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A DOUBLE INVERSION FORMULA

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Let G be an abelian group and suppose $\{a_n\}$ and $\{b_n\}$, $n \geqq 1$, are sequences in G. Let p be an odd prime and set $\eta_e = (e_1/p)$, the Legendre symbol, where $e = p^s e_1$, $s \geqq 0$, $p \nmid e_1$. Also, let $\chi_e^{\pm} = (1 \pm \eta_e)/2$. Define the sequence $\{c_n\}$ and $\{d_n\}$, $n \geqq 1$, by

$$c_n = \sum_{ef=n} (\chi_e^+ a_f + \chi_e^- b_f) \tag{1}$$

and

$$d_n = \sum_{ef=n} (\chi_e^- a_f + \chi_e^+ b_f) \; . \tag{2}$$

THEOREM. For $n \geqq 1$ and μ the Möbius function,

$$a_n = \sum_{ef=n} \mu(e)(\chi_e^+ c_f + \chi_e^- d_f) \tag{3}$$

and

$$b_n = \sum_{ef=n} \mu(e)(\chi_e^- c_f + \chi_e^+ d_f) \; . \tag{4}$$

Proof of the Theorem. Using (1) and (2) in (3) we obtain

$$\begin{aligned}
&\sum_{ef=n} \mu(e)(\chi_e^+ c_f + \chi_e^- d_f) \\
&\quad = \sum_{ef=n} \mu(e) \sum_{rs=f} [(\chi_e^+ \chi_r^+ + \chi_e^- \chi_r^-) a_s + (\chi_e^+ \chi_r^- + \chi_e^- \chi_r^+) b_s] \\
&\quad = \sum_{ef=n} \mu(e) \sum_{s|f} (\chi_{n/s}^+ a_s + \chi_{n/s}^- b_s) \\
&\quad = \sum_{s|n} (\chi_{n/s}^+ a_s + \chi_{n/s}^- b_s) \sum_{e|n/s} \mu(e) = a_n \; .
\end{aligned}$$

Formula (4) is proven similarly.

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