$. Note: $u(r) = o[K(r)]$ as $r \to \infty$. The equation satisfied by u is

$$(r^{n-1}u_r q)_r = r^{n-1}u \tag{68}$$

with

$$q(r) = \int_0^1 \{1 + [w_{2,r} + t(w_{2,r} - w_{1,r})]^2\}^{-3/2} dt \; .$$

By the corollary to Theorem 16:

$$q(r) = 1 + O[K^2(r)] \qquad \text{and} \qquad q_r(r) = O[K^2(r)]$$

as $r \to \infty$. Rewriting equation (68):

$$u_{rr} + u_r\left(\frac{n-1}{r} + \frac{q_r}{q}\right) = \frac{1}{q}u \; . \tag{69}$$

We construct a subsolution for equation (69). Let

$$U = \frac{Ae^{-r}}{r^{(n-1)/2}}\left(1 + \frac{a}{r}\right)$$

then

$$\Delta U = U + \frac{Ae^{-r}}{r^{(n+3)/2}}\left\{\frac{[8a - (n-1)(n-3)]}{4} - \frac{a(n+1)(n-5)}{4r}\right\}$$

$$U_r = -\frac{Ae^{-r}}{r^{(n-1)/2}}\left[1 + \left(a + \frac{n-1}{2}\right)\frac{1}{r} + \frac{n+1}{2}\frac{a}{r^2}\right] .$$

Choose a: $a > (n-1)(n-3)/4$. For R_1 sufficiently large

$$U_{rr} + U_r\left(\frac{n-1}{r} + \frac{q_r}{q}\right) > \frac{1}{q}U \qquad \text{for} \quad r \geqq R_1$$

(independent of A because the equation is linear) and $1/q(r) > 0$ for $r \geqq R_1$. Choose

$$A = (\exp R_1)R_1^{(n-1)/2}\left(1 + \frac{a}{R_1}\right)^{-1} u(R_1) \; .$$

Let $V(r) = u(r) - U(r)$; V satisfies

$$V_{rr} + V_r\left(\frac{n-1}{r} + \frac{q_r}{q}\right) < \frac{1}{q}V \qquad \text{for} \quad r \geqq R_1$$

and $V(R_1) = 0$. By the weak maximum principle [11] $\min_{R_1 \leqq r \leqq R_2} V(r) \geqq \min[0, V(R_2)]$. Letting $R_2 \to \infty$ yields $V(r) \geqq 0$ for $r \geqq R_1$. Thus $u(r) \geqq U(r)$ for $r \geqq R_1$, but

$$u = o[K(r)] = o\left[\frac{e^{-r}}{r^{(n-1)/2}}\right]$$

a contradiction as $r \to \infty$.

Chapter III
Solutions in Unsymmetrical Domains

10. General estimates, exponential decay. General estimates are given that apply to any solution of equation (1a):

THEOREM 18. *Suppose* $u(x) \in C^2(\Omega)$ *satisfies* $Nu = u$ *in* Ω.
(i) *If* $B_R(y) \subset \Omega$ *then* $|u(x)| \leqq v(|x - y|, R, 0)$ *for* $x \in B_R(y)$.
(ii) *If* $B_R^c(y) \subset \Omega$ *then* $|u(x)| \leqq w(|x - y|, R, 0)$ *for* $x \in B_R^c(y)$.
(iii) *If* $H = \{x: \bar{n} \cdot x > b, |\bar{n}| = 1\} \subset \Omega$ *then* $|u(x)| \leqq z(\bar{n} \cdot x - b, 0)$ *for* $x \in H$.

Proof.

Part (i). This follows directly from CP (cf. the corollory to Theorem 1) since $\limsup (Tu - Tv) \cdot \nu \leqq 0$ on $\partial B_R(y)$ implies $u(x) \leqq v(x)$ in $B_R(y)$; for the same reason $-u(x) \leqq v(x)$ in $B_R(y)$.

Part (ii). From Part (i)

$$|u(x)| \leqq v(0, d, 0) \tag{70}$$

where $d = \operatorname{dist}(x, \partial\Omega)$. By Theorem 10 $\lim_{d\to\infty} v(0, d, 0) = 0$; thus $|u(x)| = o(1)$ as $|x| \to \infty$. Also $w(|x - y|) = o(1)$ as $|x| \to \infty$, thus $w(|x-y|) - u(x) = o(1)$ as $|x| \to \infty$. On $\partial B_R^c(y)$, $\limsup (Tu - Tw) \cdot \nu \leqq 0$; hence by CP, $u(x) \leqq w(|x - y|)$ in $B_R^c(y)$. The same reasoning gives $-u(x) \leqq w(|x - y|)$ in $B_R^c(y)$.

Part (iii). For $x \in H$, let $y = x + (R - \bar{n} \cdot x + b)\bar{n}$, then $B_R(y) \subset H$. By Part (i) $|u(x)| \leqq v(|x - y|, R, 0) = v(R - \bar{n} \cdot x + b, R, 0)$. By Theorem 9 $\lim_{R\to\infty} v(R - \bar{n} \cdot x + b, R, 0) = z(\bar{n} \cdot x - b, 0)$. This gives Part (iii).

