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AN INTRINSIC CHARACTERIZATION FOR PI FLOWS

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An intrinsic characterization for a minimal flow (X, T) to be a PI flow is given. This characterization is then combined with some recent techniques of R. Ellis to prove the general PI and HPI versions of the Veech Structure Theorem.

0. Introduction. A pointed minimal flow (X, x_0, T) is PI if there is an ordinal Λ, a collection of pointed minimal flows $\{(X_\lambda, x_\lambda): \lambda \leqq \Lambda\}$, and a homomorphism π, $\pi: (X_\Lambda, x_\Lambda) \to (X, x_0)$ such that

(i) X_0 is the trivial flow

(ii) for each $\lambda < \Lambda$ there is a homomorphism $\varphi_\lambda^{\lambda+1}$, $\varphi_\lambda^{\lambda+1}: (X_{\lambda+1}, x_{\lambda+1}) \to (X_\lambda, x_\lambda)$ which is either proximal or almost periodic,

(iii) for each limit ordinal $\lambda_0 \leqq \Lambda$, $(X_{\lambda_0}, x_{\lambda_0})$ is inv lim $\{(X_\lambda, x_\lambda): \lambda < \lambda_0\}$, and

(iv) π is proximal.

The collection of flows $\{(X_\lambda, x_\lambda): \lambda \leqq \Lambda\}$ and the associated maps $\{\varphi_\lambda^{\lambda+1}\}_{\lambda<\Lambda}$ are called a PI tower for (X, x_0, T). (X, x_0, T) is *strictly* PI if π is the identity map. For a discussion of the role of PI flows in topological dynamics, see part 2 of Veech's article [9].

With the exception of the definition, the only condition equivalent to PI in the literature is that the group for the flow, $G(X, x_0)$, contain the group G_∞. (See [10] for a characterization of PD flows.) In this paper we give an intrinsic characterization of PI. Section 1 is devoted to this characterization.

In a recent paper, [3], Ellis proved that the Furstenberg structure theorem holds for any distal flow. Modifying Ellis's technique and applying our characterization we show, in § 2, that for a large class of properties of flows, all flows with one of these properties are PI iff all metric flows with the same property are PI. As a corollary, we show that every point-distal flow is PI. Using this fact we establish that every point-distal flow is actually HPI (the proximal maps in the PI tower are highly proximal) thus proving the General Veech Structure Theorem.

We assume throughout this paper that the reader is familiar with the general theory of PI flows as contained in [5] or [6]. Our notation is primarily that of Glasner's book, [6], with the obvious modification that our group actions are written on the right. In particular, for a fixed topological group T, $M(T)$, or just M, is a fixed minimal right ideal in βT with the usual semi-group structure. $J(T)$, or just J, is the set of idempotents in $M(T)$. If $U \subseteqq T$ then

$[U, U] = \{p \in M(T) \mid$ there is a $t \in U$ such that $pt \in \bar{U}\}$ where $\bar{U}$ is the closure of U in βT.

If (X, x_0) and (Y, y_0) are pointed minimal sets and $\varphi: (X, x_0) \to (Y, y_0)$ then $R(\varphi) = \{(x_1, x_2) \in X \times X \mid \varphi(x_1) = \varphi(x_2)\}$, $Q(\varphi)$ is the relative regionally proximal relation, $S(\varphi)$ is the relative equicontinuous structure relation, $D(\varphi)$ is the set of almost periodic points in $R(\varphi)$. $G(X, x_0)$ and $G(Y, y_0)$ are the groups for (X, x_0) and (Y, y_0) relative to a fixed $u \in J(T)$. We denote the map from βT to X given by $p \to x_0 p$ by e_{x_0}. We use c_X, or just c to denote closure in X.

We wish to thank the referee for pointing out an error in an earlier version of Lemma 1.1 of this paper and for suggesting shorter proofs of (1) implies (2) of Theorem 1.1 and of Lemma 2.3.

1. **PI Flows.** The characterization of PI flows we present involves the nice behavior of a certain class of closed subsets of the flow. The nice behavior involves a relation which looks like a strong localization of the regionally proximal relation. (See [9] where a nonlocal version of this relation is mentioned.) To be specific, if (X, T) is a flow, K a subset of X, and x and y belong to K, we say x is *strongly regionally proximal to y in K*, which we abbreviate $x \in SRP(K, y)$, if there are nets $\{k_n\}$ in K and $\{t_n\}$ in T such that $\lim k_n = y$, $\lim k_n t_n = x$, and $\lim x t_n = x$.

The easiest way to describe the class of closed subsets we are interested in is by using $J(T) = J$. The closed subsets we need are those sets K contained in X which contain at least two points (we will call them nontrivial) and for which Kw is dense in K for some $w \in J$.

The referee noted that by 2.2 of [5], if $x, y \in Ku$, then $x \in SRP(Ku, y)$ iff y is in the X closure of $Ku \cap U$ for every F-neighborhood U of x. In particular, $x \in SRP(Ku, y)$ implies y is in the F closure of $Ku \cap U$ for every F-neighborhood U of x. A proof of (5) implies (1) in the following theorem could be given by exploiting these ideas.

THEOREM 1.1. *Suppose (X, T) is minimal. Then the following are equivalent*:

(1) *(X, T) is not* PI.

