PACIFIC JOURNAL OF MATHEMATICS
Vol. 89, No. 2, 1980

EFFECTIVE DIVISOR CLASSES AND BLOWINGS-UP OF P^2

JEFFREY A. ROSOFF

Let $X_n \xrightarrow{\pi} P^2$ be the monoidal transformation of the (complex) projective plane centered at distinct points $P_1, \cdots, P_n$ of P^2. We recall that the Néron-Severi group of X_n is freely generated by the divisor class $[L]$ of the proper transform L of a line in P^2 and by the classes $[E_i]$ of the "exceptional" fibers E_i over P_i; the intersection pairing is given by

$$[L]^2=1\,; \quad [L]\cdot[E_i]=0\,; \quad [E_i]\cdot[E_j]=-\delta_{i,j}\,.$$

Let $\mathscr{M}(X_n)$ denote the monoid of elements F in the Néron-Severi group with the property that F contains an effective divisor. In this paper we

(1) construct a finite generating set for $\mathscr{M}(X_n)$ for $n \leqq 8$, and give a particularly simple geometric description of the generators when $P_1 \cdots P_n$ are in "general position";

(2) show that, for $n \geqq 9$, $\mathscr{M}(X_n)$ need not be finitely generated, despite the finite generation of the whole Néron-Severi group;

(3) prove the related result that if a nonsingular surface X contains an infinite number of exceptional curves of the first kind, then X is necessarily rational.

We will let K_{X_n} denote the cannonical class on X_n; it is given by $K_{X_n} = \pi^* K_{P^2} + \Sigma[E_i] = -3[L] + \Sigma[E_i]$. We observe that, for $n \leqq 9$, the anti-cannonical class $-K_{X_n}$ contains an effective divisor (which will also be denoted by $-K_{X_n}$ when no confusion is possible), since $H^0(X_n, \check{\omega}_{X_n})$ can be regarded as the (complex) vector space of homogeneous forms in 3 variables of degree 3 vanishing at the points $P_1 \cdots P_n$.

LEMMA 1. *Let X be any nonsingular rational surface, and let C be a curve on X with $p_a(C) \geqq 1$. Then $[C] + K_X$ is an effective class.*

Proof. The short exact sequence of $\mathscr{O}_X$-modules

$$0 \longrightarrow \mathscr{O}_X(-C) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_C \longrightarrow 0$$

yields, using Serre-duality and the rationality of X, $\dim H^0(X, \mathscr{O}_X(C) \otimes \omega_X) = \dim H^2(X, \mathscr{O}_X(-C)) = \dim H^1(C, \mathscr{O}_C) = p_a(C)$.

Recall that, for $n \leqq 8$, the points $P_1 \cdots P_n$ of P^2 are in *general* *position* if no three P_i are collinear and if no six of them lie on a conic.

THEOREM 1. *Let $X_n \to \boldsymbol{P}^2$ be the monoidal transformation of $\boldsymbol{P}^2$ centered at $P_1 \cdots P_n$, with $n \leqq 8$ and $P_1 \cdots P_n$ in general position. Then $\mathscr{M}(X_n)$ is finitely generated, the generators being the classes of divisors on the following list:*
(Note: $g(n)$ = number of generators of $\mathscr{M}(X_n)$).

