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ON THE LATTICE OF ALL CLOSED SUBSPACES OF A HERMITIAN SPACE

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The purpose of the paper is to prove the following

THEOREM: Let E be a vector space over a field K with char $K \neq 2$, and let ϕ be a nondegenerate hermitian form on E. Then the lattice of all orthogonally closed subspaces of (E, ϕ) is modular if and only if E is finite dimensional.

Introduction. It is well known that the lattice of all orthogonally (=topologically) closed subspaces of a Hilbert space H is modular only if H has finite dimension (see Birkhoff—Von Neumann [1]). We shall prove here that this is true generally for vector spaces E over commutative fields K with char $K \neq 2$, supplied with nondegenerate hermitian forms ϕ: The lattice of all orthogonally closed subspaces of (E, ϕ) is modular if and only if E is finite dimensional. Nonmodularity in the infinite dimensional case is due to the fact that then there are always two closed subspaces with nonclosed sum. In a Hilbert space one can exhibit such pairs of subspaces in a constructive way (see [3]); our general case is much more involved, and their existence will follow from an indirect proof.

1. Denotations. Let E be a (left-) vector space over a commutative field K, and $\phi: E \times E \to K$ a hermitian form with respect to an automorphism $\alpha \mapsto \bar{\alpha}$ of period 2 of K. We always assume that char $K \neq 2$. We usually write (x, y) instead of $\phi(x, y)$, and we write $x \perp y$ if $(x, y) = 0$, $x, y \in E$. Let F be a subspace of (E, ϕ). The orthogonal space of F is $F^{\perp} = \{x \in E: x \perp y \text{ for all } y \in F\}$, and the radical of F is $\operatorname{rad} F = F \cap F^{\perp}$. F is called semisimple if $\operatorname{rad} F = 0$. In particular, E is semisimple if $E^{\perp} = 0$, i.e., if ϕ is nondegenerate. A subspace F is called orthogonally closed if $F = F^{\perp\perp}(=(F^{\perp})^{\perp})$. All bases of vector spaces are algebraic. F is termed euclidean if it is semisimple and admits an orthogonal basis. Semisimple subspaces of countable dimension are always euclidean (see [2]). Every $x \in E$ induces a linear form ϕ_x on F, given by $\phi_x(z) = \phi(z, x)$, $z \in F$. We let F^* denote the antispace of the dual space of F, i.e., the K-space of all linear forms $f: F \to K$, where $(f + g)(z) = f(z) + g(z)$ and $(\alpha f)(z) = \bar{\alpha} \cdot f(z)$, $f, g \in F^*$, $\alpha \in K$. If $F^{\perp} = 0$ then E can be considered as a subspace of F^*, identifying $x \in E$ with ϕ_x.

If $E = \bigoplus_{i \in I} E_i$, and $E_i \perp E_j$ for $i \neq j$, we write $E = \bigoplus^{\perp}_{i \in I} E_i$.

2. The lattice $\mathscr{L}(E, \phi)$. Let (E, ϕ) be a semisimple hermitian space over K. The orthogonally closed subspaces of (E, ϕ) form a lattice $\mathscr{L} = \mathscr{L}(E, \phi)$ under the operations $F \wedge G = F \cap G$ and $F \vee G = (F + G)^{\perp\perp}$. This lattice is modular iff for all $F, G \in \mathscr{L}$ we have $F \vee G = F + G$ (see [4], Theorem 33.4). Thus modularity of $\mathscr{L}(E, \phi)$ is equivalent to the following property of (E, ϕ):

(A) The sum of two orthogonally closed subspaces is always closed .

If $\dim E < \infty$ then (A) holds trivially. We now prove the converse.

3. Nonmodularity of $\mathscr{L}(E, \phi)$ in case of infinite dimension. We start with two technical lemmas. Their importance for our problem will become evident later (cf. the proof of Lemma 3 below).

LEMMA 1. *Let (E, ϕ) be semisimple. Let F be a subspace with $\dim F = \aleph_0$ such that for all subspaces $U, V \subset F$ we have: If $U + V = F$ then $U^{\perp\perp} + V^{\perp\perp} = E$. Then $F = E$.*

Proof. Taking $U = V = F$ we get $F^{\perp\perp} = E$ and $F^{\perp} = 0$. Therefore E may be considered as a subspace of F^*. Let $F = \bigoplus_{s \in S}^{\perp} F_s$ be an orthogonal decomposition of F into finite dimensional subspaces and let $y \in F^*$. y is determined by the restrictions $y|_{F_s}$. Every F_s is semisimple since $F^{\perp} = 0$, thus $y|_{F_s}$ is induced by a unique $y_s \in F_s$. This allows us to represent y as a formal sum $y = \sum_{s \in S} y_s$, and we call the y_s's the components of y with respect to the decomposition $F = \bigoplus_S^{\perp} F_s$. In particular every $x \in E$ has the form $x = \sum_s x_s$.