(2) *For some $w \in J$ there is a closed, nontrivial set $K \subseteqq X$ such that $K = c_X(Kw)$ and such that for each $x \in K$, $x \in SRP(K, y)$ for all $y \in K$.*

(3) *For some $w \in J$ there is a closed, nontrivial set $K \subseteqq X$ such that $K = c_X(Kw)$ and such that for some $x \in K$, $x \in SRP(K, y)$ for all $y \in K$.*

(4) *For some* $w \in J$ *there is a closed, nontrivial set* $K \subseteqq X$ *such that* $K = c_X(Kw \cap K)$ *and such that for each* $x \in K$, $x \in SRP(K, y)$ *for all* $y \in K$.

(5) *For some* $w \in J$ *there is a closed, nontrivial set* $K \subseteqq X$ *such that* $K = c_X(Kw \cap K)$ *and such that for some* $x \in K$, $x \in SRP(K, y)$ *for all* $y \in K$.

Proof. Clearly (2) implies (3), (2) implies (4), (4) implies (5), and (3) implies (5). We show that (1) implies (2) and that not (1) implies not (5).

(1) implies (2).

Throughout this proof we use the notation of [6] with the modification mentioned in the introduction and all references are to [6].

Let (X, T) be a flow which is not PI. Let M be the universal minimal flow and let $u \in J$. Consider $G_\infty(u) = G_\infty$ for the pointed minimal flow (M, u) as defined on pages 135, 138. Fix $u \in J$, let $F = G_\infty(u)$, and let $w \in J$ for which Fw is an F minimal flow with F acting on the left (see page 142). Consider (M, w) and note that $G_\infty(w) = Fw$. Applying 6.2, page 143, with u replaced by w, F by $G_\infty(w)$, and $A = \{w\}$, and noting that $H(G_\infty(w)) = G_\infty(w)$, we have that, for each neighborhood U of w, the set $\{p \in M: wt \in U$ and $pt \in U\}$ is dense in $c_M(G_\infty(w))$. Now suppose $p \in c_M(G_\infty(w))$, then $pc_M(G_\infty(w)) = pwc_M(G_\infty(w)) = c_M(pwG_\infty(w)) = c_M(G_\infty(w))$ since $pw \in G_\infty(w)$ by Lemma 1.5, page 115.

Now fix x_0 in X with $x_0 w = x_0$ and take $K = x_0 c_M(G_\infty(w))$ which is $c_X(x_0 G_\infty(w))$. Let $x \in K$ and note that $xc_M(G_\infty(w)) = K$. Then clearly $x \in SRP(K, y)$ for all $y \in K$. Finally note that Kw is a dense subset of K and thus (1) implies (2).

Not (1) *implies not* (5).

Let (X, T) be PI and let $\{X_\lambda: \lambda \leqq \Lambda\}$ and $\{\varphi_\lambda^{\lambda+1}\}_{\lambda<\Lambda}$ be a PI tower for (X, T). Let $\pi: (X_\Lambda, T) \to (X, T)$ be the proximal map. We will use c_λ for closure in X_λ, c for closure in X. Let w be any element of J and let K be any nontrivial closed set in X such that $c(Kw \cap K) = K$. (Note if $c(Ku \cap K)$ is trivial for all $K \subseteqq X$, then Not (1) implies Not (5) immediately). Let $K^* = c_\Lambda(\pi^{-1}[K]w \cap \pi^{-1}[K])$. Note that π maps K^* onto K.

For each $\lambda < \Lambda$, let $\varphi_\lambda: X_\Lambda \to X_\lambda$ be the obviously induced map. Let $\beta < \Lambda$ be the smallest ordinal for which $\varphi_\beta(K^*)$ is not a single point. We note that, because of condition (iii) of the definition of PI flows, β is not a limit ordinal. Therefore, $\beta - 1$ is also an ordinal.

We first show that $\varphi^{\beta}_{\beta-1}: X_\beta \to X_{\beta-1}$ is not a proximal map. To see this, note that if it were, all points in $\varphi_\beta(K^*)$ would be proximal since $\varphi^{\beta}_{\beta-1}\varphi_\beta(K^*)$ is a single point. Since $\pi^{-1}[K]w \cap \pi^{-1}[K]$ is dense in K^*, $\varphi_\beta(\pi^{-1}[K]w \cap \pi^{-1}[K])$ is dense in $\varphi_\beta(K^*)$. Let x^* and y^* be in $\pi^{-1}[K]w \cap \pi^{-1}[K]$. Then $x^*, y^* \in \pi^{-1}[K]w$ and therefore $x^*w = x^*$ and $y^*w = y^*$. Thus $\varphi_\beta(x^*)w = \varphi_\beta(x^*)$ and $\varphi_\beta(y^*)w = \varphi_\beta(y^*)$. So, if all points in $\varphi_\beta(K^*)$ are proximal, $\varphi_\beta(x^*) = \varphi_\beta(y^*)$ and $\varphi_\beta(K^*)$ has a dense subset of only one point, contradicting the choice of β. Since $\varphi^{\beta}_{\beta-1}$ is not proximal, it must be almost periodic.

We show that K does not satisfy (5) of the statement of the theorem by demonstrating that for every $x \in K$ there is a $y \in K$ such that $x \notin SRP(K, y)$. With this in mind, let $x \in K$ and choose $x^* \in K^*$ such that $\pi(x^*) = x$. Let y^* be any element of K^* such that $\varphi_\beta(y^*) \neq \varphi_\beta(x^*)$ and let $y = \pi(y^*)$. If x and y are equal, then x^* and y^* are proximal and thus $\varphi_\beta(x^*)$ and $\varphi_\beta(y^*)$ are proximal. But $\varphi^{\beta}_{\beta-1}$ is almost periodic and hence does not identify any proximal points. Therefore, since $\varphi_\beta(x^*) \neq \varphi_\beta(y^*)$ and $\varphi^{\beta}_{\beta-1}\varphi_\beta(x^*) = \varphi^{\beta}_{\beta-1}\varphi_\beta(y^*)$, we conclude that $x \neq y$.