n	$g(n)$	Divisor	Description
1	2	E_1	Exceptional curve
		$L-E_1$	Proper transform of a line through P
2, 3, 4	2, 6, 10	$E_i (1 \leqq i \leqq n)$	Exceptional curve
	Respt.	$L-E_i-E_j (1 \leqq i<j \leqq n)$	Proper transform of the line through P_i and P_j
5	16	$E_i \quad (1 \leqq i \leqq 5)$	Exceptional curve
		$L-E_i-E_j (1 \leqq i<j \leqq 5)$	Proper transform of the line through P_i and P_j
		$2L-\sum E_i$	Proper transform of the conic through all $\{P_i\}$
6	27	$E_i \quad (1 \leqq i \leqq 6)$	Exceptional curve
		$L-E_i-E_j (1 \leqq i<j \leqq 6)$	Proper transform of the line through P_i and P_j
		$2L-\sum_{i \neq k} E_i (1 \leqq k \leqq 6)$	Proper transform of the conic through all $\{P_i\}$ except P_k
7	56	$E_i \quad (1 \leqq i \leqq 7)$	Exceptional curve
		$L-E_i-E_j (1 \leqq i<j \leqq 7)$	Proper transform of the line through P_i and P_j
		$2L-\sum_{i \neq k, l} E_i (1 \leqq k<l \leqq 7)$	Proper transform of the conic through all points $\{P_i\}$ except P_k and P_l
		$3L-2E_j-\sum_{i \neq j} E_i (1 \leqq j \leqq 7)$	Proper transform of a cubic through all P_i and with a double point at P_j
8	241	$E_i \quad (i=1 \cdots 8)$	Exceptional curve
		$L-E_i-E_j (1 \leqq i<j \leqq 8)$	Proper transform of the line through P_i and P_j
		$2L-\sum_{i \neq j, k, l} E_i (1 \leqq j<k<l \leqq 8)$	Proper transform of the conic through all $\{P_i\}$ except P_j, P_k and P_l
		$3L-2E_k-\sum_{i \neq j, k} E_i (1 \leqq j, k \leqq 8, j \neq k)$	Proper transform of a cubic through all points $\{P_i\}$ except P_j, and with a double point at P_k
		$4L-2E_j-2E_k-2E_l - \sum_{i \neq j, k, l} E_i (1 \leqq j<k<l \leqq 8)$	Proper transform of a quartic through all $\{P_i\}$ with double points at P_j, P_k and P_l
		$5L-E_j-E_k-2 \sum_{i \neq j, k} (1 \leqq j<k \leqq 8)$	Proper transform of a quintic through all $\{P_i\}$ and with double points at all but P_j and P_k
		$6L-3E_k-2 \sum_{i \neq k} E_i (1 \leqq k \leqq 8)$	Proper transform of a sextic with a triple point at P_k and with double points at P_i, $\forall i \neq k$
		$3L-\sum_{i=1}^{8} E_i$	Anti-cannonical curve

REMARK. For $n = 6$, we see that the generators of the monoid for the cubic hypersurface in P^3 are the classes of the classical twenty-seven lines on X_6. More generally, the classes of the divisors listed above are, for $2 \leqq n \leqq 7$, precisely the classes of all rational curves on X_n with self-intersection -1. [1, Th. 26.2].

Before proving the theorem, we will first prove

LEMMA 2. *Let X_n be as in the theorem. Suppose that C is any curve on X_n for $1 \leqq n \leqq 7$, or that C is a curve on X_8 whose class is not represented above for $n = 8$. Then for any divisor $\mathscr{L}$ on the above list,* $\dim H^2(X_n, \mathscr{O}_{X_n}(C - \mathscr{L})) = 0$.

Proof. [*Case* 1: $n \leqq 7$]. A look at the proposed generating set of $\mathscr{M}(X_n)$ shows that, given $\mathscr{L}$ as above, there is an effective nontrivial divisor D such that $-K_{X_n} = [\mathscr{L}] + [D]$. Therefore $0 = \dim H^0(X_n, \omega_{X_n} \otimes \mathscr{O}_{X_n}(\mathscr{L})) = \dim H^0(X_n, \omega_{X_n} \otimes \mathscr{O}(\mathscr{L} - C))$, and the result follows by duality.

[*Case* 2: $n = 8$]. Again, we will use duality and show that $\dim H^0(X_8, \omega_{X_8} \otimes \mathscr{O}_{X_8}(\mathscr{L} - C)) = 0$. Suppose the contrary. Then $K_{X_8} + [\mathscr{L}]$ must be an effective class for some $\mathscr{L}$, and we may clearly assume that $[\mathscr{L}] \neq -K_{X_8}$. Then either

$$[\mathscr{L}] = < \begin{cases} [4L - 2E_i - 2E_j - 2E_K - \sum_{l \neq i,j,k} E_l] \text{ some } i, j, k, \text{ or} \\ [5L - E_i - E_j - 2\sum_{l \neq i,j} E_l] \text{ some } i, j, \text{ or} \\ [6L - 3E_k - 2\sum_{i \neq k} E_i] \text{ some } k\,. \end{cases}$$

But by the general position of $P_1 \cdots P_8$, the first two choices for $\mathscr{L}$ do not yield effective classes $[\mathscr{L}] + K_{X_8}$; hence $K_{X_8} + [\mathscr{L}]$ is of the form $[3L - 2E_k - \sum_{i \neq k} E_i]$.