COROLLARY 1. *Let* $d = \operatorname{dist}(x, \partial\Omega)$, *then*

$$|u(x)| \leqq C \frac{d^{(n-1)/2}}{\exp d} \qquad \textit{for} \quad d \geqq d_0 > 0$$

where C depends on d_0.

Proof. By inequality (70) $|u(x)| \leqq v(0, d, 0)$; by Corollary 1 to Theorem 11 $v(0, d, 0) \sim H(d, 0)I(0)$ as $d \to \infty$, where

$$H(d, 0) = \sqrt{2\pi}C_1(0) \frac{d^{(n-1)/2}}{\exp d} .$$

Combining estimates gives the result.

COROLLARY 2. *If $B_R^c(0) \subset \Omega$ then*

$$|u(x)| \leqq C \frac{\exp(-|x|)}{|x|^{(n-1)/2}} \qquad \text{for} \quad |x| \geqq R$$

where C depends on R.

Proof. By Part (ii) $|u(x)| \leqq w(|x|, R, 0)$; by Theorem 16 $w(|x|, R, 0) \sim C(R, 0)K(|x|)$ as $|x| \to \infty$. By § 4

$$K(|x|) \sim \sqrt{\pi/2} \frac{\exp(-|x|)}{|x|^{(n-1)/2}} \qquad \text{as} \quad |x| \to \infty .$$

Combining estimates gives the result.

COROLLARY 3. *If $H \subset \Omega$ then $|u(x)| \leqq \sqrt{2} \exp(b - \bar{n} \cdot x)$ for $x \in H$.*

Proof. By Part (iii) $|u(x)| \leqq z(\bar{n} \cdot x - b, 0)$ for $x \in H$. By Theorem 5 $z(\bar{n} \cdot x - b, 0) \leqq \sqrt{2} \exp(b - \bar{n} \cdot x)$ for $x \in H$.

Corollaries 1, 2, and 3 give different exponential rates of decay for solutions to equation (1a) away from the boundary of the domain. The estimate of Corollary 3 is best possible because if $\Omega = H$ and $u(x) = z(\bar{n} \cdot x - b, 0)$ then $u = z(0, 0) = \sqrt{2}$ on ∂H. Gerhardt [8] gives an estimate of the same form with $\sqrt{2}$ replaced by $(n + 1)$.

COROLLARY 4. *If Ω is an exterior domain then problem* (1) *has a unique solution.*

Proof. Two solutions u_1 and u_2 to problem (1) must satisfy $u_1(x) = o(1)$ and $u_2(x) = o(1)$ as $|x| \to \infty$; thus $u_1(x) - u_2(x) = o(1)$ as $|x| \to \infty$. CP gives that $u_1 \equiv u_2$.

The derivatives of any solution to equation (1a) can be estimated as follows:

LEMMA 15. *Suppose* $u(x) \in C^2(\Omega)$ *and* $Nu = u$ *in* Ω. *If* $B_{4\delta}(x_0) \subset \Omega$ *then* $|u|_{2,\alpha,B_\delta} < C|u|_{0,B_{4\delta}}$ *where* C *depends only on* δ *and* n.

Proof. See Step 3 of the proof of Theorem 4.

Lemma 15 in conjunction with Corollaries 1, 2, and 3 shows that the derivatives of a solution to equation (1a) decay at the same rate at which the solution decays:

COROLLARY.

$$|\nabla u(x)| < C\frac{d^{(n-1)/2}}{\exp d} \qquad \text{for} \quad d \geqq d_0\,, \quad d = \operatorname{dist}(x, \partial\Omega)$$

where C *depends on* d_0.

Proof. Combining Corollary 1 and Lemma 15 (with $8\delta = d_0$) gives

$$|\nabla u(x)| < \bar{C}\frac{(d + d_0/2)^{(n-1)/2}}{\exp(d - d_0/2)}$$

for $d \geqq d_0$, where $\bar{C}$ depends on d_0. Thus the corollary holds with a different constant C.

REMARK. Similar estimates hold for higher derivatives, and if $B_R^c \subset \Omega$ or $H \subset \Omega$, the analogous estimates are based upon Corollaries 2 and 3. These estimates improve upon the derivative estimate given by Gerhardt [8].

11. Special estimates. We now put restrictions on $\gamma(\sigma)$ and Ω to obtain special estimates on solutions to problem (1). For simplicity we assume that Σ is piecewise smooth.

We first extend CPS of Chapter I.

THEOREM 19. *Suppose* $Nu = u$ *in* Ω *and* $Tu \cdot \nu = \cos\gamma$ *on* Σ^*, *with* $0 \leqq \gamma(\sigma) \leqq \gamma_0 \leqq \pi/2$. *If* $\Omega \subset B_R^c(y)$ *and* $u \geqq o(1)$ *as* $|x| \to \infty$, *then* $w(r, R, \gamma_0) \leqq u(x)$ *where* $r = |x - y|$.

Proof. By the proof of Theorem 2, $Tw \cdot \nu \leqq \cos\gamma_0 \leqq Tu \cdot \nu$ on Σ^*. By hypothesis $w - u \leqq o(1)$ as $|x| \to \infty$. Thus CP gives the conclusion.