Now suppose $x \in SRP(K, y)$. Then we can find nets $\{k_n\}$ in K and $\{t_n\}$ in T such that $\lim k_n = y$ and $\lim (x, k_n)t_n = (x, x)$. For each n, let $k^*_n \in K^*$ such that $\pi(k^*_n) = k_n$. Since X_A is compact there are subnets $\{k^*_m\}$ and $\{t_m\}$ such that the three nets $\{k^*_m\}$, $\{k^*_m t_m\}$, and $\{x^* t_m\}$ all converge, say to z^*, b^*, and a^* respectively. Since π is proximal, z^* and y^* are proximal, and a^*, b^*, and x^* are proximal. Now, since $\varphi_{\beta-1}(k^*)$ is a singleton, $\varphi_{\beta-1}(x^*) = \varphi_{\beta-1}(k^*_m)$ and $\varphi_{\beta-1}(x^* t_m) = \varphi_{\beta-1}(k^* t_m)$. Thus $\varphi_{\beta-1}(x^*) = \varphi_{\beta-1}(y^*)$ and $\varphi_{\beta-1}(a^*) = \varphi_{\beta-1}(b^*)$. Therefore, since $\varphi_{\beta-1}$ is almost periodic, $\varphi_\beta(x^*) = \varphi_\beta(y^*)$ and $\varphi_\beta(a^*) = \varphi_\beta(b^*)$.

Now consider $\varphi_\beta(x^*)$ and $\varphi_\beta(k^*_m)$. Clearly the net $\{\varphi_\beta(k^*_m)\}$ converges to $\varphi_\beta(z^*)$ and hence to $\varphi_\beta(y^*)$; $(\varphi_\beta(x^*), \varphi_\beta(k^*_m)) \in R(\varphi^{\beta}_{\beta-1})$; and $\lim (\varphi_\beta(x^*), \varphi_\beta(k^*_m))t_m = (\varphi_\beta(a^*), \varphi_\beta(a^*))$. Thus, we must conclude that $(\varphi_\beta(x^*), \varphi_\beta(y^*))$ is in $Q(\varphi^{\beta}_{\beta-1})$. Since $\varphi_\beta(x^*) \neq \varphi_\beta(y^*)$ by construction and $\varphi^{\beta}_{\beta-1}$ is almost periodic, this is a contradiction. We therefore must conclude that $x \notin SRP(K, y)$ completing the proof of the theorem.

The following lemmas contain useful characterizations of the conditions of Theorem 1.1.

LEMMA 1.1. *Suppose (X, T) is minimal and $K \subsetneqq X$ is closed and nontrivial, then the following are equivalent.*

(1) *There is a $w \in J(T)$ such that $Kw \cap K$ is dense in K.*

(2) *There is a cartesian product $\times_\alpha(X_\alpha, T)$ where $(X_\alpha, T) = (X, T)$ for each α and a point $\langle x_\alpha \rangle$ in $\times_\alpha X_\alpha$ such that*

(a) *the range of $\langle x_\alpha \rangle$, that is, $\{x_\alpha\}$, is a dense subset of K, and*

(b) *$\langle x_\alpha \rangle$ is an almost periodic point of $\times_\alpha(X_\alpha, T)$.*

The proof is straightforward and is omitted.

LEMMA 1.2. *Suppose (X, T) is minimal. Suppose $K \subsetneq X$ and $x \in K$, then the following are equivalent.*

(1) *$x \in SRP(K, y)$ for all $y \in K$.*

(2) *For every index α for X, $\alpha T(x) \cap K$ is dense in K.*

The proof involves the iterated limit theorem for nets and is also omitted.

As an application, combining Theorem 1.1, Lemma 1.2, and the general Furstenberg structure theorem for distal flows [3], we have the following characterization for distal flows.

THEOREM 1.2. *A minimal flow (X, T) is distal iff for every nontrivial closed K and for every $x \in K$, there is an index α such that $\alpha T(x) \cap K$ is not dense in K.*

Proof. Suppose (X, T) is distal. By the general Furstenberg structure theorem, (X, T) is PI. Then, by Theorem 1.1 and Lemma 1.2, for every $w \in J$ and every nontrivial K such that $K = c(Kw)$, and for every $x \in K$, there is an index α such that $\alpha T(x) \cap K$ is not dense in K. Since (X, T) is distal, $K = Kw$ for all $w \in J$, completing the first half of the proof.

Suppose for every nontrivial closed K and for every $x \in K$ there is an index α such that $\alpha T(x) \cap K$ is not dense in K. We show that every pair of distinct points are distal. Let $x, y \in X$ and let $K = \{x, y\}$. Then either $x = y$ or there is an index α such that $\alpha T(x) \cap K = \{x\}$. Thus, either $x = y$ or x and y are not proximal; (X, T) is distal.