Now, since C is unequal to any E_i, $C \cdot E_i \geqq 0$ and we may write $[C] = m[L] - \sum_{i=1}^{8} b_i[E_i]$, with $m \geqq 1$ and $b_i \geqq 0$. If $K_{X_8} + [\mathscr{L} - C]$ is to be effective, we must have $m = 1, 2$ or 3. If $m = 1$, the general position of the $\{P_i\}$ forces all but two of the b_i to be 0 and the nonzero b_i to be 1, making $[K_{X_8} + \mathscr{L} - C] = [2L - \sum c_i E_i]$ with $\sum c_i \geqq 6$. This class is not effective since no six of the $\{P_i\}$ lie on a conic. An analogous proof works for $m = 2$. If $m = 3$ we have, since $[C] \cdot [L - E_i - E_j] \geqq 0$ for all i, j, three possibilities:

(a) some $b_i = 3$, all others 0, or

(b) all b_i are 0 or 1, or

(c) some $b_i = 2$, all others are 0 or 1.

Neither (a) nor (b) can occur, as in these cases $K_{X_8} + [\mathscr{L} - C] = \sum c_i[E_i]$ with some $c_i < 0$, violating the effectiveness of $K_{X_8} + [\mathscr{L} -$

$C]$. Similarly, (c) can be dismissed unless $[C]$ is of the form $[3L-2E_i - \sum_{k\neq i,j} E_k]$, some i, j, which violates the hypothesis that $[C]$ not be represented on the list of divisors in the theorem.

Proof of Theorem 1. Fix a projective embedding of X_n into $\boldsymbol{P}^N$, some $N \geqq 3$. Then we may speak of the "degree" of a divisor on X_n with respect to this embedding. It suffices to show that, for C an effective divisor on X_n, $[C - \mathscr{L}]$ is an effective class for some divisor $\mathscr{L}$ listed in the theorem; the result will then follow by induction on "degree". Furthermore, for $n = 1, \cdots, 7$ we note that $-K_{X_n}$ is a sum of classes of divisors listed, while for $n = 8$ the anti-cannonical class is included on the list of proposed generators. Hence, by Lemma 1, we may assume that C is a curve with $p_a(C) = 0$. Finally, we may assume that C is an irreducible curve whose class is not represented on the list in the theorem.

By Riemann-Roch, together with Lemma 2 and the rationality of X_n, we have, for $\mathscr{L}$ any divisor on the above list except $-K_{X_8}$, $\dim H^0(X_n, \mathscr{O}_{X_n}(C - \mathscr{L})) - \dim H^1(X_n, \mathscr{O}_{X_n}(C - \mathscr{L})) = 1/2(C^2 - 2\mathscr{L}\cdot C - K_{X_n}\cdot C)$. Since $p_a(C) = 0$, the adjunction formula applied to C yields $C^2 = -K_{X_n}\cdot C - 2$, so we have, for all divisors $\mathscr{L}$ on the list in the theorem except for $-K_{X_8}$,

$$\begin{aligned}\dim H^0(X_n, \mathscr{O}_{X_n}(C - \mathscr{L})) - \dim H^1(X_n, \mathscr{O}_{X_n}(C - \mathscr{L}))\\ = (-K_{X_n}\cdot C) - 1 - (\mathscr{L}\cdot C)\ .\end{aligned}$$

Thus, it suffices to show that for some divisor $\mathscr{L}$ in the above list except for $-K_{X_8}$,

$$(*)\quad -K_{X_n}\cdot C > \mathscr{L}\cdot C + 1\ .$$

The proof of the validity of (*) is, for $n = 1, \cdots, 5$, a simplified version of the cases $n = 6, 7, 8$; hence we include only the later cases.