Now suppose that $E \neq F$.

(1) We first show that then $E = F^*$. Let $x \in E$ with $x \notin F$. One readily constructs a decomposition $F = \bigoplus_S^{\perp} F_s$ such that $\dim F_s = 2$ and $x_s \neq 0$ for all $s \in S$ (choose an orthogonal basis $\{e_i : i \in I\}$ of F and observe that $\operatorname{card}\{i \in I : (e_i, x) \neq 0\} = \aleph_0 = \operatorname{card} I$). Now let $y \in F^*$. We write $y = \sum_s y_s$, where $y_s \in F_s$, and suppose first that $\{x_s, y_s\}$ is linearly independent for all s. Let U and V be the subspaces spanned by $\{y_s : s \in S\}$ and $\{x_s - y_s : s \in S\}$ respectively. We have $U + V = F$, thus $U^{\perp\perp} + V^{\perp\perp} = E$. Write $x = u + v$, where $u = \sum_s u_s \in U^{\perp\perp}$ and $v = \sum_s v_s \in V^{\perp\perp}(u_s, v_s \in F_s)$. Pick $a_s \in F_s$ with $a_s \neq 0$ and $a_s \perp y_s$. Then $a_s \in U^{\perp}$, hence $0 = (a_s, u) = (a_s, u_s)$. Since $\dim F_s = 2$ it follows that $u_s = \lambda_s y_s$ for some $\lambda_s \in K$. In the same way we get $v_s = \mu_s(x_s - y_s)$, $\mu_s \in K$. Since $u_s + v_s = x_s$ we have $\lambda_s = \mu_s = 1$. Thus $y_s = u_s$ for all s, hence $y = u$ and in particular $y \in E$ in this case. Next we consider $y = \sum_s y_s$ in F^* with $y_s \neq 0$ for all s. For every s choose $z_s \in F_s$ such that $\{x_s, z_s\}$ and $\{z_s, y_s\}$ are both linearly independent. Applying the above reasoning to x and $z = \sum_s z_s \in F^*$ as well as to z and y we get first $z \in E$ and

then $y \in E$. The y's in F^* with all components $\neq 0$ generate F^*. Since all these y's are in E we have $E = F^*$.

(2) Suppose $F = \bigoplus_S^{\perp} F_s$, where $\dim F_s < \aleph_0$ for all s. Let $x = \sum_s x_s$, $y = \sum_s y_s$ be in E, $x_s, y_s \in F_s$. We claim: if $(x_s, y_s) = 0$ for all s, then $(x, y) = 0$. To prove this let U and W be the subspaces generated by $\{x_s : s \in S\}$ and $\{y_s : s \in S\}$ respectively. We have $U \perp W$, hence $U^{\perp\perp} \perp W^{\perp\perp}$. Therefore it is enough to show that $x \in U^{\perp\perp}$ and $y \in W^{\perp\perp}$. Choose linear complements V_s of (x_s) in F_s, $F_s = V_s \oplus (x_s)$, and put $V = \bigoplus_S^{\perp} V_s$. Then $U + V = F$, hence $U^{\perp\perp} + V^{\perp\perp} = E$. Write $x = u + v$, where $u = \sum_s u_s \in U^{\perp\perp}$, $v = \sum_s v_s \in V^{\perp\perp}$. For every $z_s \in F_s$ with $z_s \perp x_s$ we have $z_s \in U^{\perp}$ and so $0 = (z_s, u) = (z_s, u_s)$. This gives $u_s \in (x_s)^{\perp\perp} = (x_s)$. In the same way we get $v_s \in V_s^{\perp\perp} = V_s$. Since $u_s + v_s = x_s$ it follows that $v_s = 0$ and $u_s = x_s$. Thus $x = u \in U^{\perp\perp}$. In the same way we see that $y \in W^{\perp\perp}$.