2. The Veech structure theorem. In a recent paper [3] Ellis proves that all distal minimal sets satisfy the Furstenberg structure theorem, that is, all distal minimal sets are PI. Using a modification of Ellis's technique and Theorem 1.1, we show that, for a large class of properties, all minimal flows with a given one of these properties are PI if and only if all metric minimal flows with the given property are PI. One of these properties is "there is a point with countable proximal cell". It then follows from known results that any flow with a point with countable proximal cell is PI. Since all point-distal flows satisfy this property, it follows that all point-distal flows are PI.

Suppose P is a property of transformation groups. We will call P a *transferable property* if

(1) P is preserved by transformation group homomorphisms onto minimal sets, and

(2) if (X, T) has property P and S is a subgroup of T then there is a point $x^* \in X$ such that $(c(x^*S), S)$ has property P.

As will be seen in the proof of the main theorm of this section (Theorem 2.1), conditions (1) and (2) are what is needed to transfer property P from a minimal (X, T) to a constructed minimal metric (Y, H).

For the purposes of the following discussion we assume (X, T) is minimal with a transferable property P and that (X, T) is not a PI flow. Then by Theorem 1.1 and Lemma 1.2 there is a nontrivial closed subset $K \subseteqq X$, an idempotent $u \in J(T)$, and an $x_0 \in K$ such that $c(Ku) = K$ and $\alpha T(x_0) \cap K$ is dense in K for every index α on X. We assume that d is a continuous pseudo-metric for X for which $\sup\{d(x_0, x) \mid x \in K\} \neq 0$ and for which $d(x, y) \leqq 1$ for all $x, y \in X$. Proofs of some of the following can be found in [3].

Suppose H is a countable subgroup of T. Define $R(H)$ to be $\{(x_1, x_2) \in X \times X \mid d(x_1h, x_2h) = 0$ for all $h \in H\}$. Then $R(H)$ is a closed, H invariant, equivalence relation. We denote the quotient map from X to $X/R(H)$ by φ. Suppose H is indexed by $H = \{h_i\}_{i=1}^\infty$. For every $a, b \in X/R(H)$ let $\sigma(a, b) = \sum_{i=1}^\infty 2^{-i} d(x_1h_i, x_2h_i)$ where $\varphi(x_1) = a$ and $\varphi(x_2) = b$. Then σ is a metric whose topology is the quotient topology on $X/R(H)$. Thus $X/R(H)$ is a compact metric space and is therefore 2nd countable. We let $\mathscr{B} = \mathscr{B}(H)$ be a countable basis for the topology of $X/R(H)$. If $\{H_i\}_{i=1}^\infty$ is a sequence of countable subgroups of T such that $H_i \subseteqq H_{i+1}$ and $H = \bigcup_{i=1}^\infty H_i$ and if for $i \leqq j$, $\psi_i^j\colon X/R(H_j) \to X/R(H_i)$ is the cannonical map, then $X/R(H) = \text{inv lim}\{X/R(H_i), \psi_i^j\}$.

The basic idea is that used by Ellis in Proposition 1.6 of [3]. We would like to find a subgroup H of T so that (a) $(X/R(H), H)$ is metric; (b) $\varphi(x_0)$ is an almost periodic point with dense orbit; (c) $\varphi(K) = K^*$ has the properties which insure that $(X/R(H), H)$ is not a PI flow; and (d) $(X/R(H), H)$ is a P-flow. In general we can insure (a) by choosing H countable and (d) comes for free if we have (b). If H is any countable subgroup of T we can find another countable subgroup H', $H \subseteqq H'$, so that, relative to the new group H', (b) and (c) are true in $X/R(H)$. Since, in general, $(X/R(H), H')$ is not a transformation group, it is necessary to use an induction and pass to a limit. Before stating the induction lemma we need to introduce some notation.

To show that $\varphi(x_0) \in \varphi(X) = Y$ is an almost periodic point, for any subgroup H of T, and any $V \in \mathscr{B}(H)$, we will denote by F_V a finite subset of T with the property that $x_0T \subseteqq \varphi^{-1}[V]F_V$. The existence of these finite sets is guarenteed by the fact that x_0 is almost periodic in X.

To show (c), that $(X/R(H), H)$ is not a PI flow, we use Lemma

1.1 and Theorem 1.1. We would like to show the existence of a $\langle y_n \rangle \in \times_{i=1}^{\infty} Y$ that is almost periodic and has dense range $\{y_n\}$ in K^*. In actuality we are forced into $\times_{i=1}^{\infty}(\times_{j=1}^{\infty} Y)$ in order to index things properly. We do this by inductively constructing $\langle\langle y_n \rangle_m \rangle$. Suppose H is a subgroup of T and suppose we have $K_n = \{k_{i,j} \mid 1 \leqq j \leqq n, 1 \leqq i < \infty\}$ a subset of Ku. Let $\mathscr{B}_n$ be the basis for $\times_{j=1}^{n}(\times_{i=1}^{\infty} X/R(H))$ formed from $\mathscr{B}(H)$. Let $\langle\langle k_i \rangle_j \rangle$ be the element of $\times_{j=1}^{n}(\times_{i=1}^{\infty} X)$ whose jth coordinate has $k_{i,j}$ as its ith coordinate. Then $\langle\langle k_i \rangle_j \rangle$ is an almost periodic point of $\times_{j=1}^{n}(\times_{i=1}^{\infty} X)$ since $\langle\langle k_i \rangle_j \rangle u = \langle\langle k_i \rangle_j \rangle$. Then, for each $U \in \mathscr{B}_n$, we will denote by E_U a finite subset of T such that $\langle\langle k_i \rangle_j \rangle T \subseteq \hat{\varphi}^{-1}[U]E_U$ where $\hat{\varphi}$ is the map from $\times_{j=1}^{n}(\times_{i=1}^{\infty} X)$ into $\times_{j=1}^{n}(\times_{i=1}^{\infty} X/R(H))$ induced by φ.