Let $[C] = m[L] - \sum_{i=1}^n b_i[E_i]$. Since $[C]$ is not represented on the above list, we intersect C with each element on the list to get

$$n = 6:\quad \begin{array}{llll}(1) & m \geqq 1 & (3) & m - b_i - b_j \geqq 0 \forall i \neq j\\ (2) & b_i \geqq 0 \forall i & (4) & 2m - \sum_{i\neq k} b_i \geqq 0 \forall k\ .\end{array}$$

Since $-K_{X_6}\cdot C = 3m - \sum_{i=1}^6 b_i$, our condition (*) to be fulfilled becomes

$$(**) \left\{\begin{array}{l} 3m > \sum_{i=1}^{6} b_i + b_k + 1 \text{ for some } k, \text{ or}\\ 2m > \sum_{k\neq i,j} b_k + 1 \text{ for some } i, j \text{ or}\\ m > b_k + 1 \text{ for some } k\ .\end{array}\right.$$

If $m > 1$, and if the third inequality of (**) fails, then, by conditions (2) and (3) above we have $m = 2$ and $b_k = 1 \forall k$, violating (4) above. If $m = 1$, then by (2) and (3) at most one b_i can be nonzero, and the first two inequalities of (**) hold.

$n = 7$ we have

(1) $m \geqq 1$

(2) $b_i \geqq 0 \forall i$

(3) $m - b_i - b_j \geqq 0 \forall i \neq j$

(4) $2m - \sum_{i \neq j\, k} b_i \geqq 0 \forall j \neq k$

(5) $3m - \sum_{j \neq i} b_j - 2b_i \geqq 0 \forall i$,

and condition (*) becomes

$$(**) \left< \begin{array}{l} 3m > \sum_{i=1}^{7} b_i + b_k + 1 \text{ for some } k, \text{ or} \\ 2m > \sum_{i \neq j,k} b_i + 1 \text{ for some } j, k, \text{ or} \\ m > b_j + b_k + 1 \text{ for some } j, k, \text{ or} \\ b_i > 1 \text{ for some } i\ . \end{array} \right.$$

Assume that the fourth inequality of (**) fails. If all b_i are 1, and if the third inequality of (**) fails, then $m \leqq 3$. By condition (4) we have $m \geqq 3$, so $m = 3$ and $[C] = -K_{X_7}$, which we have already seen is a sum of proposed generators of $\mathscr{M}(X_7)$. If some b_i is 0, then conditions (1)$\cdots$(4) and the first three conditions of (**) become the same as in the case $n = 6$.

$n = 8$ writing condition (*) in terms of m and the $b_i (i=1, \cdots, 8)$ and assuming that (*) does not hold, we have:

(α) $|3m - b_k - \sum_{i=1}^{8} b_i| \leqq 1$ for all k

(β) $|2m - \sum_{i \neq j,k} b_i| \leqq 1$ for all j, k

(γ) $|m - b_i - b_j - b_k| \leqq 1$ for all i, j, k

(δ) $|b_i - b_j| \leqq 1$ for all i, j.

Let $b = \min\{b_i\}$, and $B = \max\{b_i\}$. Note that by (δ), $0 \leqq B - b \leqq 1$. Let r of the b_i's have value b, and $8 - r$ of the b_i's have value B. We will obtain our contradiction on a case-by-case basis:

$r = 0$. Then by (α) $m - 3B = 0$ and $[C] = B(-K_{X_8})$, $B \in \boldsymbol{Z}$; since $p_a(C) = 0$ the adjunction formula yields $B^2 - B + 2 = 0$.

$r = 8$. Again by (α), $[C] = b(-K_{X_8})$.

$r = 1$. By (β), $m - 3B = 0$, and by (α) $|3m - 7B - 2b| \leqq 1$, contradicting $B - b = 1$.

$r = 7$. Then $m - 3b = 0$ by β, which is again impossible by (α) and the fact that $B - b = 1$ for $r \neq 0, 8$.