COROLLARY. *If* Ω *is an exterior domain,* $0 \in \Omega^c$, *and* u *is a solution to problem* (1) *with* $0 \leqq \gamma(\sigma) \leqq \gamma_0 < \pi/2$, *then*

$$C_1 \frac{e^{-|x|}}{|x|^{(n-1)/2}} < u(x) < C_2 \frac{e^{-|x|}}{|x|^{(n-1)/2}} \qquad \textit{for} \quad x \in \Omega$$

where C_1 *and* C_2 *are positive constants.*

Proof. For some positive R_1 and R_2, $B^c_{R_1}(0) \subset \Omega \subset B^c_{R_2}(0)$. By Theorem 18 and Corollary 2 to Theorem 18

$$u(x) \leqq w(|x|, R_1, 0) \leqq C \frac{\exp(-|x|)}{|x|^{(n-1)/2}}$$

and $u(x) = o(1)$ as $|x| \to \infty$. Thus by Theorem 19: $w(|x|, R_2, \gamma_0) \leqq u(x)$. By Theorem 16: $w(|x|, R_2, \gamma_0) \sim C(R_2, \gamma_0)K(|x|)$. Combining estimates gives the result.

REMARK. In this case we have determined the rate of decay.

We now improve upon the upper bounds given in § 10.

THEOREM 20. *Let* u *be a solution to problem* (1) *with* $\gamma_1 \leqq \gamma(\sigma) \leqq \pi$, $0 < \gamma_1 < \pi/2$. *Suppose* $B_R(y) \subset \Omega$ *and* $B_{\bar{R}}(y) \cap \Omega$ *is piecewise smooth and convex, where* $B_{\bar{R}}(y)$ *is the maximal domain of existence of* $v(r, R, \gamma_1)$, $r = |x - y|$, *then* $u(x) \leqq v(r, R, \gamma_1)$ *in* $B_R(y)$.

Proof. Let $\partial(B_{\bar{R}} \cap \Omega) = \Sigma^0 + \Sigma^1 + \Sigma^2$ where Σ^1 and $\Sigma^2 \in C^1$, $\Sigma^1 \subset \Sigma = \partial\Omega$ and $\Sigma^2 \subset (\partial B_{\bar{R}}) \cap \Omega$.

$$\text{For} \quad x \in \Sigma^2\,, \quad Tv \cdot \nu = 1 > Tu \cdot \nu\,.$$
$$\text{For} \quad x \in \Sigma^1\,, \quad Tv \cdot \nu = (\cos\theta)\sin\psi(r)$$

where θ is the angle between ν and $x - y$. Let π_x be the tangent plane to Σ at x and let $r_1 = \operatorname{dist}(y, \pi_x)$. We have $\cos\theta = r_1/r$ and $r \geqq r_1 \geqq R$. Thus

$$Tv \cdot \nu = r_1 \frac{\sin\psi(r)}{r} \geqq \sin\psi(r_1) \geqq \sin\psi(R) = \cos\gamma_1 \geqq Tu \cdot \nu\,.$$

The first inequality comes from Lemma 6 (ii).

Therefore CP applies, giving $u \leqq v$ in $B_{\bar{R}} \cap \Omega$.

COROLLARY 1. *Let* u *be solution to problem* (1) *with* $0 \leqq \gamma(\sigma) \leqq \pi/2$. *Suppose* $B_R(y) \subset \Omega$ *and* $B_{R_i}(y) \cap \Omega$ *is piecewise smooth and convex, with* $\lim_{i\to\infty} R_i = \infty$, *then* $u(x) \geqq 0$ *in* $B_R(y)$.

Proof. Define γ_i so that $v(r, R_i, \pi) = v(r, R, \gamma_i)$ for $0 \leqq r \leqq R$. Note: $\pi/2 < \gamma_i < \pi$. By Corollary 1 to Theorem 11 $\lim_{i\to\infty} v(r, R_i, \pi) = 0$

for $0 \leqq r \leqq R$. By § 3, $-v(r, R, \gamma_i) = v(r, R, \pi - \gamma_i)$ and $-u$ is a solution to problem (1) with boundary data $\pi - \gamma(\sigma)$. Since $0 < \pi - \gamma_i < \pi/2 \leqq \pi - \gamma(\sigma)$, Theorem 20 gives $-u \leqq -v$. Thus $v(r, R, \gamma_i) \leqq u(x)$ in $B_R(y)$. Letting $i \to \infty$ gives the result.

COROLLARY 2. *Let $H = \{x: x_1 \geqq 0\}$. There is a unique solution to $Nu = u$ in H and $Tu \cdot \nu = \cos\gamma$ on ∂H, γ constant: the one-dimensional solution, $u = z(x_1, \gamma)$.*

Proof. We need only consider $0 \leqq \gamma \leqq \pi/2$. By Corollary 1, $u(x) \geqq 0$ on H. By Theorem 19, if $H \subset B_R^c(y)$ then $w(r, R, \gamma) \leqq u(x)$. For $x \in H$, choose $y = (-R, x_2, \cdots, x_n)$, then $r = R + x_1$. Thus $w(R + x_1, R, \gamma) \leqq u(x)$. Take the limit $R \to \infty$: $z(x_1, \gamma) \leqq u(x)$.