LEMMA 2.1. *Suppose H is a countable subgroup of T and $K(n) = \{k_{i,j} \mid 1 \leqq j \leqq n, 1 \leqq i < \infty\} \subseteq Ku$. Then there is a countable subgroup H^* of T and a set $K(n+1) = \{k^*_{i,j} \mid 1 \leqq j \leqq n+1, 1 \leqq i < \infty\} \subseteq Ku$ such that*

(1) $H \subseteq H^*$.

(2) $k^*_{i,j} = k_{i,j}$ *for* $1 \leqq j \leqq n, 1 \leqq i < \infty$.

(3) $\varphi(x_0 H^*)$ *is dense in* $X/R(H)$.

(4) $\varphi(K(n+1))$ *is dense in* $\varphi(K)$.

(5) *For any* $V \in \mathscr{B}(H)$, $\varphi[(\varphi^{-1}[V] \times \varphi^{-1}[V])H^*(x_0) \cap K]$ *is dense in* $\varphi(K)$.

(6) $\cup \{F_V \mid V \in \mathscr{B}(H)\} \subseteq H^*$.

(7) $\cup \{E_U \mid U \in \mathscr{B}_{n+1}(H)\} \subseteq H^*$.

Proof. For each $V \in \mathscr{B}(H)$ let $t_V \in T$ such that $x_0 t_V \in \varphi^{-1}[V]$ and let $L_1 = \{t_V \mid V \in \mathscr{B}(H)\}$. Then $\varphi(x_0 L_1)$ is dense in $\varphi(X)$ and L_1 is countable.

For each $W \in \mathscr{B}(H)$ such that $K \cap \varphi^{-1}[W] \neq \phi$ and for each $V \in \mathscr{B}(H)$ let $t = t(V, W)$ be such that $[(\varphi^{-1}[V] \times \varphi^{-1}[V])t](x_0) \cap K \cap \varphi^{-1}[W] \neq \phi$. Let $L_2 = \{t(V, W) \mid K \cap \varphi^{-1}[W] \neq \phi, V, W \in \mathscr{B}(H)\}$. Then $\varphi[(\varphi^{-1}[V] \times \varphi^{-1}[V])L_2)(x_0) \cap K]$ is dense in $\varphi(K)$ and L_2 is countable.

Let $L_3 = \cup \{F_V \mid V \in \mathscr{B}(H)\}$. Then L_3 is countable.

Since $\varphi(Ku)$ is dense in $\varphi(K)$ we can choose a countable subset, K', of Ku such that $\varphi(K')$ is dense in $\varphi(K)$. Let K' be indexed by $K' = \{k'_i\}_{i=1}^{\infty}$. Let $\langle\langle k^*_i \rangle_j \rangle$ be the element of $\times_{j=1}^{n+1}(\times_{i=1}^{\infty} X)$ whose jth coordinate, $1 \leqq j \leqq n$, has $k_{i,j}$ as its ith coordinate and whose $(n+1)$th coordinate has k'_i as its ith coordinate. Then $\langle\langle k^*_i \rangle_j \rangle$ is an almost periodic point of $\times_{j=1}^{n+1}(\times_{i=1}^{\infty} X)$. For each $U \in \mathscr{B}_{n+1}(H)$ choose E_U as described above. Let $K(n+1) = K(n) \cup K'$ with the obvious indexing. Let $L_4 = \cup \{E_U \mid U \in \mathscr{B}_{n+1}(H)\}$. Then L_4 is countable.

Let H^* be the subgroup of T generated by $H \cup L_1 \cup L_2 \cup L_3 \cup L_4$. Then, since each of these sets is countable, H^* is countable.

It is now easily checked that $K(n+1)$ and H^* satisfy the requirements of the lemma.

THEOREM 2.1. *Suppose P is a transferable property of flows. Then every minimal P-flow is a* PI *flow iff every metric minimal P-flow is a* PI *flow.*

Proof. Clearly if every minimal P-flow is a PI flow then every metric minimal P-flow is a PI flow.

Suppose (X, T) is a minimal P-flow and (X, T) is not PI. Then, from Theorem 1.1 and Lemma 1.2, there is a nontrivial closed set $K \subseteqq X$, a point $x_0 \in X$, and an idempotent $u \in J(T)$, such that $c(Ku) = K$ and $\alpha T(x_0) \cap K$ is dense in K for all indices α on X. Choose a pseudo-metric d as in the above discussion.