$r = 2$. Since $B - b = 1$, (β) implies that $2m - 5B - b = 0$, and (γ) implies that $m - 2B - b = 0$. Thus $B - b = 0$, a contradiction.

$r = 6$. Again, (γ) and (β) imply that $B - b = 0$.

$r = 3, 4, 5$. By (γ), $|m - 3b| \leqq 1$ and $|m - 3B| \leqq 1$, so $B - b = 0$, a contradiction.

We now examine the case in which the points $P_1, \cdots, P_n$, with $n \leqq 8$, of $\boldsymbol{P}^2$ are not in general position; in this case the classes of the divisors listed in Theorem 1 may contain reducible curves. For each $n \leqq 8$, let $F_1 \cdots F_m$ be the classes of the formal sums of L and the $\{E_i\}$ listed in Theorem 1, and let $D_i \in F_i$ be an effective divisor with the property that the number of distinct components of D_i is maximal for effective divisors in F_i. (Such a divisor D_i exists since, for any effective divisor $D \in F_i$, $\#$ components of $D \leqq \deg D = \deg E$ for any $E \in F_i$.) Write $D_i = \sum_j n_{i,j} E_{i,j}$ with $n_{i,j} > 0$.

LEMMA 3. *Let $P_1, \cdots, P_8$ be distinct points of $\boldsymbol{P}^2$ in arbitrary position, and let $X_8 \rightarrow \boldsymbol{P}^2$ be the monoidal transformation centered at the $\{P_i\}$. Let $D_i \in F_i$ be as above, for $n = 8$. Then there are only a finite number of divisor classes F on X_8 with the property that F contains curve C with $p_a(C) = 0$ and with the property that* $\dim H^2(X_8, \mathcal{O}_{X_8}(C - D_i)) \geqq 1$ *for some i.*

Proof. If $\dim H^2(X_8, \mathcal{O}_{X_8}(C - D_i)) \geqq 1$, then, by duality, $K_{X_8} + [D_i] - [C]$ must contain an effective divisor, and so must $K_{X_8} + F_i$. Thus, as in the proof of Theorem 1, $K_{X_8} + F_i$ must be of the form

$$[L] - [E_i] - [E_j] - [E_k], \text{ some } i, j, k, \text{ or}$$
$$2[L] - \sum_{l \neq i,j} [E_l], \text{ some } i, j, \text{ or}$$
$$3[L] - 2[E_k] - \sum_{i \neq k} [E_i], \text{ some } k\,.$$

Hence, if $[C] = m[L] - \sum b_i[E_i]$, we must have $0 \leqq m \leqq 3$, and since $p_a(C) = 0$, the adjunction formula yields $(m^2 - 3m) - \sum_{i=1}^{8} (b_i^2 - b_i) = -2$. Clearly with $0 \leqq m \leqq 3$ there are only a finite number of solutions to this diaphantine equation.

Let $R_1 \cdots R_k$ be the divisor classes on X_8 referred to in Lemma 3, and let $S_i \in R_i$ be an effective divisor with maximal number of distinct components. Write $S_i = \sum_j m_{i,j} Q_{i,j}$, with $m_{i,j} > 0$.

THEOREM 2. *Let $X_n \rightarrow \boldsymbol{P}^2$ be the monoidal transformation centered at points $P_1 \cdots P_n$ of $\boldsymbol{P}^2$, with $n \leqq 8$ and with the points $\{P_i\}$ in arbitrary positions. Then $\mathscr{M}(X_n)$ is finitely generated, the generators being $\{E_{i,j}\}$ for $n \leqq 7$, and $\{[E_{i,j}]\} \cup \{[Q_{i,j}]\}$ if $n = 8$.*

Proof. [*Case* 1: $n \leqq 7$]. We will show that, for C an irreducible curve on X_n, $C - E_{i,j}$ is equivalent to an effective divisor, for some i, j. As in the proof of Theorem 1, we may assume that $p_a(C)=0$. Moreover, the proof of Lemma 2 for $n \leqq 7$ did not rely on the general position of the $\{P_i\}$; hence for any curve C on X_n, $n \leqq 7$, $\dim H^2(X_n, \mathcal{O}_{X_n}(C - D_i)) = 0$ for all i. Thus it suffices to show that

(a) if $p_a(C) = 0$, C irreducible and $[C] \neq [E_{i,j}]$ for all i, j, then $\chi(\mathcal{O}_{X_n}(C - D_i)) \geqq 1$ for some i, and

(b) $[E_{i,j}]$ cannot be written nontrivially as a sum of effective divisor classes.