By Theorem 20, if $B_R(y) \subset H$ then $u(x) \leqq v(r, R, \gamma)$. For $x \in H$ choose $y = (R, x_2, \cdots, x_n)$, then $r = R - x_1$. Thus $u(x) \leqq v(R - x_1, R, \gamma)$. Take the limit $R \to \infty$: $u(x) \leqq z(x_1, \gamma)$. Therefore $u(x) = z(x_1, \gamma)$.

REMARK. Uniqueness also holds for $n = 1$, since a solution to problem (1) in H is a solution to problem (1) in H for $n = 2$.

For an interior domain, we define the "interior rolling number":

$$\mathscr{R}_1 = \max\{R: \text{for each } x \in \Sigma \text{ there is a ball } B_R \subset \Omega, \text{ with } x \in \bar{B}_R\}.$$

THEOREM 21. *Let Ω be bounded and $\Sigma \in C^1$. Let $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ be a solution to problem (1), with $\gamma_0 \leqq \gamma(\sigma) < \pi$, $0 < \gamma_0 \leqq \pi/2$. Suppose $\mathscr{R}_1$ exists, then*

(i) $\max u(x) \leqq v(\mathscr{R}_1, \mathscr{R}_1, \gamma_0)$.

(ii) *if $B_R(y) \subset \Omega$ and $R \leqq \mathscr{R}_1$, then $u(x) \leqq v(r, R, \gamma_0)$ in $B_R(y)$, with $r = |x - y|$.*

Proof.

Part (i). If $u \leqq 0$ there is nothing to prove. If $u(x_1) > 0$ for $x_1 \in \Omega$, then by the maximum principle, $\max u(x)$ occurs on Σ, say at x_0. Suppose $u(x_0) > v(\mathscr{R}_1, \mathscr{R}_1, \gamma_0)$. By Theorems 6 and 9, $v(R, R, \gamma_0)$ is continuous and strictly decreasing in R. By Theorem 7, $\lim_{R\to 0} v(R, R, \gamma_0) = \infty$. We can thus choose $R < \mathscr{R}_1$ so that $v(R, R, \gamma_0) = u(x_0)$. There is a ball $B_R \subset \Omega$ with $x_0 \in \bar{B}_R$. By CP: $u(x) < v(r, R, \gamma_0)$ in B_R. Thus $(\partial u/\partial\nu)(x_0) \geqq (\partial v/\partial\nu)(x_0)$. However, $(\partial v/\partial\nu)(x_0) = \cot\gamma_0$ and $(\partial u/\partial\nu)(x_0) = \cot\gamma(x_0)$, since at x_0:

$$\cos\gamma(x_0) = \frac{\nabla u \cdot \nu}{\sqrt{1 + |\nabla u|^2}} = \frac{|\nabla u|}{\sqrt{1 + |\nabla u|^2}}$$

and $(\partial u/\partial \nu)(x_0) = |\nabla u|$. Therefore $(\partial u/\partial \nu)(x_0) = (\partial v/\partial \nu)(x_0)$, contradicting Lemma 1.

Part (ii). This follows immediately from Part (i) and CP.

REMARK. The proof shows that equality holds in Part (i) only if Ω is a ball and $\gamma(\sigma) \equiv \gamma_0$.

For an exterior domain, we define the "exterior rolling number":

$$\mathscr{R}_2 = \min\{R\colon \text{for each } x \in \Sigma \text{ there is a ball } B_R \supset \Omega^c, \text{ with } x \in \bar{B}_R\}.$$

THEOREM 22. *Let Ω be an exterior domain, $\Sigma \in C^1$ and Ω^c convex. Let $u(x) \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ be a solution to problem* (1) *with $\gamma_0 \leqq \gamma(\sigma) < \pi$, $0 < \gamma_0 \leqq \pi/2$. Then*

(i) $\max u(x) < z(0, \gamma_0)$.

(ii) *if $\mathscr{R}_2$ exists, then* $\max u(x) \leqq w(\mathscr{R}_2, \mathscr{R}_2, \gamma_0)$.

(iii) *if $\mathscr{R}_2$ exists and $B_R^c(y) \subset \Omega$, $R \geqq \mathscr{R}_2$, then $u(x) \leqq w(r, R, \gamma_0)$ in $B_R^c(y)$, with $r = |x - y|$.*

Proof.

Part (i). If $u \leqq 0$ there is nothing to prove. If $u(x_1) \geqq 0$ for $x_1 \in \Omega$, then by the maximum principle and the fact that $u(x) = o(1)$ as $|x| \to \infty$: $\max u(x)$ occurs on Σ, say at x_0. Suppose $u(x_0) \geqq z(0, \gamma_0) = \sqrt{2(1 - \sin \gamma_0)}$. Choose $\gamma_1 \leqq \gamma_0$ so that $u(x_0) = z(0, \gamma_1)$. For convenience suppose $x_0 = 0$, that the tangent plane to Σ at x_0 is given by $\{x\colon x_1 = 0\}$, and that $H = \{x\colon x_1 > 0\} \subset \Omega$. Since $u - z \leqq o(1)$ as $|x| \to \infty$, by CP: $u(x) < z(x_1, \gamma_1)$ in H. Hence $(\partial u/\partial \nu)(0) \geqq (\partial z/\partial \nu)(0)$. However, $(\partial z/\partial \nu)(0) = \cot \gamma_1$ and $(\partial u/\partial \nu)(0) = \cot \gamma(0)$; thus $(\partial u/\partial \nu)(0) = (\partial v/\partial \nu)(0)$, contradicting Lemma 1.