Let $k_1 \in Ku$, $K_1 = \{k_1\}$ and $\langle k_{i,1} \rangle \in \times_{i=1}^{\infty} X$ so that $k_{i,1} = k_1$. Let $H_1 = \{e\}$ where e is the group identity. Using Lemma 2.1 construct a countable subgroup $H_2 \subseteqq T$ and a countable set $K_2 = \{k_{i,j} \mid j = 1, 2; 1 \leqq i < \infty\} \subseteqq Ku$ satisfying the lemma. Proceed by induction to construct sequences $\{H_n\}_{n=1}^{\infty}$ and $\{K_n = \{k_{i,j} \mid 1 \leqq j \leqq n, 1 \leqq i < \infty\}\}$ such that $H_n \subseteqq H_{n+1}$ for $n \geqq 1$, $K_n \subseteqq K_{n+1}$ for $n \geqq 1$, and for each n, letting $H = H_n$, $K(n) = K_n$, $H^* = H_{n+1}$ and $K(n+1) = K_{n+1}$, the conditions of Lemma 2.1 are satisfied. For each n, let $X/R(H_n) = X_n$, $\varphi_n: X \to X_n$. For $i \leqq j$ let $\psi_i^j: X_j \to X_i$ be the map induced by $H_i \subseteqq H_j$. Let $Y = \operatorname{inv\,lim}\{X_n, \psi_i^j\}$ and $H = \bigcup_{n=1}^{\infty} H_n$. Then (Y, H) is a transformation group, Y is a compact metric space, and (Y, H) is $(X/R(H), H)$. Let $\varphi: X \to Y$ be the quotient map and $\pi_n: Y \to X_n$ the projection map, $n \geqq 1$.

REMARK 1. (Y, H) is minimal.

We prove this remark by showing that $y_0 = \varphi(x_0)$ is a transitive point which is almost periodic. The collection of sets of the form $\pi_n^{-1}[V_n]$ where n ranges over the positive integers and V_n ranges over $\mathscr{B}_n$ is a basis for Y (see [2]). Let $\pi_n^{-1}[V_n]$ be a basic open set. Since $\varphi_n(x_0 H_{n+1})$ is dense in X_n, there is a $t \in H_{n+1}$ such that $\varphi_n(x_0 t) \in V_n$. But, since $H_{n+1} \subseteqq H$, $\varphi(x_0 t) \in \varphi(x_0)H \cap \pi_n^{-1}[V_n]$ and y_0 is transitive.

To see that y_0 is almost periodic, let $V_n \in \mathscr{B}(H_n)$. We claim $y_0 H \subseteqq \pi_n^{-1}[V_n]F_{V_n}$. Indeed, by construction $x_0 T \subseteqq \varphi_n^{-1}[V_n]F_{V_n}$; so for $h \in H$, $x_0 h \in \varphi_n^{-1}[V_n]F_{V_n}$ and thus $\varphi(x_0 h) \in \varphi[\varphi_n^{-1}[V_n]F_{V_n}] = \varphi[\varphi_n^{-1}[V_n]]F_{V_n} = \pi_n^{-1}[V_n]F_{V_n}$. Now since $F_{V_n} \subseteqq H_{n+1} \subseteqq H$ and F_{V_n} is finite, y_0 is almost periodic.

REMARK 2. (Y, H) is a P-flow

Since P is a transferable property there is an $x^* \in X$ such that $(c(x^*H), H)$ is a P-flow, and since (Y, H) is minimal, $\varphi: (c(x^*H), H) \to (Y, H)$ is surjective. Since $(c(x^*H), H)$ is a P-flow and P is transferable, (Y, H) is a P-flow.

REMARK 3. Let $K' = \bigcup_{n=1}^{\infty} K_n$. Then $\varphi(K')$ is dense in $\varphi(K)$. We leave the proof of this remark to the reader.

REMARK 4. There is an idempotent $u^* \in J(H)$ such that $\varphi(K')u^* = \varphi(K')$.

Define $\langle\langle a_i \rangle_j\rangle \in \times_{j=1}^{\infty} (\times_{i=1}^{\infty} Y)$ by $a_{i,j} = \varphi(k_{i,j}) \in \varphi(K')$. By Lemma 1.1, we can complete the proof of this remark by demonstrating that $\langle\langle a_i \rangle_j\rangle$ is almost periodic in $\times_{j=1}^{\infty} (\times_{i=1}^{\infty} Y)$.

A basic open set in $\times_{j=1}^{\infty} (\times_{i=1}^{\infty} Y)$ is of the form $f_n^{-1}[V_n]$ where $f_n: \times_{j=1}^{\infty} (\times_{i=1}^{\infty} Y) \to \times_{j=1}^{n} (\times_{i=1}^{\infty} X_n)$ is the obvious projection map and V_n is a basic open set in $\times_{j=1}^{n} (\times_{i=1}^{\infty} X_n)$. By construction, for every such V_n, $\langle\langle k_i \rangle_j\rangle T \subseteqq \hat{\varphi}_n^{-1}[V_n]E_{V_n}$ where $E_{V_n} \subseteqq H$, $1 \leqq i < \infty$, $1 \leqq j \leqq n$, and $\hat{\varphi}_n: \times_{j=1}^{n} (\times_{i=1}^{\infty} X) \to \times_{j=1}^{n} (\times_{i=1}^{\infty} X_n)$ is the homomorphism induced by φ_n. Let g_n be the projection from $\times_{j=1}^{\infty} (\times_{i=1}^{\infty} X)$ onto $\times_{j=1}^{n} (\times_{i=1}^{\infty} X)$. Then, $\langle\langle k_i \rangle_j\rangle H \subseteqq \langle\langle k_i \rangle_j\rangle T \subseteqq g_n^{-1}\hat{\varphi}_n^{-1}[V_n]E_{V_n}$, $1 \leqq i < \infty$, $1 \leqq j < \infty$. Therefore, $\langle\langle a_i \rangle_j\rangle H \subseteqq f_n^{-1}[V_n]E_{V_n}$ and since E_{V_n} is finite and a subset of H, $\langle\langle a_i \rangle_j\rangle$ is almost periodic, completing the proof of Remark 4.

REMARK 5. For every index β on Y, $\beta H(y_0) \cap \varphi(K)$ is dense in $\varphi(K)$.