Part (b) follows from the maximality of the number of components of D_i for effective divisors in F_i. For part (a) we note that, since the intersection-theoretic properties of the $\{F_i\}$ are the same as in Theorem 1, it suffices to show that

$$(*) \quad -K_{X_n}\cdot C > (D_i\cdot C) + 1 \text{ for some } i,$$

with $[C] \neq [E_{i,j}]\ \forall\, i, j$. Writing $[C]=m[L] - \sum_{i=1}^{n} b_i[E_i]$ and writing (*) in terms of m and the $\{b_i\}$, the condition (*) becomes precisely the condition (**) of Theorem 1.

Since $[C] \neq [E_{i,j}]$ for all i, j, we have $C\cdot D_i \geqq 0 \forall i$, i.e., the constraints on m and the $\{b_i\}$ are the same as in the proof of Theorem 1. Since the truth of (**) depended only on these constraints, we are done.

[*Case* 2: $n = 8$]. As in the case $n \leqq 7$, it suffices to show that for C an irreducible curve on X_8 with $p_a(C) = 0$, either $C - E_{ij}$ or $C - Q_{i,j}$ is equivalent to an effective divisor. Clearly, if $C \in R_i$, for some i, then $C - Q_{i,j}$ is equivalent to an effective divisor for some i, j. If $C \notin R_i$ for any i, it suffices to show that, with $C \neq E_{i,j}$ for all i, j,

$$(*) \quad \chi(\mathcal{O}_{X_8}(C - D_i)) \geqq 1 \text{ for some } i\,.$$

Since $C\cdot D_i \geqq 0$ for all i, the verification of (*) reduces to the case $n = 8$ of Theorem 1.

In contrast with the above, if $n \geqq 9$, $\mathscr{M}(X_n)$ need not be finitely generated.

EXAMPLE. Let C_1 be a cuspidal cubic curve in P^2, and let C_2 be any cubic curve intersecting C_1 in nine distinct points, none of which is a singular point of C_1. Let Y be the surface obtained by blowing up P^2 at $C_1 \cap C_2$. *Claim*: $\mathscr{M}(Y)$ is not finitely generated.

Let $F_i(X_0, X_1, X_2)$ be the (cubic) defining polynomials of $C_i (i = 1, 2)$. Then the rational function F_1/F_2 on P^2 has its only indeterminate points on $C_1 \cap C_2$. Since C_1 and C_2 are transversal, the rational function F_1/F_2 pulls back to Y to give a holomorphic map ϕ: $Y \to \boldsymbol{P}^1$, with fibers the proper transforms under the blowing up π: $Y \to \boldsymbol{P}^2$ of the curves in the pencil generated by C_1 and C_2.

Let Y^* denote the set $Y - \bigcap_{t \in P^1} \operatorname{sing} \phi^{-1}(t)$, and let $\phi^{-1}(t_0)$ be the proper transform of the cuspidal curve C_1. The fibers of an elliptic fibering have been classified by [2, Th. 6.2 and 9.1], along with the possible group structures of the set of nonsingular points; we see by the classification that $\phi^{-1}(t_0) \cap Y^*$ has the structure of a torsion-free abelian group, with any point serving as the identity element.