Part (ii). Suppose $u(x_0) = \max u(x) > w(\mathscr{R}_2, \mathscr{R}_2, \gamma_0)$, where $x_0 \in \Sigma$. By Theorems 13 and 15: $w(R, R, \gamma_0)$ is continuous and strictly increasing in R, with $\lim_{R\to\infty} w(R, R, \gamma_0) = z(0, \gamma_0)$. We can choose $R > \mathscr{R}_2$ so that $w(R, R, \gamma_0) = u(x_0)$, since by Part (i) $u(x_0) < z(0, \gamma_0)$. There is a ball B_R, with $\Omega^c \subset B_R$ and $x_0 \in \bar{B}_R$. By CP, $u(x) < w(r, R, \gamma_0)$ in B_R^c, thus $(\partial u/\partial \nu)(x_0) \geqq (\partial v/\partial \nu)(x_0)$. However, $(\partial w/\partial \nu)(x_0) = \cot \gamma_0$ and $(\partial u/\partial \nu)(x_0) = \cot \gamma(x_0)$. Therefore $(\partial w/\partial \nu)(x_0) = (\partial u/\partial \nu)(x_0)$, contradicting Lemma 1.

Part (iii). This follows immediately from CP and Part (ii).

REMARKS. (1) The proof shows that equality can hold in Part (ii) only if Ω^c is a ball and $\gamma(\sigma) \equiv \gamma_0$.

(2) If $\Sigma \in C^2$ and Ω^c is strictly convex, then $\mathscr{R}_2$ will exist, $\mathscr{R}_2 = (\bar{k})^{-1}$ where $\bar{k} = \min k(\sigma)$ over Σ and $k =$ the minimum principal curvature of Σ at σ (see [2]).

12. Two examples. We give two examples that show the appropriateness of the convexity condition of Theorems 20 and 22 and the "rolling number" condition of Theorem 21.

EXAMPLE 1. We construct a domain $\Omega = B_R(y) \cup \mathscr{K}$ where $\mathscr{K}$ is part of a cone $\mathscr{K} = \{x: x_1 > |x| \cos \alpha,\ x_1 < \bar{x}_1\}$, $0 < \alpha < \pi/2$, and $y = (R + \varepsilon, 0, \cdots, 0)$; here $\varepsilon > 0$ and

$$\bar{x}_1 = \varepsilon \cdot \frac{2R + \varepsilon}{R + \varepsilon} \left\{1 + \left[1 - \varepsilon \sec^2 \alpha \frac{2R + \varepsilon}{(R + \varepsilon)^2}\right]^{1/2}\right\}^{-1} \tag{71}$$

where $\bar{x}_1$ is the x_1 coordinate of the first intersection of $\mathscr{K}$ with B_R (see Figure 1).

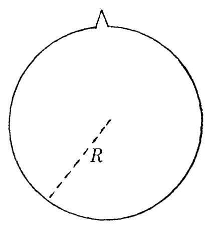

FIGURE 1

Let $v = v(|x - y|, R, \gamma)$ and let $u(x)$ be the solution to problem (1) in Ω, where $\gamma = \pi/2 - \alpha$. For ε sufficiently small we will show that $u \not< v$ in B_R.

Let w_n be the volume of the unit ball in n-dimensions.

We modify an argument of Finn [5]. By integration by parts:

$$\int_\Omega u dx = \int_{\partial\Omega} Tu \cdot \nu d\sigma = (\cos \gamma) |\partial \Omega|$$

$$\int_{B_R} v dx = (\cos \gamma) |\partial B_R| .$$

Thus

$$\int_{B_R} (u - v) dx = \cos \gamma (|\partial \Omega| - |\partial B_R|) - \int_{\Omega - B_R} u dx .$$

Now $|\partial\Omega| - |\partial B_R| = |\Sigma^1| - |\Sigma^2|$ where $\Sigma^1 \subset \partial\mathscr{K}$ and $\Sigma^2 \subset \partial B_R$. We calculate:

$$|\Sigma^1| = w_{n-1} (\tan^{n-2} \alpha) \sec \alpha \bar{x}_1^{n-1}$$

$$|\Sigma^2| = (n - 1) w_{n-1} \int_0^a r_1^{n-2} ds \leqq w_{n-1} a^{n-1}$$

[where r_1 is the distance from the x_1 axis along a profile curve and s is arc length, thus $r_1(s) \leqq s$]

$$a = R\sin^{-1}\left(\frac{\bar{x}_1\tan\alpha}{R}\right) = \bar{x}_1\tan\alpha + O(\bar{x}_1^3)$$

$$|\Omega - B_R| \leqq |\mathscr{K}| = w_{n-1}\frac{\tan^{n-1}\alpha}{n}x_1^n .$$

Let $0 < R_1 < R$ and $y_1 = (R_1, 0, \cdots, 0)$.