(3) Let $\{e_i : i \in I\}$ be an orthogonal basis of F. According to $F = \bigoplus_I^{\perp}(e_i)$ every $x \in E = F^*$ can be written in the form $x = \sum_i \xi_i \cdot e_i$ with $\xi_i = \overline{(e_i, x)}(e_i, e_i)^{-1}$. For $T \subset I$ we put $x_T = \sum_i \xi_i' e_i$, where $\xi_i' = \xi_i$ for $i \in T$ and $\xi_i' = 0$ for $i \notin T$. We consider $a = \sum_i \alpha_i e_i$ and $b = \sum_i \beta_i e_i$, where $\alpha_i = (e_i, e_i)^{-1}$ and $\beta_i = 1$ for all i. $a, b \in E$ by (1). Let $I = S \cup T$ be a partitioning with $\operatorname{card} S = \operatorname{card} T$. We show that

$$(a_S, b_S) = (a_T, b_T) .$$

We observe that $(a_S, b_T) = (a_T, b_S) = 0$ by (2). Thus it suffices to show that $a = a_S + a_T$ and $c = b_S - b_T$ are orthogonal. Let $\sigma: S \to T$ be a bijection. For $s \in S$ put $F_s = K(e_s, e_{\sigma s})$. Then $F = \bigoplus_S^{\perp} F_s$. The corresponding components of a and c are $a_s = (e_s, e_s)^{-1} \cdot e_s + (e_{\sigma s}, e_{\sigma s})^{-1} \cdot e_{\sigma s}$ and $c_s = e_s - e_{\sigma s}$. We find $(a_s, c_s) = 0$, and by (2) this implies $(a, c) = 0$, as claimed.

We now choose $t \in T$ and put $S' = S \cup \{t\}$ and $T' = T - \{t\}$. We have $\operatorname{card} S' = \operatorname{card} T'$, hence $(a_{S'}, b_{S'}) = (a_{T'}, b_{T'})$. On the other hand, from the relations $a_{S'} = a_S + (e_t, e_t)^{-1} \cdot e_t$, $a_{T'} = a_T - (e_t, e_t)^{-1} \cdot e_t$ and $b_{S'} = b_S + e_t$, $b_{T'} = b_T - e_t$ we get

$$(a_{S'}, b_{S'}) = (a_S, b_S) + 1 , \quad (a_{T'}, b_{T'}) = (a_T, b_T) - 1 .$$

It follows that $+1 = -1$, a contradiction since $\operatorname{char} K \neq 2$. This completes the proof.

We can easily generalize the statement of Lemma 1.

LEMMA 2. *Let (E, ϕ) be semisimple. Let F be a euclidean subspace such that whenever $U + V = F$ it follows that $U^{\perp\perp} + V^{\perp\perp} = E$. Then $F = E$.*

Proof. Since $F^{\perp\perp} = E$ we may suppose that $\dim F \geqq \aleph_0$. Let

$\{e_i: i \in I\}$ be an orthogonal basis of F. Suppose that there exists a $x \in E$ with $x \notin F$. Then there exists a subset $L \subset I$ with card $L = \aleph_0$ and such that $(e_i, x) \neq 0$ for all $i \in L$. Put $Q = K(e_i)_{i \in L}$ and $R = K(e_i)_{i \in I-L}$; then $F = Q \oplus^{\perp} R$ and so $E = Q^{\perp\perp} \oplus^{\perp} R^{\perp\perp}$. Write $x = q + r$ where $q \in Q^{\perp\perp}$, $r \in R^{\perp\perp}$. One easily verifies that the hypotheses of Lemma 1 are satisfied for $(Q^{\perp\perp}, \phi|_{Q^{\perp\perp}})$ and Q (in lieu of (E, ϕ) and F). Hence $Q = Q^{\perp\perp}$ and in particular $q \in Q$. But this is a contradiction since $(e_i, q) = (e_i, x) \neq 0$ for all $i \in L$.

We now pass to study spaces (E, ϕ) with property (A).

LEMMA 3. *Suppose that the semisimple space (E, ϕ) has property* (A). *Then for every euclidean subspace F we have $F^{\perp\perp} = F \oplus^{\perp} \operatorname{rad} F^{\perp}$.*

Proof. We have $\operatorname{rad} F^{\perp\perp} = \operatorname{rad} F^{\perp}$, and $F \cap F^{\perp} = 0$. Hence there is a decomposition $F^{\perp\perp} = Q \oplus^{\perp} \operatorname{rad} F^{\perp}$ with $F \subset Q$. The space Q with the induced form $\Psi = \phi|_Q$ (restriction) is semisimple. We shall show that the hypotheses of Lemma 2 are satisfied for (Q, Ψ) and F (in place of (E, ϕ) and F); then it will follow that $F = Q$, proving our lemma. For $U \subset Q$ we let U° denote the orthogonal space of U formed in (Q, Ψ). Thus $U^{\circ} = \{x \in Q: x \perp y \text{ for all } y \in U\} = U^{\perp} \cap Q$. Now let U, V be subspaces of F with $U + V = F$; we must show that $U^{\circ\circ} + V^{\circ\circ} = Q$. It is immediate that $U^{\circ\circ} \oplus \operatorname{rad} F^{\perp} \supset U^{\perp\perp}$ and $V^{\circ\circ} \oplus \operatorname{rad} F^{\perp} \supset V^{\perp\perp}$. By (A), $U^{\perp\perp} + V^{\perp\perp}$ is closed in (E, ϕ), thus $U^{\perp\perp} + V^{\perp\perp} = (U^{\perp\perp} + V^{\perp\perp})^{\perp\perp} = (U + V)^{\perp\perp} = F^{\perp\perp}$. It follows that $(U^{\circ\circ} + V^{\circ\circ}) \oplus \operatorname{rad} F^{\perp} \supset U^{\perp\perp} + V^{\perp\perp} = F^{\perp\perp}$, hence $U^{\circ\circ} + V^{\circ\circ} = Q$, as claimed.