Let Γ denote the set of sections of ϕ (which necessarily map into Y^*); then after choosing some element of Γ (such as one of the nine exceptional curves lying over a point of $C_1 \cap C_1$) as an identity element, Γ has the structure of an abelian group under pointwise addition (the addition being the group operations on the nonsingular sets of the fibers of ϕ). We have, for each $t \in \boldsymbol{P}^1$, a natural evaluation homomorphism

$$\psi_t\colon \Gamma \longrightarrow \phi^{-1}(t) \cap Y^*, \text{ defined by } \sigma \longrightarrow \sigma(t) .$$

Since Γ contains at least nine disjoint sections (i.e., the nine exceptional curves lying over $C_1 \cap C_2$), the map ψ_{t_0} maps Γ nontrivially into a torsion-free group, so Γ must be infinite.

By [2, Th. 9.2], each $\eta \in \Gamma$ induces a fiber-preserving automorphism

$$L_\eta\colon Y^* \longrightarrow Y^*, \text{ defined by } L_\eta(z) = z + \eta \circ \phi(z), \text{ which}$$

actually extends to an automorphism of Y. Thus, any two elements of Γ differ by an automorphism of Y.

Hence, the orbits of the exceptional curves lying over $C_1 \cap C_2$ under the action of Aut (Y) yield an infinite number of exceptional curves of the first kind on Y. The following fact shows that $\mathscr{M}(Y)$ is not finitely generated, while of course N. S. $(Y) \approx PIC(Y) \approx Z \oplus^{10}$.

Fact. Let Y be any surface containing an infinite number of curves of negative self-intersection. Then $\mathscr{M}(Y)$ is not finitely generated.

Proof. Suppose to the contrary that $\mathscr{L}_1, \cdots, \mathscr{L}_n$ is a (finite) generating set of $\mathscr{M}(Y)$. To obtain a contradiction it suffices to show that if C_i is a fixed curve in the algebraic equivalence class $\mathscr{L}_i$, and if E is a curve on Y with negative self-intersection, then

E must be a component of C_i, for some i. For the curves C_i and E as stated, write

$$[E] = \sum_{i=1}^{n} m_i \mathscr{L}_i = \sum_{i=1}^{n} m_i [C_i], \text{ with } m_i \geqq 0 .$$

Therefore $E^2 = \sum_{i=1}^{n} m_i (C_i \cdot E)$. If E is not a component of C_i for any i, then the right-hand side of the above equation is nonnegative, which is a contradiction.

REMARK. The elliptic surface constructed above is only one of a large number of known examples of surfaces which contain an infinite number of rational curves with self-intersection -1 and which are obtained by blowing up the projective plane at nine points. For other examples, see [5, p. 164], or [1, p. 407].

REMARK. It is not hard to show, using the projection formula [1, p. 426 A. 4] that if $X \to Y$ is a monoidal transformation of surfaces, and if $\mathscr{M}(X)$ is finitely generated, then $\mathscr{M}(Y)$ is also finitely generated. Hence $\mathscr{M}(X_n)$ need not be finitely generated for $n \geqq 9$.

In view of the *fact* used above, the question naturally arises as to which surfaces can contain an infinite number of curves with negative self-intersection. A partial answer is given by a conjecture of A. Kas, a proof of which is provided below:

THEOREM 3. *Let X be nonsingular algebraic surface over C which contains an infinite number of exceptional curves of the first kind. Then X is rational.*

Proof. Let $\phi_1, \cdots, \phi_n$ be a basis of holomorphic 1-forms on X, for $n \geqq 0$. We will first reduce to the case $n = 0$.

Case 1. $n \geqq 2$ and $\phi_i \wedge \phi_j \neq 0$, some i, j.

We write the cannonical map $\pi: X \to \mathrm{Alb}(X)$, given by

$$z \longrightarrow \left[\int_P^z \phi_1, \cdots, \int_P^z \phi_n\right]$$

modulo the lattice in C^n generated by the $2n$ vectors

$$\left[\int_{\Gamma_i} \phi_1, \cdots, \int_{\Gamma_i} \phi_n\right], \quad i = 1, \cdots, 2n ,$$

where P is a fixed point of X and $\Gamma_1, \cdots, \Gamma_{2n}$ are 1-cycles whose homology classes generate the free subgroup of $H_1(X, \boldsymbol{Z})$.