Let β be an index on Y. Then there is a basic open set $\pi_n^{-1}[V_n]$ such that $(\pi_n^{-1}[V_n] \times \pi_n^{-1}[V_n])H \subseteqq \beta H$. Let $\pi_m^{-1}[W_m]$ be any basic open set for which $\pi_m^{-1}[W_m] \cap \varphi(K) \neq \phi$. Let $n^* = \max(m, n)$ and let $V_{n^*} = (\psi_n^{n^*})^{-1}[V_n]$ and $W_{n^*} = (\psi_m^{n^*})^{-1}[W_m]$. Then $\pi_m^{-1}[W_m] = \pi_{n^*}^{-1}[W_{n^*}]$ and $\pi_n^{-1}[V_n] = \pi_{n^*}^{-1}[V_{n^*}]$. By construction, $\varphi_{n^*}[(\varphi_{n^*}^{-1}[V_{n^*}] \times \varphi_{n^*}^{-1}[V_{n^*}])H_{n^*}(x_0) \cap K] \cap W_{n^*}$ is not empty, say it contains $\varphi(x^*)$ where $x^* \in (\varphi_{n^*}^{-1}[V_{n^*}] \times \varphi_{n^*}^{-1}[V_{n^*}])H_{n^*}(x_0) \cap K$. Then, $\varphi(x^*)$ is in $\pi_{n^*}^{-1}[W_{n^*}]$ and in $\varphi(K)$. There is an $h \in H_{n^*}$ such that $(x^*, x_0)h^{-1} \in \varphi_{n^*}^{-1}[V_{n^*}] \times \varphi_{n^*}^{-1}/[V_{n^*}]$ and hence $(\varphi(x^*), y_0)h^{-1} \in \pi_{n^*}^{-1}[V_n] \times \pi_{n^*}^{-1}[V_n]$. Since $h \in H_{n^*} \subseteqq H$, $\beta H(y_0) \cap \varphi(K) \cap \pi_m^{-1}[W_m]$ contains $\varphi(x^*)$. Thus, $\beta H(y_0) \cap \varphi(K)$ is dense in $\varphi(K)$.

REMARK 6. Combining Remarks 1, 3, 4, and 5 we conclude that, since K^* is nontrivial by the original choice of d, (Y, H) is not PI. By Remark 2, (Y, H) is a P-flow. Thus, if there is a non-PI minimal P-flow there is a non-PI metric minimal P-flow, completing the proof of the theorem.

It is well known that any metric minimal flow that contains a point x_0 for which $P[x_0] = \{x \in X \mid x \text{ is proximal to } x_0\}$ is countable is a PI flow [5, 6]. In particular, if (X, T) has a distal point this is the PI version of the Veech structure theorem. (The HPI version states that point-distal flows which are quasi-separable are HPI, the proximal maps in the PI tower are highly-proximal [1].) The property "has a point whose proximal cell is countable" is transferable and thus, by the above theorem and known results, any minimal flow with this property is PI. Thus, the PI version of the Veech structure theorem holds in general. We will also show that the HPI version holds in general.

LEMMA 2.2. *The property*: (X, T) *contains a point* x_0 *such that* $P[x_0] = \{x \in X \mid x \text{ is proximal to } x_0\}$ *is countable; is transferable.*

Proof. Suppose (X, T) has this property. Let $\varphi: (X, T) \to (Y, T)$ be a transformation group homomorphism and (Y, T) be minimal. Then $\varphi(P[x_0]) = P[\varphi(x_0)]$ and hence (Y, T) has the property.

Suppose S is a subgroup of T. Consider $(c(x_0S), S)$. Clearly if y is S-proximal to x_0 then y is T-proximal to x_0 and hence $(c(x_0S), S)$ has the property. Thus it is transferable.

COROLLARY 2.1. *Any minimal set that has a point with countable proximal cell is* PI.

Proof. Use X.7.2 of [6], Theorem 2.1, and Lemma 2.2.

COROLLARY 2.2. (*General Veech structure theorem*-PI *Version*) *A point-distal flow is* PI.

Note that Corollary 2.1 also implies the general Furstenberg structure theorem. We conclude this paper by proving the HPI version of the general Veech structure theorem. The following lemma is needed.

LEMMA 2.3. *Suppose* (X, T) *and* (Y, T) *are minimal and* φ: $(X, T) \to (Y, T)$ *is a homomorphism. Suppose* (X, T) *is* PI *and* $D(\varphi)$ *is dense in* $R(\varphi)$. *Then either* φ *is an isomorphism or* $S(\varphi) \neq R(\varphi)$.

Proof. Fix $u \in J$, and let x_0, y_0 be points in X and Y respectively such that $x_0u = x_0$, $y_0u = y_0$, and $\varphi(x_0) = y_0$. Since $D(\varphi)$ is dense in $R(\varphi)$, φ is full by 4.3 of [5]. Thus there exists a flow Z such that the diagram

$$\begin{array}{ccc} X & & \\ {\scriptstyle\alpha}\downarrow & \searrow^{\varphi} & \\ Z & \xrightarrow[\beta]{} & Y \end{array}$$

commutes, β is almost periodic, and $G(Z) = G(X)H(G(Y))$ where $G(X) = G(X, x_0)$, $G(Y) = G(Y, y_0)$, $G(Z) = G(Z, \alpha(x_0))$ and H is as defined in 1.9 of [6], page 117.