Let (H, Ψ) be any hermitian, euclidean space over K. We denote by H_0^* the set of all linear forms f on H with the property that $\ker(f)$, as a subspace of (H, Ψ), admits an orthogonal basis. Let $\{h_i: i \in I\}$ be an othogonal basis of H, and let f be any linear form on H. Put $J = \{i \in I: f(h_i) \neq 0\}$. f is induced by some $x \in H$ iff J is finite. In this case, of course, $f \in H_0^*$. Suppose J is infinite. Then $\ker(f)$ is semisimple and we have $f \in H_0^*$ iff card $J = \aleph_0$ ([2], Satz 1). We now see that $f \in H_0^*$ if and only if there is a decomposition $H = Q \oplus^{\perp} R$ with $\dim R \leqq \aleph_0$ and $f|_Q = 0$. In such a decomposition Q is always euclidean (cf. [2]). We also see that H_0^* is a subspace of H^*.

LEMMA 4. *Suppose (E, ϕ) is semisimple and has property* (A). *Let F be a euclidean subspace. Then every $f \in F_0^*$ is induced by some $y \in E$.*

Proof. If f is not induced by a $x \in F$ then $G = \ker(f)$ is semisimple and thus, by definition of F_0^*, euclidean; furthermore $\dim F/G = 1$. We have $F^\perp \neq G^\perp$, for otherwise by Lemma 3 we would have

$$G \oplus \operatorname{rad} F^\perp = G \oplus \operatorname{rad} G^\perp = G^{\perp\perp} = F^{\perp\perp} = F \oplus \operatorname{rad} F^\perp ,$$

which is impossible. Hence there is a $y \in G^\perp$ with $y \notin F^\perp$, and it is clear that f is induced by a suitable multiple $\lambda y (\lambda \in K)$.

We are ready to prove our main result.

THEOREM. *Let (E, ϕ) be a semisimple hermitian space over a commutative field K with* $\operatorname{char} K \neq 2$. *The lattice of all orthogonally closed subspaces of (E, ϕ) is modular if and only if E is finite dimensional.*

Proof. One half of the statement is clear. Suppose $\mathscr{L}(E, \phi)$ is modular. Then (A) holds for (E, ϕ). Let $M = \{v_i : i \in I\}$ be a maximal set of pairwise orthogonal anisotropic vectors of E ($x \in E$ is anisotropic if $(x, x) \neq 0$). The subspace F spanned by the v_i's is euclidean. By the maximality of M we have $\phi|_{F^\perp} = 0$, hence $\operatorname{rad} F^\perp = F^\perp$. Thus $F^{\perp\perp} = F \oplus F^\perp$ by Lemma 3. Now suppose that $\dim E \geqq \aleph_0$. Then $\dim F \geqq \aleph_0$ since (E, ϕ) is semisimple. Hence there exists an element $f \in F_0^*$ which is not induced by a $x \in F$. By Lemma 4, f is induced by some $y \in E$. Clearly $y \notin F \oplus F^\perp$; since $F \oplus F^\perp = F^{\perp\perp}$ there exists $v \in F^\perp$ with $(v, y) \neq 0$. Put $G = F \oplus (y) \oplus (v)$. One readily verifies that G is semisimple. Since $f \in F_0^*$ there is a decomposition $F = Q \oplus^\perp R$ such that $f|_Q = 0$ and $\dim R = \aleph_0$; here Q is euclidean. We have $y \in Q^\perp$ and so $G = Q \oplus^\perp (R \oplus (y) \oplus (v))$ which shows that G is euclidean. We define a linear form g on G by $g|_F = f$, $g(y) = 0$, $g(v) = (v, y) + 1$. The above decomposition of G shows that $g \in G_0^*$. Hence g is induced by some $z \in E$. Since $g|_F = f$ we have $z - y \in F^\perp$, i.e., $z = y + w$ with $w \in F^\perp$. Now $(v, y) + 1 = g(v) = (v, z) = (v, y) + (v, w)$, hence $(v, w) = 1$. But this is a contradiction since $v, w \in F^\perp$ and ϕ vanishes on $F^\perp$. This completes the proof.

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