Claim. For ε sufficiently small $\Omega - B_R \subset B_{R_1}(y_1)$ and $u(x) < h(x)$ in $B_{R_1} \cap \Omega$, where

$$h(x) = \frac{n}{R_1} + R_1 + \sqrt{R_1^2 - |x - y_1|^2} .$$

Assuming the claim:

$$\begin{aligned}\int_{B_R}(u - v)dx \geqq \cos\gamma w_{n-1}\tan^{n-2}\alpha(\sec\alpha - \tan\alpha)\bar{x}_1^{n-1}\\ - \left(\frac{n}{R_1} + R_1\right)\frac{w_{n-1}}{n}\tan^{n-1}\alpha\bar{x}_1^n\\ + O(\bar{x}_1^{n+2}) .\end{aligned}$$

For $\bar{x}_1$ sufficiently small

$$\int_{B_R}(u - v)dx > 0$$

by equation (71) this is true if ε is sufficiently small. Therefore $u \nless v$ in B_R.

Proof of claim. Since $Nh \leqq h$ (cf. Corollary 1 to Theorem 1) we need show only that $Th\cdot\nu \geqq \cos\gamma$ on $[\partial(\Omega \cap B_{R_1})]^* = \Sigma^1 + \Sigma^3 + \Sigma^4$, where $\Sigma^3 \subset \partial B_{R_1}$ and $\Sigma^4 \subset \partial B_R$. On Σ^1, $Th\cdot\nu = \cos\gamma$ and on Σ^3, $Th\cdot\nu = 1$. For $x \in \Sigma^4$ we must show

$$\cos\theta \geqq \frac{R_1}{r}\cos\gamma \tag{72}$$

where $r = |x - y_1|$ and θ is the angle between $x - y_1$ and ν.

By changing the x_1 coordinate, $x \to R_1 - x_1$, $-(R_1, 0, \cdots, 0)$, we calculate

$$\cos\theta = \frac{1}{R}\left[r + \frac{x_1}{r}(R - R_1 + \varepsilon)\right] . \tag{73}$$

We check that $\cos\theta$ is smallest for $x_1 = \hat{x}_1$ determined by $\partial B_R \cap \partial B_{R_1}$:

$$\hat{x}_1 = \frac{(R - R_1)(R_1 - \varepsilon) - \varepsilon^2/2}{R - R_1 + \varepsilon} .$$

Putting this value into equation (73) gives

$$\cos\theta \geqq 1 - \varepsilon\frac{(R - R_1)}{RR_1} - \frac{\varepsilon^2}{2RR_1}\,. \tag{74}$$

For ε sufficiently small, so that

$$1 - \varepsilon\frac{(R - R_1)}{RR_1} - \frac{\varepsilon^2}{2RR_1} > \frac{R_1\cos\gamma}{R_1 - \varepsilon}$$

then condition (72) will be satisfied.

REMARKS. (1) The domain Ω is not convex.

(2) The example can be modified so that Σ is smooth $B_R \subset \Omega$, $R > \mathscr{R}_1$, and $u(x) \not< v(r, R, \gamma)$ in B_R.

EXAMPLE 2. We construct a domain $\Omega = B_R^c(y) \cup \mathscr{K}$ (see Figure 2) by the method of Example 1 so that $u(x) \not< w(r, R, \gamma)$ in $B_R^c(y)$.

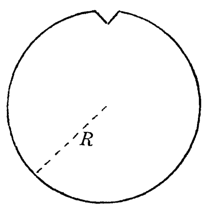

FIGURE 2

REMARKS. (1) Ω^c is not convex.

(2) The example can be modified so that Σ is smooth, Ω^c is not convex, $B_R^c \subset \Omega$, and $u(x) \not< w(r, R, \gamma)$ in B_R^c.

13. The infinite wedge. We consider solutions to problem (1) in

$$\mathscr{K}_\alpha = \{(x_1, x_2)\colon x_1 > |x_1|\cos\alpha\}$$

with $0 < \alpha < \pi$, and γ constant, $0 < \gamma \leqq \pi/2$.

THEOREM 23. *For $0 < \gamma < \pi/2$ problem* (1) *has a unique positive solution in $\mathscr{K}_\alpha$. Furthermore, for $\alpha < \pi/2$: on Σ^*, $u(x) \geqq z(0, \gamma)$ and $\lim_{|x|\to\infty} u(x) = z(0, \gamma)$. For $\alpha > \pi/2$: on Σ^*, $u(x) \leqq z(0, \gamma)$ and $\lim_{|x|\to\infty} u(x) = z(0, \gamma)$. Here $\Sigma^* = \partial\mathscr{K}_\alpha - \{(0, 0)\}$.*

Proof. Existence comes from Theorem 4. Let $d = \operatorname{dist}(x, \Sigma)$.