Now suppose $S(\varphi) = R(\varphi)$. Then β is an isomorphism and thus $G(Y) = G(X)H(G(Y))$. For each ordinal λ let $H_{\lambda+1}(G(Y)) = H(H_\lambda(G(Y)))$ and if λ is a limit ordinal then $H_\lambda(G(Y)) = \cap \{H_\gamma(G(Y)) : \gamma < \lambda\}$. It then follows by induction and X.4.1 of [6] that $G(Y) = G(X)H_\lambda(G(Y))$ for all ordinals λ. Let η be the least ordinal for which $G_\eta = G_\infty$. Then $H_\eta(G(Y)) \subseteqq G_\infty$. We then have

$$G(Y) = G(X)H_\eta(G(Y)) \subseteqq G(X)G_\infty = G(X)$$

since (X, T) is PI.

Therefore $G(Y) = G(X)$ and φ is proximal. Thus $D(\varphi) \subseteqq P(\varphi)$ and hence, since $D(\varphi)$ is dense in $R(\varphi)$, $R(\varphi) = \Delta$ and φ is an isomorphism.

Suppose (X, T) is a PI flow. To construct the cannonical PI tower for (X, T) we begin with the map $X \rightarrow \{x\}$ from X to the one point flow, and use the prime construction [5, 6] to form the diagram

$$\begin{array}{ccc} X' & \xrightarrow{\beta} & X \\ {\scriptstyle \varphi'}\downarrow & & \downarrow \\ Y' & \xrightarrow[\alpha]{} & \{x\} \end{array} .$$

Since (X, T) is PI, either φ' is an isomorphism or $S(\varphi') \neq R(\varphi')$ and a nontrivial Z' can be inserted so that the following diagram commutes.

$$\begin{array}{ccccc} X' & & \xrightarrow{\quad\beta\quad} & & X \\ {\scriptstyle \gamma}\downarrow & \searrow^{\varphi'} & & & \downarrow \\ Z' & \xrightarrow[\delta]{} & Y' & \xrightarrow[\alpha]{} & \{x\} \end{array}$$

The prime construction is then applied to γ. The process, inductively, yeilds a PI tower for X.

An alternative to the prime construction of Ellis, Glasner, and Shapiro is the star construction originally due to Veech [8] (see also [9]). In general, the star construction yields a diagram

$$\begin{array}{ccc} X^* & \xrightarrow{\beta} & X \\ {\scriptstyle \varphi^*}\downarrow & & \downarrow{\scriptstyle \varphi} \\ Y^* & \xrightarrow[\alpha]{} & Y \end{array}$$

where α and β are highly proximal and φ^* is open, but in general $S(\varphi^*)$ may be equal to $R(\varphi^*)$ without φ^* being an isomorphism. When the star construction can be used, the constructed tower is an HPI tower [1]. We show that for point-distal flows the star construction can be used to construct an HPI tower and hence point-distal flows are HPI.

THEOREM 2.2. (*General Veech structure theorem*-HPI *version*) *Every point-distal flow is* HPI.

Proof. Recall that if $\varphi: X \to Y$ then $x_0 \in X$ is a φ-distal point iff $\varphi(x) = \varphi(x_0)$ implies x and x_0 are distal. We first note that if $\varphi: X \to Y$ is open and x_0 is a φ-distal point, then $D(\varphi)$ is dense in $R(\varphi)$, and that if $\varphi: X \to Y$ has x_0 as a φ-distal point then $\varphi^*: X^* \to Y^*$ has a φ^* distal point (see, for example, [4] for proofs).

We then proceed by transfinite induction, using pointed flows, to construct a diagram

$$\begin{array}{ccccccccccccc}
X = X_0 & \longleftarrow & & & X_1 & \longleftarrow & & & X_2 \cdots X_\alpha & \longleftarrow & & & X_{\alpha+1} \longleftarrow \\
{\scriptstyle c_0}\downarrow & & & \swarrow{\scriptstyle d_0} & \downarrow{\scriptstyle c_1} & & & \swarrow{\scriptstyle d_1} & \downarrow{\scriptstyle c_2} \quad \downarrow{\scriptstyle c_\alpha} & & & \swarrow{\scriptstyle d_\alpha} & \downarrow{\scriptstyle c_{\alpha+1}} \\
\{x\} = Z_0 & \underset{a_0}{\longleftarrow} & Y_1 & \underset{b_0}{\longleftarrow} & Z_1 & \underset{a_1}{\longleftarrow} & Y_2 & \underset{b_1}{\longleftarrow} & Z_2 \cdots Z_\alpha & \underset{a_\alpha}{\longleftarrow} & Y_{\alpha+1} & \underset{b_\alpha}{\longleftarrow} & Z_{\alpha+1} \longleftarrow
\end{array}$$

where for any nonlimit ordinal i, $X_i = X_{i-1}^*$, $Y_i = Z_{i-1}^*$, $d_i = c_{i-1}^*$. The existence of Z_i, b_{i-1}, and c_i are guarenteed by the remarks above, the fact that X is point-distal and hence PI by Corollary 2.2, and by Lemma 2.3. If i is a limit ordinal $X_i \xrightarrow{c_i} Z_i$ is the inverse limit of the diagram for all $\lambda < i$ and hence $X_i \xrightarrow{c_i} Z_i$ will have a c_i distal point. If η is the least ordinal for which $R(d_\eta) = S(d_\eta)$, then d_η is an isomorphism and the diagram yields an HPI tower for X, completing the proof.

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