The hypothesese imply that the Jacobian of the Albanese map π has rank 2; hence π is generically finite-to-one in the sense that there are only a finite number of points $p \in \mathrm{Alb}(X)$ such that $\dim \pi^{-1}(p) = 1$. Let $\{p_1, \cdots, p_k\}$ be this finite set, and let $\pi^{-1}(p_i)$ be the divisor $\sum n_{ij}D_j$, with $n_{ij} > 0$ and D_{ij} irreducible. If C is a rational curve on X, then $\pi(C)$ is a single point; hence the number of rational curves on X is bounded by $\sum n_{ij}$. (Actually it is not hard to see that a rational curve on X must be a component of a fixed divisor in the cannonical class of X.)

Case 2. $n = 1$, or $n \geqq 2$ and $\phi_i \wedge \phi_j = 0 \forall i, j$.

If $n = 1$, then $\dim \pi(X) = \dim \mathrm{Alb}(X) = 1$. If $n \geqq 2$, the fact that $\phi_i \wedge \phi_j = 0 \forall i, j$ implies that the Jacobian matrix of π has rank 1, and $\dim \pi(X) = 1$ in this case as well.

Let Δ be the curve $\pi(X) \subset \mathrm{Alb}(X)$, and let $\{a_1 \cdots a_r\} \subset \Delta$ be the (finite) set of points such that $\forall t \in \Delta, \pi^{-1}(t)$ is singular if and only if $t = a_i$, some i. Let C be a rational curve on X with nonzero self-intersection. Then $\pi(C)$ is a point of Δ, so C is a component of $\pi^{-1}(t_0)$, some $t_0 \in \Delta$. Since $(\pi^{-1}(t))^2 = 0 \forall t$, and since $C^2 \neq 0$, $t_0 \in \{a_1 \cdots a_r\}$. Thus the number of rational curves on X with nonzero square is bounded by $\sum_{i,j} n_{i,j}$, where $\pi^*(a_i)$ is the effective divisor $\sum_j n_{i,j}D_j$. Therefore, we have reduced to

Case 3. X has no (global) holomorphic 1-forms. For C an exceptional curve of the first kind on X, the adjunction formula yields $C \cdot K_x = -1$, and so $C \cdot mK_x < 0 \forall m > 0$.

Case 3a. $2K_x$ contains an effective divisor D. Then since $D \cdot C < 0$, C must be a component of D, and the number of exceptional curves of the first kind on X is bounded by $\sum n_i$, where $D = \sum n_i D_i$, with D_i integral and $n_i > 0$.

Case 3b. $2K_x$ does not contain an effective divisor, i.e., $P_2(X) = 0$. Since X has no global holomorphic 1-forms, $q(X) = \dim H^1(X, \mathcal{O}_x) = 0$. Since $q(X) = P_2(X) = 0$, X is rational by the classification theorem of Castelnuovo [3. Th. 49]).

REMARK. Among the standard surface types, it is also known that certain $K3$ surfaces contain an infinite number of -2 curves. In addition, it seems to be a part of the folklore that, for each positive integer n, there is an elliptic surface containing an infinite number of curves with self-intersection $-n$.

We end this paper with a conjecture, a discussion of which is to appear in the near future:

Conjecture. Let X be a nonsingular algebraic surface of general type. Then $\mathscr{M}(X)$ is finitely generated.

REFERENCES

1. R. Hartshorne, *Algebraic Geometry,* Graduate Texts in Math., Springer-Verlag, New York Inc., 1977.
2. K. Kodaira, *On compact analytic surfaces II,* Annals of Math., 77, No. **3** (1963).
3. ———, *On the structure of complex analytic surfaces IV,* Amer. J.Math., **90** (1968), 1048-1066.
4. Yu I. Manin, *Cubic Forms: Algebra, Geometry, Arithmetic;* North Holland Pub. Co., Amsterdam, 1974.
5. I. R. Saferevic, *Algebraic surfaces*, Proc. Steklov Inst. Math., **75** (1965), Transl. by Amer. Math. Soc., (1967).

Received September 17, 1979 and in revised form December 16, 1979.

ILLINOIS STATE UNIVERSITY
NORMAL, IL 61761.