Case $\alpha < \pi/2$. Any solution $u(x)$ to problem (1) must be nonnegative since $\mathscr{K}_\alpha$ is convex, by Corollary 1 to Theorem 20. By

Theorem 19, if $\Omega \subset B_R^c$ then $w(r, R, \gamma) < u(x)$ in B_R^c. By a limit argument (cf. the proof of Corollary 2 to Theorem 20) we obtain

$$z(d, \gamma) \leqq u(x) \quad \text{for} \quad x \in \mathscr{K}_\alpha . \tag{75}$$

If $B_R \subset \mathscr{K}_\alpha$, then $u(x) < v(r, R, \gamma)$ in B_R, by Theorem 20. Given $x \in \mathscr{K}_\alpha$ we choose $B_R \subset \mathscr{K}_\alpha$ with center on the x_1 axis so that the radius through x is perpendicular to Σ. This determines $R = x_1 \tan\alpha \sec\alpha - d \tan^2\alpha$. Thus,

$$u(x) \leqq v(R - d, R, \gamma) . \tag{76}$$

Suppose there are two solutions $u_1(x)$ and $u_2(x)$ to problem (1). We show that $u_1 - u_2 = o(1)$ as $|x| \to \infty$.

Given $\varepsilon > 0$, choose d_1 so that $v(0, d_1, 0) < \varepsilon/2$. For $d > d_1$: $|u_1(x) - u_2(x)| < |u_1(x)| + |u_2(x)| < \varepsilon$. For $x_1 > \bar{x}_1$ and $d \leqq d_1$, with $\bar{x}_1$ sufficiently large: $0 < v(R - d, R, \gamma) - z(d, \gamma) < \varepsilon$, by Theorem 9. Thus, $|u_1(x) - u_2(x)| < \varepsilon$ for $x_1 > \bar{x}_1$, by estimates (75) and (76). Therefore $u_1 - u_2 = o(1)$ as $|x| \to \infty$. By CP, $u_1 \equiv u_2$.

Taking $d = 0$ in estimates (75) and (76) gives the limit statement and estimate on Σ^*.

Case $\alpha > \pi/2$. We first show that for any solution $u(x)$ to problem (1): $u \geqq o(1)$ as $|x| \to \infty$.

Given $\varepsilon > 0$, choose d_1 as above. For $d > d_1$: $u(x) > -\varepsilon/2$. Choose $\gamma_1 > \pi/2$ so that $v(r, d_1, \gamma_1) > -\varepsilon$ for $0 \leqq r \leqq d_1$. For $d \leqq d_1$ and $x_1 < \bar{x}_1$, $\bar{x}_1$ sufficiently small so that $x \in B_{d_1}(y) \subset \mathscr{K}_\alpha$, with dist$(y, \Sigma) = d_1$, and $B_{\bar{d}_1}(y) \cap \mathscr{K}_\alpha$ is convex, we have: $u(x) \geqq v(r, d_1, \gamma) > -\varepsilon$ (cf. the proof of Corollary 1 to Theorem 20). Therefore $u \geqq o(1)$ as $|x| \to \infty$.

Now, if $\mathscr{K}_\alpha \subset B_R^c$ then $w(r, R, \gamma) \leqq u(x)$ in B_R^c, by Theorem 19. This shows $u(x) > 0$ in $\mathscr{K}_\alpha$. Furthermore, given $x \in \mathscr{K}_\alpha$ with $x_1 < 0$, we choose $B_{R_x}(y) \subset \mathscr{K}_\alpha^c$, with center on the x_1 axis so that segment joining x and y is perpendicular to Σ. This determines $R_x = -x_1 \sec\alpha \tan\alpha + d \tan^2\alpha$. We have

$$w(R_x + d, R_x, \gamma) \leqq u(x) . \tag{77}$$

Given $\varepsilon > 0$, choose d_1 as above. Thus,

$$0 < u(x) < \varepsilon/2 \quad \text{for} \quad d > d_1 . \tag{78}$$

Choose $\bar{x}_1$ so that $|w(R_x + d, R_x, \gamma) - z(d, \gamma)| < \varepsilon/2$ for $x_1 < \bar{x}_1$ and $0 \leqq d \leqq d_1$. Choose R so that $|v(R - d, R, \gamma) - z(d, \gamma)| < \varepsilon/2$ for $d \leqq d_1$. For $x \in \mathscr{K}_\alpha$ with $d \leqq d_1$, there is a ball $B_R(y) \subset \mathscr{K}_\alpha$, dist$(y, \Sigma) = R$, the radius through x is perpendicular to Σ, and $B_{\bar{R}}(y) \cap \mathscr{K}_\alpha$ convex, if $x_1 < \hat{x}_1$, $\hat{x}_1$ sufficiently small. Thus

$$u(x) \leqq v(R - d, R, \gamma) \quad \text{for} \quad d \leqq d_1 \quad \text{and} \quad x_1 < \hat{x}_1 . \tag{79}$$

Suppose there are two solutions $u_1(x)$ and $u_2(x)$ to problem (1). They must both satisfy estimates (77), (78), and (79). Thus $|u_1(x) - u_2(x)| < \varepsilon$ for $x_1 < \min(\bar{x}_1, \hat{x}_1)$. This implies $u_1 - u_2 = o(1)$ as $|x| \to \infty$; by CP, $u_1 \equiv u_2$.

The limit statements come from estimates (77) and (79) with $d = 0$.

We now obtain the estimate on Σ^*. Let $\Omega = \mathscr{K}_\alpha \cap \{x: x_2 < 0\}$. Let $B_R(y) \subset \mathscr{K}_\alpha$ be tangent to Σ^* at a point $x_0 \in \partial\Omega$. Let $v = (r, R, \gamma)$ and $[\partial B_{\bar{R}}(y)]^* = \Sigma^1 + \Sigma^2 + \Sigma^3$, where $\Sigma^1 \subset \Sigma^*$, $\Sigma^2 \subset \{x: x_1 > 0, x_2 = 0\}$ and $\Sigma^3 \subset \partial B_{\bar{R}}(y)$. On Σ^1, $Tv \cdot \nu \geqq \cos\gamma$, by the proof of Theorem 20. On Σ^2, $Tv \cdot \nu > 0$. On Σ^3, $Tv \cdot \nu = 1$. Because of uniqueness, $u(x_1, x_2) = u(x_1, -x_2)$. Thus $\partial u/\partial x_2 = 0$ on Σ^2, hence $Tu \cdot \nu = 0$ on Σ^2. Therefore $Tv \cdot \nu \geqq Tu \cdot v$ on $[\partial B_{\bar{R}}(y)]^*$; by CP, $u(x) < v(r, R, \gamma)$ in $B_R(y)$. Taking $R \to \infty$, with $B_R(y)$ tangent at x_0, gives:

$$\begin{aligned} u(x) &< z(\bar{n}_1 \cdot x, \gamma) \quad \text{in} \quad H_1 = \{x: \bar{n}_1 \cdot x > 0\} \\ u(x) &< z(\bar{n}_2 \cdot x, \gamma) \quad \text{in} \quad H_2 = \{x: \bar{n}_2 \cdot x > 0\} \end{aligned} \tag{80}$$

where $\bar{n}_1 = (\sin\alpha, -\cos\alpha)$ and $\bar{n}_2 = (\sin\alpha, \cos\alpha)$. Thus on Σ^*, $u(x) \leqq z(0, \gamma)$.

We now study the behavior at the vertex.

THEOREM 24. *For $\alpha + \gamma \geqq \pi/2$ and $0 < \alpha < \pi/2$, the solution to problem* (1) *in $\mathscr{K}_\alpha$ satisfies*

(i) $$\liminf_{x \to 0} u(x) \geqq \sqrt{2}(1 - \sqrt{1 - k^2})^{1/2}$$

(ii) $$\limsup_{x \to 0} u(x) \leqq 2\sqrt{2}(1 - \sqrt{1 - k^2})^{1/2}$$

where $k = \cos\gamma/\sin\alpha$.

Proof.

Part (i). As a comparison surface choose $z(x_1, \hat{\gamma})$, with $\hat{\gamma} = \cos^{-1} k$. On $(\partial\mathscr{K}_\alpha)^*$, $Tz \cdot \nu = -\sin\psi_1(x_1)\sin\alpha \leqq \cos\hat{\gamma}\sin\alpha = \cos\gamma = Tu \cdot \nu$. Since $u > 0$, then $z - u \leqq o(1)$ as $|x| \to \infty$, in $\mathscr{K}_\alpha$. Hence by CP, $z < u$ in $\mathscr{K}_\alpha$. Therefore $\liminf_{x\to 0} u \geqq z(0, \hat{\gamma})$. This gives the result.

Part (ii). As a comparison surface choose

$$h(x) = \frac{2}{R} + R - \sqrt{\mathrm{R}^2 - |x - y|^2}$$

where $y = (kR, 0)$. We have $Nh \leqq h$ in $B_R(y)$. We check: $Th \cdot \nu = \cos\gamma$ on $(\partial\mathscr{K}_\alpha)^*$ and $Th \cdot \nu = 1$ on ∂B_R. Thus, $Th \cdot \nu \geqq Tu \cdot \nu$ on $[\partial(B_R \cap \mathscr{K}_\alpha)]^*$; by CP, $u(x) < h(x)$ in $B_R \cap \mathscr{K}_\alpha$. Therefore,

$$\limsup_{x\to 0} u(x) \leqq h(0) = \frac{2}{R} + R(1 - \sqrt{1-k^2}) .$$

Minimize this bound by choosing

$$R = \sqrt{2}(1 - \sqrt{1-k^2})^{-1/2} .$$

This gives the result.

REMARKS. In n-dimensions, with $\mathscr{K}_\alpha = \{x: x_1 > |x| \cos \alpha\}$, $0 < \alpha < \pi/2$, Theorems 23 and 24 still hold with $2\sqrt{2}$ replaced by $2\sqrt{n}$ in Part (ii) of Theorem 24. We note

$$\sqrt{2}(1 - \sqrt{1-k^2})^{1/2} > z(0, \gamma) .$$

This shows that there is a "rise" at the vertex. For the case $\alpha + \gamma < \pi/2$, Concus and Finn [3] show that the solution to problem (1) is unbounded at the vertex.

For

$$\mathscr{K}_\alpha = \{x: x_1 > \sqrt{x_1^2 + x_2^2} \cos \alpha\} , \qquad 0 < \alpha < \pi .$$

Theorems 23 and 24 still hold, with the same modification.

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